On a more constraint of knots yielding lens spaces

Motoo Tange

Department of Mathematics, Kyoto University Kyoto 606-8224, Japan
e-mail: tange@math.kyoto-u.ac.jp

November 13, 2006

Abstract

Suppose that lens space $L(p,q)$ is obtained from p-surgery of a knot in S^3. Let d be the degree of the Alexander polynomial $\Delta_K(x)$. We show that the coefficients of the degree x^d and x^{d-1} are 1 and -1 when $d \leq \frac{p}{2}$. Furthermore we also generalize that result. [1] [2]

1 Introduction

Ozsváth and Szabó have studied a constraints of the Alexander polynomial of a knot yielding a lens space in [8]. The main theorem is the following

Theorem 1.1 (P. Ozsváth-Z. Szabó [8]) *Let $K \subset S^3$ be a knot for which there is an integer p for which $S^3_p(K)$ is an L-space and the Alexander polynomial is not 1. Then the Alexander polynomial of K has the form*

$$\Delta_K(x) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(x^{n_j} + x^{-n_j}),$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < \cdots < n_k = d$.

This theorem says that the coefficients of Alexander polynomials of knots admitting lens surgery are 0 or ± 1 and the non-zero terms of the polynomial alternate in sign. In this paper we always use symmetrized Alexander polynomial.

From Theorem 1.1 the top coefficient of such an Alexander polynomial is one. In the present paper we show that the second term of the Alexander polynomial can be determined naturally by means of Theorem 1.1 and the main result of [9]. We state the main result.

Theorem 1.2 *Let $K \subset S^3$ be a knot for which there is an integer p for which $S^3_p(K)$ is a lens space. The notations n_k, n_{k-1}, and d are the same as Theorem 1.1. Then we have*

$$n_k - n_{k-1} = \begin{cases} 1 & d < \frac{p+1}{2} \\ 1 \text{ or } 2 & p \text{ is odd and } d = \frac{p+1}{2}. \end{cases}$$

[1]Keyword: lens surgery, Heegaard Floer homology, Alexander polynomial
[2]MSC: 57M25,57M27,57R58

This result solves Question 1.2 in [3] except $2d-1=p$.

Alexander polynomial of K admitting lens surgery is as either

$$\Delta_K(x) = 1 + \sum_{i=1}^{r}(x^{p_i} + x^{-p_i}) - \sum_{j=1}^{r}(x^{m_j} + x^{-m_j}), \tag{1}$$

where $0 = p_0 < m_1 < p_1 < m_2 < p_2 < \cdots < p_{r-1} < m_r < p_r = d$, or

$$\Delta_K(x) = -1 + \sum_{i=1}^{r}(x^{p_i} + x^{-p_i}) - \sum_{j=2}^{r}(x^{m_j} + x^{-m_j}), \tag{2}$$

where $0 = m_1 < p_1 < m_2 < p_2 < m_3 \cdots < p_{r-1} < m_r < p_r = d$. Theorem 1.2 asserts $m_r = d-1$ holds if $d < \frac{p+1}{2}$.

We denote by $\tilde{a}_i(K)$ the coefficient of $\Delta_K(x) \bmod x^p - 1$, i.e.,

$$\tilde{a}_i(K) := \sum_{j \equiv i \bmod p} a_j(K).$$

The coefficient $\tilde{a}_i(K)$ is

$$\tilde{a}_i(K) = \begin{cases} 1 + a_{\frac{p-1}{2}}(K) & p \text{ is odd and } d = \frac{p+1}{2}, \text{ and } i = \pm d, \ \pm(d-1) \\ 2 & p \text{ is even, } d = \frac{p}{2}, \text{ and } i = \pm d \\ a_i(K) & \text{otherwise} \end{cases} \tag{3}$$

from the estimate $2d-1 \le p$ proven in [5].

