Remarks on lens space surgery

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Abstract

Let q be a fixed integer. We show that the set of lens spaces whose second parameter is q and which are homeomorphic to $S^3_{-p}(K)$ for a knot K is finite if and only if q is a non-square number. We partially solve Teragaito's conjecture, which is that lens spaces that a Klein bottle cannot be constructed by any hyperbolic knot. Moreover, we directly show the correction term coincides with Fukumoto and Furuta's w-invariant by using a correction term formula, in an appendix. This coincidence has been shown by M. Ue recently.

1 Introduction

In this paper we define $L(p,q)$ to be $S^3_{-p/q}(U)$, which is $-p/q$-Dehn surgery of the unknot $U \subset S^3$. For any lens space $L(p,q)$ we call the parameters p and q the *first parameter* and the *second parameter* respectively. Throughout this paper we assume that the first parameter p is always positive. When a lens space $L(p,q)$ is homeomorphic to $S^3_{-p}(K)$ for a knot K, we say that K admits negative lens surgery (or simply lens surgery).

P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó in [6] have shown that if $L(p,1)$ is homeomorphic to $S^3_{-p}(K)$ for a knot K, then K must be isotopic to the unknot. J. Rasmussen in [13] proved that if $L(p,2)$ or $L(p,3)$ is homeomorphic to $S^3_{-p}(K)$ for a knot K then the lens space must be $L(7,2)$, $L(11,3)$, or $L(13,3)$. The author proved in the previous paper [12] that if $L(p_1,-1)$, $L(p_2,-2)$, or $L(p_3,-3)$ is obtained by a Dehn surgery of a knot K in S^3, then the p_i's ($i=1,2,3$) satisfy $p_1 = 2,5$, $p_2 = 3,9,11$, or $p_3 = 4,7,13,14,19$. Combining this classification with the result in [5] we can prove that the knots K which yield $L(p,-1)$ are either the unknot and $p=2$ or the trefoil knot and $p=5$. If q is a square number r^2, then there exist infinitely many lens spaces $L(rs\pm 1, r^2)$ each of which is $-(rs\pm 1)$-surgery of the (r,s)-torus knot (where $r,s>0$). In Section 3 we shall show the converse proposition.

Theorem 1.1 *Let q be an integer. The set $\{p \in \mathbb{Z}_{\geq 0} | L(p,q) = S^3_{-p}(K)$ for a knot $K\}$ is an infinite set if and only if q is a square of an integer.*

R. Fintushel and R. Stern's theorem in [3] says that if a Dehn surgery over a homology sphere yields a lens space, then q is always the quadratic residue mod p. In Theorem 1.1 we may assume that the condition $0 \leq |q| < p$ is satisfied. Here x is said to be *reduced positively* (or *negatively*) by mod p if x is $0 \leq x < p$ (or $-p < x \leq 0$). This theorem contains the assertion that the lens spaces which have negative second parameters and which are obtained by negative Dehn surgery are finite.

The result in [6] showed that the lens space $L(2,1)$, which includes the non-orientable surface $\mathbb{RP}^2$, can be obtained by the unknot only. M. Teragaito in [15] determined genus one knots which yield manifolds containing a Klein bottle. He has also conjectured that lens spaces containing a Klein bottle cannot be constructed from Dehn surgery of any hyperbolic knot.

Conjecture 1.1 ([14]) *If a lens space obtained by Dehn surgery of a knot contains a Klein bottle, then the knot is non-hyperbolic.*

K. Ichihara and T. Saito in [11] have solved this conjecture in the case of doubly primitive knots. The notion of doubly primitive knots is defined in [1]. It is conjectured that doubly primitive knots are all knots admitting lens surgery. We shall prove the following theorem in Section 4.

Theorem 1.2 *If a lens space obtained by negative or positive Dehn surgery of a knot contains a Klein bottle, then the lens space is either* $L(4,\pm1)$,$L(16,\pm7)$ *or* $L(20,\pm9)$.