We define $\tilde{d} = \tilde{d}(K,p)$ as $\max\{i | \tilde{a}_i(K) \neq 0 \ (0 \le i \le [\frac{p}{2}])\}$. Thus if K admits lens surgery, then from Equation (3) we have

$$\tilde{d} = \begin{cases} n_{k-2} & p \text{ is odd and } d = \frac{p+1}{2}, \text{ and } n_{k-1} = d-1 \\ d-1 & p \text{ is odd and } d = \frac{p+1}{2}, \text{ and } n_{k-1} \neq d-1 \\ d & \text{otherwise.} \end{cases}$$

We define lens space $L(p,q)$ by $S^3_{p/q}(U)$, where U is unknot. We take an isomorphism $H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$ which sends the core of a handlebody of genus one Heegaard decomposition of $L(p,q)$ to 1. Suppose that $S^3_p(K)$ is homeomorphic to $L(p,q)$. Then let h be an integer with $0 < h < p$ which it corresponds to $[K^*]$ by the isomorphism. Here K^* is the dual knot of K in $L(p,q)$. Thus we have $h^2 = q^{\pm 1} \bmod p$. We consider a set $\{h, p-h, h', p-h'\}$. If $q = \pm 1 \bmod p$, then the number of the components of the set is two. We assume that $q \neq \pm 1 \bmod p$ and permute the elements in this set to define as $0 < h_2 < h_1 < p - h_1 < p - h_2 < p$. Here we state a more extended theorem than Theorem 1.2. It is proven in Section 3.

Theorem 1.3 *Suppose that $S^3_p(K)$ is homeomorphic to $L(p,q)$. The notations m_i and p_i are the same as above. Let M be a set $\{m_i | m_i \ge \tilde{d} - h_1 + 1\}$. Then we have $p_i = m_i + 1$ for $m_i \in M$. Namely $\Delta_K(x)$ can be expanded as the following: if $d < \frac{p+1}{2}$, then*

$$\Delta_K(x) = x^{p_r} - x^{p_r - 1} + x^{p_{r-1}} - x^{p_{r-1}-1} + \cdots + x^{p_s} - x^{p_s - 1} + \cdots \tag{4}$$

and if $d = \frac{p+1}{2}$, then

$$\Delta_K(x) = \begin{cases} x^{p_r} - x^{p_r-1} + x^{p_{r-1}} - x^{p_{r-1}-1} + \cdots + x^{p_s} - x^{p_s-1} + \cdots & a_{d-1}(K) \neq 0 \\ x^{p_r} - x^{p_r-2} + x^{p_{r-1}} - x^{p_{r-1}-1} + \cdots + x^{p_s} - x^{p_s-1} + \cdots & a_{d-1}(K) = 0 \end{cases}$$

where $s := \min_i \{i | p_i - 1 \ge \tilde{d} - h_1 + 1\}$.

In particular since $h_1 \ge 2$, we have $s \ge 1$, this leads to Theorem 1.2.

Hence we can prove the following easily.

Corollary 1.1 *Suppose that K admits lens surgery. If $d < h_1$, then the Alexander polynomial $\Delta_K(x)$ is either*

$$\Delta_K(x) = 1 + \sum_{i=1}^{r}(x^{p_i} + x^{-p_i}) - \sum_{j=1}^{r}(x^{p_j-1} + x^{-p_j+1}),$$

or

$$\Delta_K(x) = -1 + \sum_{i=1}^{r}(x^{p_i} + x^{-p_i}) - \sum_{j=2}^{r}(x^{p_j-1} + x^{-p_j+1}),$$

where $p_i + 1 < p_{i+1}$.

For example let K be $(-2,3,7)$ pretzel knot. A homeomorphism $S^3_{18}(K) \cong L(18,5)$ induces a set $\{5,7,11,13\}$. Since $d = 5$, the non-zero coefficients of $\Delta_K(x)$ are everywhere adjacent in pairs. In fact $\Delta_K(x)$ is

$$x^5 - x^4 + x^2 - x + 1 - x^{-1} + x^{-2} - x^{-4} + x^{-5}.$$

Conversely Alexander polynomials of knots yielding lens spaces do not satisfy this condition generally. For example the Alexander polynomial of $(4,7)$-torus knot is

$$x^9 - x^8 + x^5 - x^4 + x^2 - 1 + x^{-2} - x^{-4} + x^{-5} - x^{-8} + x^{-9}.$$

The Alexander polynomials of knots yielding lens surgery has been studied in [4, 6, 8, 9, 10]. Here we review the formulae used in this paper. Suppose that K yields lens space $L(p,q)$ and K^* is the dual knot of K. Let h be the integer corresponding to homology class $[K^*]$ and g a positive integer satisfying $hg = 1 \bmod p$ and $\gcd(h,g) = 1$. Then the Alexander polynomial satisfies the following:

$$\Delta_K(x) = x^{-\frac{(h-1)(g-1)}{2}}\frac{(x^{hg}-1)(x-1)}{(x^h-1)(x^g-1)} \quad \bmod x^p - 1.$$

The right hand side is the famous Alexander polynomial of (h,g)-torus knot.

We denote by c the following.

$$c = c(p,h) := \begin{cases} \frac{q-1}{2} & pq = 1 \bmod 2 \\ \frac{p+q-1}{2} & p(p-q) = 1 \bmod 2 \\ \frac{h^2-1}{2} & p = 0 \bmod 2. \end{cases} \tag{5}$$

Let p be a positive number. We denote by the symbol $[\alpha]_p$ the representative satisfying $0 \leq [\alpha]_p < p$ and $\alpha \equiv [\alpha]_p \bmod p$. For any class $x \in \mathbb{Z}/p\mathbb{Z}$ we define x' by $[x^{-1}]_p$.

Definition 1.1 *For $h, x \in \mathbb{Z}$ we define a $\{0,1\}$-valued function $\delta_h(x)$ as follows:*

$$\delta_h(x) = \begin{cases} 1 & [x]_p \in \{1,2,3,\cdots,[h]_p\} \\ 0 & \textit{otherwise} \end{cases}$$

From Theorem 1.5 in [9] and [10] we have

$$\tilde{a}_i(K) := -m + \Phi^{hi+c}_{p,q}(h), \tag{6}$$

where h is the class corresponding to homology class $[K^*]$ and $\Phi^{\ell}_{p,q}(h) := \#\{j \in \{1,2,\cdots,h\} |\ \delta_h(qj-\ell) = 1\}$ and $m = \frac{hh'-1}{p}$. From here we abbreviate (K) in any coefficient $\tilde{a}_i(K)$.

2 Proof of Main theorem

First of all we prove the five lemmas to prove the main theorem.

Lemma 2.1 *If $S^3_p(K)$ is homeomorphic to $L(p,q)$, the following is satisfied for any i:*

$$\tilde{a}_i + \tilde{a}_{i-1} + \cdots + \tilde{a}_{i-h+1} = \delta_h(q - hi - c),$$

where h is the same integer defined before.

Proof) From Equation (6)

$$\begin{aligned}
\tilde{a}_i + \tilde{a}_{i-1} + \cdots + \tilde{a}_{i-h+1} &= -mh + \sum_{l=0}^{h-1} \#\{j \in \{1,2,\cdots,h'\} | \delta_h(qj - (h(i-l)+c)) = 1\} \\
&= -mh + \#\{j \in \{1,2,\cdots,hh'\} | \delta_h(qj - hi - c) = 1\} \\
&= -mh + mh + \#\{j \in \{1\} | \delta_h(qj - hi - c) = 1\} \\
&= \delta_h(q - hi - c)
\end{aligned}$$

□

From Lemma 2.1 and symmetry of Alexander polynomial we have

$$\begin{aligned}
\tilde{a}_i + \tilde{a}_{i-1} + \cdots + \tilde{a}_{i-h+1} &= \tilde{a}_{-i} + \tilde{a}_{-i+1} + \cdots + \tilde{a}_{-i+h-1} \\
&= \delta_h(q - h(-i+h-1) - c) \\
&= \delta_h(h(i+1) - c).
\end{aligned}$$

Thus, by subtracting the equation replaced i by $i-1$,

$$\tilde{a}_i - \tilde{a}_{i-h} = \delta_h(h(i+1) - c) - \delta_h(hi - c)$$

Hence

$$|\tilde{a}_i - \tilde{a}_{i-h}| \leq 1. \tag{7}$$

We replace h with $p-h$ to be $0 < h < \frac{p}{2}$, if not.