These lens spaces are the same as ones obtained in [11]. But the author does not know whether these lens spaces, except $L(4,\pm1)$, can be obtained from non-doubly primitive, hyperbolic knots.

As the appendix we will show that the correction term by P. Ozsváth and Z. Szabó, using the formulae proven in [12], coincides with the theta divisor invariant [10] by P. Ozsváth and Z. Szabó and the w-invariant [16] by Y. Fukumoto and M. Furuta for all lens spaces.

2 The computations of correction term and lens surgery

We shall review Heegaard Floer homology and correction terms of lens spaces and results in [12]. We define the notations used here. The floor function $\lfloor\alpha\rfloor$ is the largest integer less than or equal to α. The bracket $[\beta]_p$ stands for the positive reduction of $\beta \bmod p$. We denote by x' the inverse of $x\in(\mathbb{Z}/p\mathbb{Z})^\times$ and always reduce x' positively as mod$p$.

Let Y be an oriented closed 3-manifold. P. Ozsváth and Z. Szabó have defined in [8] Heegaard Floer homologies $HF^\infty(Y,\mathfrak{s}),HF^+(Y,\mathfrak{s})$ and $HF^-(Y,\mathfrak{s})$, which are topological invariants with respect to spin^c-structure $\mathfrak{s}$. They also gave the long exact sequence

$$\cdots\to HF^-(Y,\mathfrak{s})\to HF^\infty(Y,\mathfrak{s})\overset{\pi_*}{\to}HF^+(Y,\mathfrak{s})\to HF^-(Y,\mathfrak{s})\to\cdots.$$

Moreover, for a rational homology sphere Y they introduced a grading of these homologies as in [7]. The *correction term* $d(Y,\mathfrak{s})$ is defined to be the minimal grading of the image $HF^\infty(Y,\mathfrak{s})\overset{\pi_*}{\to}HF^+(Y,\mathfrak{s})$. If Y is a lens space, then we can identify $\mathrm{Spin}^c(Y)$ with $\mathbb{Z}/p\mathbb{Z}$. This is called the canonical ordering in [7] and is determined by genus one Heegaard decomposition of the lens space. For any integer i the correction term $d(L(p,q),i)$ can be computed by using the following recursive formula for $0\le i<p+q$ in [7]:

$$d(L(p,q),i)=\frac{pq-(2i+1-p-q)^2}{4pq}-d(L(q,r),j),$$

where $0<q<p$, $r:=[p]_q$ and $j:=[i]_q$ are satisfied.

On the other hand, the author defined the non-recursive formula of $d(L(p,q),i)$ in [12] as follows:

$$d(L(p,q),i)=-3s(q,p)+\frac{1-p}{2p}+\frac{[i]_p}{p}-2\sum_{j=1}^{i}\left(\left(\frac{q'j}{p}\right)\right),\tag{1}$$

$$((\alpha)):=\begin{cases}\alpha-[\alpha]-\frac{1}{2} & \alpha\notin\mathbb{Z}\\ 0 & \alpha\in\mathbb{Z},\end{cases}$$

where $s(q,p)$ is the Dedekind sum.

For each pair of four positive integers p,q,h, and k we denote $\Phi^k_{p,q}(h)$ by

$$\Phi^k_{p,q}(h):=\#\{j\in\{1,2,\cdots,h'\}\,|\,0<[qj-k]_p\le h\},$$

where h and h' are positively reduced integers satisfying $hh'=1 \bmod p$. The Alexander polynomial $\Delta_K(t)$ of K is symmetrized as $\Delta_K(t)=\Delta_K(t^{-1})$ and the i-th coefficient of $\Delta_K(t)$ is denoted by $a_i(K)$. We put $\tilde{a}_i(K):=\sum_{j\equiv i \mod p}a_i(K)$.