Lemma 2.2 *Thus it follows that*

$$\tilde{a}_i - \tilde{a}_{i-h} = \begin{cases} 1 & \Leftrightarrow \ \delta_h(h(i+1)-c) = 1 \text{ and } \delta_h(hi-c) = 0 \\ -1 & \Leftrightarrow \ \delta_h(h(i+1)-c) = 0 \text{ and } \delta_h(hi-c) = 1 \\ 0 & \Leftrightarrow \ \delta_h(h(i+1)-c) = 0 \text{ and } \delta_h(hi-c) = 0. \end{cases} \tag{8}$$

From this lemma the following lemma holds

Lemma 2.3 *Suppose that $S^3_p(K)$ is homeomorphic to $L(p,q)$. Let h be the same as stated above. Then*

$$\tilde{a}_i - \tilde{a}_{i-h} = 1, \text{ if and only if } \tilde{a}_{i+1} - \tilde{a}_{i-h+1} = -1.$$

Proof) By using Equation (8) and $0 < h < \frac{p}{2}$, the assertion demanded is obtained easily.

□

Lemma 2.4 *Suppose that $L(p,q) = S^3_p(K)$ and $\tilde{d} \neq \frac{p}{2}$. Let h be the same as defined previously and $0 < h < \frac{p}{2}$. Then we have*

$$\tilde{a}_{\tilde{d}+h} = 0$$

and $p - m_{\mu+1} < \tilde{d} + h < p - p_\mu$ for μ with $1 \leq \mu \leq r$.

If $\tilde{d} = \frac{p}{2}$, then $\tilde{a}_{\tilde{d}+h} = 1$.

Proof) From Theorem 1.1 $|\tilde{a}_{\tilde{d}+h}|$ is 0 or 1. If $\tilde{a}_{\tilde{d}+h} = -1$, then $\tilde{a}_{\tilde{d}} - \tilde{a}_{\tilde{d}+h} = 2$. This is inconsistent with Inequality (7)

We assume that $\tilde{a}_{\tilde{d}+h} = 1$. Then there exists an integer ν with $1 \le \nu \le r$ such that $\tilde{d}+h = p - p_\nu$. Hence we have $\tilde{a}_{\tilde{d}} - \tilde{a}_{\tilde{d}+h} = 0$. In the case where $p_\nu - m_\nu < p - 2\tilde{d}$, we have $\tilde{a}_{p-p_\nu+i} = \tilde{a}_{p-p_\nu+i-h} = 0$ $(1 \le i < p_\nu - m_\nu)$ when $p_\nu > m_\nu + 1$ and $\tilde{a}_{p-p_\nu+1} = -1$, $\tilde{a}_{p-p_\nu+1-h} = 0$ when $p_\nu = m_\nu + 1$. Anyway

$$\tilde{a}_{p-m_\nu} - \tilde{a}_{p-m_\nu-h} = -1 - 0 = -1.$$

This is inconsistent with Lemma 2.3. In the case where $p - 2\tilde{d} < p_\nu - m_\nu$, similarly since we have

$$\tilde{a}_{p-\tilde{d}+h} - \tilde{a}_{p-\tilde{d}} = 0 - 1 = -1,$$

this is inconsistent with Lemma 2.3. In the case where $p - 2\tilde{d} = p_\nu - m_\nu$, since we have

$$\tilde{a}_{p-m_\nu} - \tilde{a}_{p-\tilde{d}} = -1 - 1 = -2$$

this is inconsistent with Inequality (7). Therefore $\tilde{a}_{\tilde{d}+h} = 0$.

If $p - p_\mu < \tilde{d} + h < p - m_\mu$, then by the same argument as above this turns out inconsistencies. Therefore $p - m_{\mu+1} < \tilde{d} + h < p - p_\mu$.