Proposition 2.1 ([12]) *Suppose that a lens space $L(p,q)$ is homeomorphic to $S^3_{-p}(K)$. Then there exists an integer $h\in(\mathbb{Z}/p\mathbb{Z})^\times$ satisfying $h^2=q \mod p$ such that*

$$\tilde{a}_i(K)=-m+\Phi_{p,q}^{hi+c}(h)$$

holds for any i, where $c:=\frac{(h+1+p)(h-1)}{2}$ and $m=\frac{hh'-1}{p}$.

The h has four choices among $\{h,h',p-h,p-h'\}$. If we replace h with one of them, then the same formula with respect to the new choice holds. We will prove the following proposition. The second equality was also proven in [12].

Proposition 2.2 *Let p and q be a pair of coprime integers with $0<q<p$. Suppose that the integer $0<h<p$ is one of the solutions to $x^2=q \mod p$. Let w be the integer with $qh'=h+pw$. Then we have*

$$\Phi_{p,q}^{-1}(h)=-2\sum_{j=1}^{w}\left[\frac{pj}{q}\right]+(w+1)(h'-1). \tag{2}$$

On the the hand, suppose that the integer $0<h<p$ is one of the solutions to $x^2=-q \mod p$. Let w be the integer with $qh'+h=pw$. Then we have

$$\Phi_{p,-q}^{-1}(h)=2\sum_{j=1}^{w-1}\left[\frac{pj}{q}\right]-(w-1)(h'-1). \tag{3}$$

Proof. We prove the first part of the proposition. Let w be the integer with $qh'=h+pw$. Then we have

$$\begin{aligned}
\Phi_{p,q}^{-1}(h) &= \#\{j\in\{1,2,\cdots,h'\}|0\le[qj]_p<h\}\\
&= \left[\frac{h}{q}\right]-\left[\frac{0}{q}\right]+\left[\frac{h+p}{q}\right]-\left[\frac{p}{q}\right]+\cdots+\left[\frac{h+pw}{q}\right]-\left[\frac{pw}{q}\right]-1\\
&= \sum_{j=0}^{w}\left(\left[\frac{h+pj}{q}\right]-\left[\frac{pj}{q}\right]\right)-1\\
&= \sum_{j=0}^{w}\left(\left[h'-\frac{p(w-j)}{q}\right]-\left[\frac{pj}{q}\right]\right)-1\\
&= \sum_{j=0}^{w}\left(h'-\left[\frac{p(w-j)}{q}\right]-1-\left[\frac{pj}{q}\right]\right)\\
&= -2\sum_{j=1}^{w}\left[\frac{pj}{q}\right]+(w+1)(h'-1).
\end{aligned}$$

We now prove the second part. Let w be the integer with $qh'+h=pw$. Then we have

$$\begin{aligned}
\Phi_{p,-q}^{-1}(h) &= \#\{j\in\{1,2,\cdots,h'\}|0\le[-qj]_p<h\}\\
&= \#\{j\in\{1,2,\cdots,h'\}|p-h<[qj]_p\}\\
&= \left[\frac{p}{q}\right]-\left[\frac{p-h}{q}\right]+\left[\frac{2p}{q}\right]-\left[\frac{2p-h}{q}\right]+\cdots+\left[\frac{p(w-1)}{q}\right]-\left[\frac{p(w-1)-h}{q}\right]\\
&= \sum_{j=1}^{w-1}\left(\left[\frac{pj}{q}\right]-\left[\frac{pj-h}{q}\right]\right)\\
&= \sum_{j=1}^{w-1}\left(\left[\frac{pj}{q}\right]-\left[\frac{qh'-p(w-j)}{q}\right]\right)\\
&= 2\sum_{j=1}^{w-1}\left[\frac{pj}{q}\right]-(w-1)(h'-1).
\end{aligned}$$

□

Let a be the quotient of p divided by q and b the remainder. Then we have

$$\begin{aligned}
\Phi^{-1}_{p,q}(h) &= -2\sum_{j=1}^{w}\left[\frac{pj}{q}\right] + (w+1)(h'-1) \\
&= -2\sum_{j=1}^{w}\left[aj+\frac{bj}{q}\right] + (w+1)(h'-1) \\
&= -aw(w+1) - 2\sum_{j=1}^{w}\left[\frac{bj}{q}\right] + (w+1)(h'-1) \\
&= -aw(w+1) + (w+1)h' + \alpha_1,
\end{aligned}$$

where $\alpha_1 = -2\sum_{j=1}^{w}\left[\frac{bj}{q}\right] - w - 1$. In the same way

$$\begin{aligned}
\Phi^{-1}_{p,-q}(h) &= 2\sum_{j=1}^{w-1}\left[\frac{pj}{q}\right] - (w-1)(h'-1) \\
&= 2\sum_{j=1}^{w-1}\left[aj+\frac{bj}{q}\right] - (w-1)(h'-1) \\
&= aw(w-1) + 2\sum_{j=1}^{w-1}\left[\frac{bj}{q}\right] - (w-1)(h'-1) \\
&= aw(w-1) - (w-1)h' + \alpha_2,
\end{aligned}$$

where $\alpha_2 = 2\sum_{j=1}^{w-1}\left[\frac{bj}{q}\right] + (w-1)$.

3 Proof of Theorem 1.1

We shall prove Theorem 1.1 in the cases where the fixed parameter q is positive reduction and q is negative reduction separately.

Lemma 3.1 *Let q be a fixed positive integer. Then the number of the lens spaces whose second parameters are q and which are obtained by a negative $(-p)$-Dehn surgery is finite, unless q is a square number.*

Proof. Let q be a fixed integer. We just have to consider lens spaces satisfying $0 < q < p$, where p is the first parameter and q is the second parameter, because the choices of p are clearly finite otherwise. Suppose that the lens space $L(p,q)$ is homeomorphic to $S^3_{-p}(K)$. Then by Proposition 2.1 and 2.2 there exist integers h, c, m, w such that

$$\begin{aligned}
\tilde{a}_{-h'c-h'}(K) &= -m + \Phi^{-1}_{p,q}(h) \\
&= -m - aw(w+1) + (w+1)h' + \alpha_1.
\end{aligned}$$

From the bound $|a_i(K)| \le 1$ in [9] and the genus bound $2g(K)-1 \le p$ in [6], $\tilde{a}_{-h'c-h'}(K)$ has $-1, 0, 1$ or 2. We set $\alpha_3 := q(\alpha_1 - \tilde{a}_{-h'c-h'}(K))$ for the same parameter α_1 as in Section 2. By the definition the value α_3 is bounded for the fixed integer q. Thus we have

$$\begin{aligned}
mq &= -aqw(w+1) + (w+1)h'q + \alpha_3 \\
&= -(p-b)w(w+1) + (w+1)(pw+h) + \alpha_3 \\
&= (w+1)h + bw(w+1) + \alpha_3.
\end{aligned}$$

We set $\alpha_4 := bw(w+1) + \alpha_3$. Thus we have

$$mqp = (w+1)hp + \alpha_4 p.$$

On the other hand, by the definition of m and w, we have

$$\begin{aligned} mqp &= q(hh' - 1) \\ &= h(pw + h) - q. \end{aligned}$$

Thus we obtain a quadratic equation $h^2 - hp - \alpha_4 p - q = 0$, and the discriminant of the quadratic equation is $(p + 2\alpha_4)^2 - 4(\alpha_4^2 - q)$. This value must be a square number X^2. Now suppose that q is not a square number, in particular $q \neq \alpha_4^2$. Then the solutions (p, X) to the equation $(p + 2\alpha_4)^2 - 4(\alpha_4^2 - q) = X^2$ are finite. Therefore the choices of p are finite, unless q is a square number. □