If $\tilde{d} = \frac{p}{2}$, then from Inequality (7) $\tilde{a}_{\tilde{d}+h} = 1$. □

Lemma 2.5 *Suppose that $L(p,q) = S^3_p(K)$. Let h be the same as defined previously and $0 < h < \frac{p}{2}$. Then we have*

$$\tilde{a}_{\tilde{d}+h-1} = 0.$$

Proof) In the case where $\tilde{d} \ne \frac{p}{2}$, the coefficient $\tilde{a}_{\tilde{d}+h-1}$ is 1 or 0. From Lemma 2.4 $\tilde{a}_{\tilde{d}+h-1} = 0$.

In the case where $\tilde{d} = \frac{p}{2}$, from Lemma 2.4 the coefficient $\tilde{a}_{\tilde{d}+h-1}$ is -1 or 0. From Theorem 1.1, $\tilde{a}_{\tilde{d}+h-1} = 0$. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2) Since $\tilde{a}_{\tilde{d}+h} - \tilde{a}_{\tilde{d}} = -1$ holds, from Lemma 2.3 and 2.5, $\tilde{a}_{\tilde{d}-1} = -1$ holds.

If $d = \frac{p+1}{2}$ and $n_{k-1} = d - 1$, then $p_1 = m_1 + 1$ holds.

If $d = \frac{p+1}{2}$ and $n_{k-1} \ne d - 1$, then

$$\begin{aligned}\Delta_K(x) &= x^{\frac{p+1}{2}} - x^{m_1} + \cdots \\ &= x^{\frac{p+1}{2}} - x^{\frac{p-1}{2}} + x^{\frac{p-1}{2}} - x^{m_1} + \cdots\end{aligned}$$

Hence $\tilde{d} = \frac{p-1}{2}$ and $m_1 = \frac{p-3}{2}$, then $p_1 - m_1 = 2$.

If $d \ne \frac{p+1}{2}$, then $\tilde{d} = d$, then $m_1 = d - 1$. □

3 Proof of Theorem 1.3

Suppose that $L(p,q) = S^3_p(K)$. Let h, p_i, m_i are the same integers as before.

Proof of Theorem 1.3) If $p - 2\tilde{d} > h$, then since $\tilde{a}_{\tilde{d}+h} = \tilde{a}_{\tilde{d}+h-1} = \cdots = \tilde{a}_{\tilde{d}+1} = 0$, from Lemma 2.3, any $\tilde{a}_i, \tilde{a}_{i-1}$ in $\{\tilde{a}_{\tilde{d}-h+i} | 1 \le i \le h\}$ are satisfied that if $\tilde{a}_i = 1$ if and only if $\tilde{a}_{i-1} = -1$. Therefore in this case this theorem holds.

We assume that $p - 2\tilde{d} \le h$ and $d \ne \frac{p}{2}$. Then

$$|\tilde{a}_i| \le 1 \tag{9}$$

holds for any i. If $\tilde{d}+h-p+m_{\mu+1}-1=m_r-p_{r-1}$, then

$$\tilde{a}_{p-m_{\mu+1}}-\tilde{a}_{p_{r-1}}=-1-1=-2$$

This is inconsistent with Inequality (7).

If $m_\nu+h-p+m_{\lambda+1}=m_\nu-p_{\nu-1}$ for $\lambda,\ \nu$, then

$$\tilde{a}_{m_\nu+h-(m_\nu+h-p+m_{\lambda+1})}-\tilde{a}_{m_\nu-(m_\nu-p_{\nu-1})}=\tilde{a}_{p-m_{\lambda+1}}-\tilde{a}_{p_{\nu-1}}=-1-1=-2.$$

This is inconsistent with Inequality (9). Therefore $m_\nu+h-p+m_{\lambda+1}\neq m_\nu-p_{\nu-1}$ holds. Here we prove the following claims.