Lemma 3.2 *Let $-q$ be a fixed negative integer. Then the number of lens spaces whose second parameters are $-q$ and which are obtained by a negative $(-p)$-Dehn surgery is finite.*

Proof. For the same reason as in Lemma 3.1 we may assume that $0 < q < p$. Suppose that the lens space $L(p, -q)$ is homeomorphic to $S^3_{-p}(K)$. Then by Proposition 2.1 and 2.2, there exist integers h, c, m, w such that

$$\tilde{a}_{-h'c-h'}(K) = -m + aw(w-1) - (w-1)h' + \alpha_2.$$

Let a, b, and α_2 be the integers defined in Section 2. Setting $\alpha_5 := q(\alpha_2 - \tilde{a}_{-h'c-h'}(K))$, we have that α_5 is bounded for a fixed integer q by the same argument as for Lemma 3.1.

$$\begin{aligned} mq &= (p-b)w(w-1) - (w-1)h'q + \alpha_5 \\ &= pw(w-1) - (w-1)(pw-h) + \alpha_5 - bw(w-1) \\ &= (w-1)h + \alpha_6. \end{aligned}$$

Setting $\alpha_5 - bw(w-1)$ as α_6, we have

$$mqp = (w-1)hp + \alpha_6 p.$$

Thus we have

$$\begin{aligned} mqp &= q(hh' - 1) \\ &= h(pw - h) - q. \end{aligned}$$

Thus we get a quadratic equation $h^2 - hp + \alpha_6 p + q = 0$. The discriminant of the quadratic equation is $(p - 2\alpha_6)^2 - 4(\alpha_6^2 + q)$. This value must be a square number. Such p is bounded for a fixed q by the same reason as in Lemma 3.1. □

We shall prove Theorem 1.1 here.

Proof of Theorem 1.1. The positive q case in Theorem 1.1 follows from Lemma 3.1 and the negative q case follows from 3.2. □

4 Proof of Theorem 1.2

It is known that lens spaces which contain a Klein bottle have the form of $L(4n, 2n \pm 1)$ for some integer n (see [2]).

Lemma 4.1 *If $L(4n, 2n \pm 1)$ $(n > 0)$ is homeomorphic to $S^3_{-4n}(K)$ for a knot K, then such lens spaces are $L(4,1), L(20,9)$ or $L(16,9)$.*

Proof. Applying $p = 4n$ and $q = 2n - 1$ to Proposition 2.1, we can find the integer h satisfying the condition in Proposition 2.1. By definition of p and q we get the equality $2(q+1) = 4n = p \equiv 0 \bmod p$. Dividing $2(q+1) = 0 \bmod p$ by $2h$, we have $h + h' = 0 \bmod p$ or $h + h' = 2n \bmod p$. If $h + h' = 0 \bmod p$, then $q = h^2 = -hh' = -1 \bmod p$, hence this contradicts $n > 0$. Hence we have $h + h' = 2n \mod p$. By replacing, if necessary, h, h' with $p - h, p - h'$ we have $h + h' = 2n$ in $\mathbb{Z}$. Then $\Phi_{p,q}^{-1}(h) = \#\{j \in \{1, 2, \cdots, h'\} | 0 < [(2n-1)j + 1]_p \leq h\}$. The sequence $[(2n-1)j + 1]_p$ $(j = 1, 2, \cdots, h')$ falls into either of the two sequences:

$$2n,\ 4n-1,\ 2n-2,\ 4n-3,\ \cdots,\ 2n-h'+2,\ 4n-h'+1,$$

or

$$2n,\ 4n-1,\ 2n-2,\ 4n-3,\ \cdots,\ 4n-h'+2,\ 2n-h'+1.$$

The sequences correspond to the even h' case and the odd h' case respectively. None of 1, 2, $\cdots$, and h is contained in these sequences. Hence we get $\Phi_{p,q}^{-1}(h) = 0$. The coefficient $\tilde{a}_{-h'c-h'}(K) = -m$ is 0 or -1 by [12]. Thus we have $m = 0$ or 1. If $m = 0$, then $h + h' = 2n$ and $hh' = 1$. This implies $n = 1$. If $m = 1$, then $h + h' = 2n$ and $hh' = 4n + 1$. This implies $n = 5$.

Second, applying $p = 4n$ and $q = 2n + 1$ to Proposition 2.1, we can find the integer h satisfying the condition in Proposition 2.1. By definition of p and q we get $2(q-1) = 4n = p \equiv 0 \bmod p$. We have $h' - h = 2n$ and $0 < h < 2n$ by the same argument as above. Then we have $\Phi_{p,q}^{-1}(h) = \#\{j \in \{1, 2, \cdots, h'\} | 0 < [(2n+1)j + 1]_p \leq h\}$. If h' is even, then the sequence $[(2n+1)j + 1]_p$ is the following:

$$[2n+2]_p,\ 3,\ [2n+4]_p,\ 5\ \cdots,\ [2n+h']_p,\ h'+1.$$

Renumbering the sequence, we have

$$3,\ 5,\ 7,\ \cdots\ h'+1,\ 2n+2,\ 2n+4, \cdots,\ 4n-2,\ 0,\ 2,\ 4,\ \cdots,\ h.$$

Then the intersection with $1, 2, \cdots, h$ is

$$\{2,\ 3,\ 4,\ 5,\ \cdots, h\}.$$

If h' is odd, then the sequence $[(2n+1)j + 1]_p$ is the following:

$$[2n+2]_p,\ 3,\ [2n+4]_p,\ 5\ \cdots,\ h',\ [2n+h'+1]_p.$$

Renumbering the sequence, we have

$$3,\ 5,\ 7,\ \cdots\ h',\ 2n+2,\ 2n+4, \cdots,\ 4n-2,\ 0,\ 2,\ 4,\ \cdots,\ h+1.$$

Then the intersection with $1, 2, \cdots, h$ is

$$\{2,\ 3,\ 4,\ 5,\ \cdots, h\}.$$

Hence in the both cases $\Phi_{p,q}^{-1}(h) = h - 1$. The coefficient $\tilde{a}_{-h'c-h'}(K) = -m + h - 1$ is 0 or -1 by [12]. Thus we have $m = h - 1$ or $m = h$. If $m = h$, then $h' - h = 2n$ and $hh' = 4nh + 1$. This implies $n = 0$. If $m = h - 1$, then $h' - h = 2n$ and $hh' = 4n(h-1) + 1$. This implies $n = 4$.

Similarly, when $L(4n, 2n \pm 1) = S_p^3(K)$ for a knot K, such lens spaces are $L(4, 3)$, $L(20, 11)$, and $L(16, 7)$. □

Lens spaces $L(4, \pm 1)$, $L(20, \pm 9)$, and $L(16, \pm 7)$ can be constructed by torus knots. It is an open problem whether these lens spaces are obtained by Dehn surgery of knots other than torus knots.

Appendix: the representation of the correction term by trigonometric functions

In this appendix we show that the theta divisor invariant and the w-invariant coincide with the correction term for any lens space, by using Formula (1). Here we introduce the following well-known formulae:

$$\left(\left(\frac{\mu}{k}\right)\right) = \frac{1}{k}\sum_{\zeta}{}' \left(\frac{\zeta}{1-\zeta} + \frac{1}{2}\right)\zeta^{\mu}, \tag{4}$$

and

$$s(h,k) = -\frac{1}{k}\sum_{\zeta}{}' \frac{1}{(\zeta^h - 1)(\zeta - 1)} + \frac{k-1}{4k}, \tag{5}$$

where the summation with the prime $'$ means that ζ runs over complex numbers satisfying $\zeta^k = 1$ and $\zeta \neq 1$.