Claim 3.1 *Let h be the same as before and $0<h<\frac{p}{2}$. Suppose that*

- $p_\nu=m_\nu+1$ *for an integer* ν *satisfying* $2\leq\nu\leq r$, *and*
- $p-m_{\lambda+1}<m_\nu+h<p_\nu+h<p-p_\lambda$, *where λ satisfies $1\leq\lambda\leq r-1$.*

Then either

- $p_{\nu-1}=m_{\nu-1}+1$ *and*
- $p-m_{\lambda+1}<m_{\nu-1}+h<p_{\nu-1}+h<p-p_\lambda$.

or

- $p_{\lambda+1}=m_{\lambda+1}+1$ *and.*
- $p_{\nu-1}<p-p_{\lambda+1}-h<p-m_{\lambda+1}-h<m_\nu$

holds.

Claim 3.2 *On the other hand suppose that*

- $p-m_\mu=p-p_\mu+1$ *for an integer μ satisfying $1\leq\mu\leq r-1$, and*
- $p_{\kappa-1}<p-p_\mu-h<p-m_\mu-h<m_\kappa$, *where κ satisfies $2\leq\kappa\leq r$.*

Then either

- $p_{\mu+1}=m_{\mu+1}+1$ *and,*
- $p_{\kappa-1}<p-p_{\mu+1}-h<p-m_{\mu+1}-h<m_\kappa$.

or

- $p_{\kappa-1}=m_{\kappa-1}+1$ *and.*
- $p-m_{\mu+1}<m_{\kappa-1}+h<p_{\kappa-1}+h<p-p_\mu$

holds.

Proof of Claim 3.1 Suppose that $p_\nu=m_\nu+1$ for an integer ν and $p-m_{\lambda+1}<m_\nu+h<p_\nu+h<p-p_\lambda$. Then we can divide this situation into the following cases (a), (b).

(a) The $m_\nu+h-p+m_{\lambda+1}<m_\nu-p_{\nu-1}$ case: When $m_\nu+h-p+m_{\lambda+1}>1$, $\tilde{a}_{m_\nu+h-i}=\tilde{a}_{m_\nu-i}=0$ $(1\leq i<m_\nu+h-p+m_{\lambda+1})$. Or when $m_\nu+h-p+m_{\lambda+1}=1$, $(\tilde{a}_{m_\nu+h-1},\tilde{a}_{m_\nu-1})=(-1,0)$. Anyway,

$$\tilde{a}_{p-m_{\lambda+1}}-\tilde{a}_{p-m_{\lambda+1}-h}=-1-0=-1.$$

If $\tilde{a}_{p-m_{\lambda+1}-1-h}=1$, then due to $\tilde{a}_{p-m_{\lambda+1}-1}-\tilde{a}_{p-m_{\lambda+1}-1-h}=1$, we have $\tilde{a}_{p-m_{\lambda+1}-1}=2$. This is inconsistent with Inequality (9). Hence $\tilde{a}_{p-m_{\lambda+1}-1-h}=0$, $\tilde{a}_{p-m_{\lambda+1}-1}=1$ namely, $p_{\lambda+1}=m_{\lambda+1}+1$. The condition $p-m_{\lambda+1}<m_\nu+h<p_\nu+h<p-p_\lambda$ holds clearly.

(b) The $m_\nu + h - p + m_{\lambda+1} > m_\nu - p_{\nu-1}$ case: Exchanging $m_\nu + h - p + m_{\lambda+1}$ and $m_\nu - p_{\nu-1}$ in (a),

$$\tilde{a}_{p_{\nu-1}+h} - \tilde{a}_{p_{\nu-1}} = 0 - 1 = -1$$

By the same way as (a) $\tilde{a}_{p_{\nu-1}+h-1} = 0$ and $\tilde{a}_{p_{\nu-1}-1} = 1$ are proven easily. Therefore $p_{\nu-1} - 1 = m_{\nu-1}$ and $p_{\nu-1} < p - p_{\lambda+1} - h < p - m_{\lambda+1} - h < m_\nu$.

□

Proof of Claim 3.2 Suppose that $p - m_\mu = p - p_\mu + 1$ for an integer μ and $p_{\kappa+1} < p - p_\mu - h < p - m_\mu - h < m_\kappa$. Then we can divide this situation into the following cases (c), (d).

(c) The $p - p_\mu - h - p_{\kappa+1} < m_{\mu+1} - p_\mu$ case: By the same way as (a),

$$\tilde{a}_{p_{\kappa+1}+h} - \tilde{a}_{p_{\kappa+1}} = 0 - 1 = -1,$$

$\tilde{a}_{p_{\kappa+1}+h-1} = 0$ and $\tilde{a}_{p_{\kappa+1}-1} = -1$ holds. Therefore $p_{\kappa-1} - 1 = m_{\kappa-1}$ and $p_{\kappa-1} < p - p_{\mu+1} - h < p - m_{\mu+1} - h < m_\kappa$.

(d) The $p - p_\mu - h - p_{\kappa+1} > m_{\mu+1} - p_\mu$ case: By the same way as (a),

$$\tilde{a}_{p-m_{\mu+1}} - \tilde{a}_{p-m_{\mu+1}-h} = -1 - 0 = -1,$$

$\tilde{a}_{p-m_{\mu+1}-h-1} = 0$ and $\tilde{a}_{p-m_{\mu+1}-1} = 1$. Therefore $p_{\mu+1} - 1 = m_{\mu+1}$ and $p - m_{\mu+1} < m_{\kappa-1} + h < p_{\kappa-1} + h < p - p_\mu$.

□

By applying Lemma 2.4 and 2.5 to Claim 3.1, we have $p_{r-1} = m_{r-1} + 1$ or $p_{\lambda+1} = m_{\lambda+1} + 1$. We apply these condition to Claim 3.1 or Claim 3.2. After here, by applying either of the two claims inductively, shifting i of $(\tilde{a}_{i+h}, \tilde{a}_i)$ per -1, i reaches $i = p - \tilde{d} - h$. When $i < p - \tilde{d} - h$, since $\tilde{a}_{p-\tilde{d}-1} = \tilde{a}_{p-\tilde{d}-2} = \cdots = \tilde{a}_{\tilde{d}+1} = 0$, this is reduced to the case of $p - 2\tilde{d} > h$. Therefore the result demanded holds.

The case of $\tilde{d} = \frac{p}{2}$ is easily proven by putting $\tilde{a}_{\tilde{d}} = 2$ and $\tilde{a}_{\tilde{d}+h} = 1$.

Since if we exchange q for q' and h for h' or $p - h'$, the same argument holds, we may assume that $h = h_1$. □

We remark that taking account of Goda and Teragaito's conjecture in [2], for general knots yielding lens surgery at least $2d+1 \leq p$ conjecturally holds. The author hopes that the cases of $d = \frac{p+1}{2}$ or $d = \frac{p}{2}$ in Theorem 1.2 1.3 are removed in the future.

Acknowledgements

The author would like express the gratitude for Professor Masaaki Ue and would thank for T. Kadokami's organizing seminars and useful communication there with N. Maruyama and T .Sakai as well. The author was supported by the Grant-in-Aid for JSPS Fellows for Young Scientists(17-1604).

References

[1] J. Berge, *Some knots with surgeries yielding lens spaces*, unpublished manuscript.

[2] H. Goda and M. Teragaito, *Dehn surgeries on knots which yield lens spaces and genera of knots*, Math. Proc. Cambridge Philos. Soc. 129 (2000), 501-515.

[3] T. Kadokami, *On the Alexander polynomial satisfying Ozsváth Szabó's condition for lens surgery*, preprint

[4] T. Kadokami, and Y. Yamada,A deformation of the Alexander polynomials of knots yielding lens spaces, Bull. of Austral. Math. Soc.

[5] P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó, Monopoles and lens space surgeries, arXiv:math.GT/0310164

[6] K. Ichihara, T. Saito, and M. Teragaito, Alexander polynomials of doubly primitive knots, Proc. Amer. Math. Soc. 135 (2007), 605-615

[7] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003), no. 2, 179–261,

[8] P. Ozsváth and Z. Szabó, *On knot Floer homology and lens surgery*,

[9] M. Tange, *Ozsváth-Szabó's correction term of lens surgery*, preprint.

[10] M. Tange, *A classification of doubly primitive knots by the Alexander polynomial* preprint.