By using Formula (4) and (5),

$$\begin{aligned} -2s(q,p) - \frac{1-p}{2p} + \frac{i}{p} - 2\sum_{j=1}^{i}\left(\left(\frac{q'j}{p}\right)\right) &= \frac{2}{p}\sum_{\zeta}{}' \frac{1}{(\zeta^q - 1)(\zeta - 1)} + \frac{i}{p} - \frac{2}{p}\sum_{j=1}^{i}\sum_{\zeta}{}' \left(\frac{1}{1-\zeta} - \frac{1}{2}\right)\zeta^{q'j} \\ &= \frac{2}{p}\sum_{\zeta}{}' \frac{1}{(\zeta^q - 1)(\zeta - 1)} + \frac{i}{p} - \frac{2}{p}\sum_{\zeta}{}' \frac{1}{1-\zeta^q}\sum_{j=1}^{i}\zeta^j + \frac{1}{p}\sum_{j=1}^{i}\sum_{\zeta}{}' \zeta^j \\ &= \frac{2}{p}\sum_{\zeta}{}' \frac{1}{(\zeta^q - 1)(\zeta - 1)} + \frac{i}{p} - \frac{2}{p}\sum_{\zeta}{}' \frac{\zeta(\zeta^i - 1)}{(1-\zeta^q)(\zeta - 1)} + \frac{1}{p}\sum_{j=1}^{i}(-1) \\ &= \frac{2}{p}\sum_{\zeta}{}' \frac{(1 + \zeta^{i+1} - \zeta)}{(\zeta^q - 1)(\zeta - 1)}. \end{aligned} \tag{6}$$

Here, using the following equality

$$\begin{aligned} \sum_{\zeta}{}' \frac{1-\zeta}{(\zeta^q - 1)(\zeta - 1)} &= \sum_{\zeta}{}' \frac{1}{1-\zeta} \\ &= \frac{p-1}{2}, \end{aligned} \tag{7}$$

from (6) and (7) we have

$$-2s(q,p) + \frac{1-p}{2p} + \frac{i}{p} - 2\sum_{j=1}^{i}\left(\left(\frac{q'j}{p}\right)\right) = \frac{2}{p}\sum_{\zeta}{}' \frac{\zeta^{i+1}}{(\zeta^q - 1)(\zeta - 1)}.$$

Thus from Formula (1),

$$\begin{aligned} d(L(p,q),i) &= -s(q,p) + \frac{2}{p}\sum_{\zeta}{}' \frac{\zeta^{i+1}}{(\zeta^q - 1)(\zeta - 1)} \\ &= -s(q,p) - \frac{1}{2p}\sum_{\ell=1}^{p-1} e^{(i-\frac{q-1}{2})\frac{2\pi\sqrt{-1}}{p}\ell}\operatorname{cosec}(\frac{\pi\ell}{p})\operatorname{cosec}(\frac{q\pi\ell}{p}) \\ &= -\frac{1}{4p}\sum_{\ell=1}^{p-1}\left(\cot(\frac{\pi\ell}{p})\cot(\frac{q\pi\ell}{p}) + 2\cos\left\{(i-\frac{q-1}{2})\frac{2\pi}{p}\ell\right\}\operatorname{cosec}(\frac{\pi\ell}{p})\operatorname{cosec}(\frac{q\pi\ell}{p})\right). \end{aligned}$$

After shifting i in this equality appropriately, the right hand side coincides with the theta divisor invariant of $L(p,q)$ in [10] and moreover, it also coincides with the w-invariant by Y. Fukumoto and M. Furuta for $L(p,q)$, which was originally defined for any homology 3-sphere in [4].

Recently M. Ue showed via various eta invariant formulae that for any spherical manifold the correction term and Fukumoto and Furuta's w-invariant are the same invariant [16].

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