On the Alexander polynomial of lens space knot

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Abstract

Ozsváth and Szabó discovered the coefficients constraints of the Alexander polynomial of lens space knot. All the coefficients are either ± 1 or 0 and the non-zero coefficients are alternating. We here show that the non-zero coefficients give a traversable curve in a plane $\mathbb{R}^2$. This determines the second top coefficient a_{g-1} easily, and can see more constraints of coefficients of a lens space knot. In particular, we determine the third and fourth non-zero coefficients for lens space knot with at least 4 non-zero Alexander coefficients. [1] [2]

1 Introduction.

1.1 Alexander polynomial of lens space knot.

Let $Y_r(K)$ denote a r-surgery along K of a homology sphere Y. We call the rational number r *slope* of the Dehn surgery. $L(p,q)$ is defined to be $-p/q$-surgery of the unknot in S^3. A knot $K \subset Y$ is called a *lens space knot* if an integral Dehn surgery of K is a lens space. Here we only consider the integeral Dehn surgery. Berge in [1] defined *double-primitive knot*, which is a class of lens space knots in S^3. Conjecturely, double-primitive knots are all lens space knots in S^3. We will define *double-primitive knot in a homology sphere*. The typical example of double-primitive knots is the dual knot of any simple 1-bridge knot in a genus 1 Heegaard splitting of a lens space. Such a knot yields a lens space by an integral Dehn surgery. These knots are parametrized by a pair of coprime integers (p,k) and we denote the knot by $K_{p,k}$. Hence the knot $K_{p,k}$ is a lens space knot in a homology sphere.

The lens space knot has interesting features as follows. Suppose that K or $K_{p,k}$ is a lens space knot in an L-space homology sphere or such a knot as explained in a homology sphere. Then those lens space knots are fibered knots, which this is proven by Ni [17] and Ozsváth-Szabó [10] respectively. Thus, the genus g coincides with the degree of the Alexander polynomial in these cases. The coefficients of Alexander polynomials of those lens space knots are studied in [10] and in [9]. The Alexander polynomials of both types of lens space knots have been also studied in [7], [10], and [12].

Throughout this paper, we deal with the symmetrized Alexander polynomial.

[1]Keyword: lens surgery, Alexander polynomial, double-primitive knot, simple 1-bridge knot
[2]MSC: 57M25,57M27

Theorem 1.1 (Ozsváth-Szabó [10], Ichihara-Saito-Teragaito [9]). *Suppose that K $K_{p,k}$ is an L-space knot in S^3, or the dual of simple 1-bridge knot in a lens space. Then the Alexander polynomial of K is of form*

$$\Delta_K(t) = (-1)^r + \sum_{j=1}^{r}(-1)^{j-1}(t^{n_j} + t^{-n_j}), \tag{1}$$

for some decreasing sequence of positive integers $d = n_1 > n_2 > \cdots > n_r > 0$.

This theorem holds even if K is a lens space knot in an L-space homology sphere. Theorem 1.1 says that any Alexander polynomial of lens space knot satisfies the following:

$$\begin{cases} \text{The absolute values of coefficients are } \leq 1. \text{ (Flat)} \\ \text{The non-zero coefficients alternate in sign. (Alternating)} \end{cases} \tag{2}$$

In particular, the top coefficient of $\Delta_K(t)$ is 1. The Alexander polynomial of (r,s)-torus knot $T(r,s)$, which is computed by $\Delta_{T(r,s)} = t^{-\frac{(r-1)(s-1)}{2}} \frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)}$, is a typical example satisfying (2), because the $(rs \pm 1)$-surgery of $T(r,s)$ is a lens space. We call this polynomial *a torus knot polynomial.* In general, we call the Alexander polynomial of a lens space knot *lens surgery polynomial.*

We call the sequence $(d = n_1, \cdots, n_r)$ *half non-zero sequence (or exponents)* and the decreasing sequence $(d = n_1, n_2, \cdots, n_{2r}, n_{2r+1} = -d)$ *(full) non-zero sequence (or exponents).* From the symmetry of the Alexander polynomial $n_{2r+2-i} = -n_i$ holds. Let K be a lens space knot in an L-space homology sphere or the dual of a simple 1-bridge knot. We denote the non-zero sequence (or the half non-zero sequence) of K by $NS(K)$ or $NS_h(K)$ respectively. For example, $NS_h(3_1) = (1,0)$ and

$$NS(Pr(-2,3,7)) = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = (5,4,2,1,0,-1,-2,-4,-5), \tag{3}$$

where $Pr(-2,3,7)$ is the $(-2,3,7)$-pretzel knot.

In this paper we reveal that there exists some 'connection' between the non-zero coefficients of the Alexander polynomial of a lens space knot. This connectivity gives some constraints about the non-zero sequences or lens space knots.

1.2 The genus and lens space surgery

Suppose that the p-surgery $Y_p(K)$ of an L-space homology sphere Y is an L-space. Then the following inequality holds:

$$2g(K) \leq p + 1. \tag{4}$$

In the lens surgery case of $Y = S^3$, actually the inequality can be made sharper as in [4]. Furthermore, there exist lens surgeries satisfying the equality when $Y = \Sigma(2,3,5)$ as in [13].

Here we define Dehn surgery realization.

Definition 1.1. *Let K be a lens space knot with $Y_p(K) = L(p,q)$ for a homology sphere Y with parameter (p,k). If there exists a knot K' in a homology sphere Y' such that $Y'_p(K') = L(p,q)$ and these lens surgeries give the same surgery parameter (p,k). Then we say that the surgery $Y_p(K) = L(p,q)$ can be realized by a lens surgery of K', or (p,k) can be realized by (Y', K').*

The surgery parameter (p,k) will be defined in Section 2.1. The second parameter k means the homology class of the dual knot of lens space surgery.

The nice realization of lens space surgery is the set of double-primitive knots, which is defined in [1]. In fact, it is proven in [5] that any lens surgery on S^3 can be realized by a double-primitive knot in S^3. It is reasonable that double-primitive knots are all lens space knots. However, in general homology sphere case, we can find a lens space knot with not double-primitive knot. If you are restricted to L-space homology spheres, any double primitive knots might be all lens space knots. In the homological level, equivalently in terms of realization of lens surgery, lens space knots whose dual knots are simple 1-bridge knots in the genus one Heegaard splitting of the lens spaces are the complete representatives. Because the knots has one-to-one correspondence a pair of coprime integers $(p, \{k, k_2\})$ up to the clear ambiguity.

Rasmussen in [11] gave the genus bound $2g(K) - 1 \leq p$ for a lens space knot in an L-space homology sphere. In [15] the author showed many $K_{p,k}$ in the Poincaré homology sphere with the usual orientation have $2g(K) \leq p$. Conjecturely, there never exist $K_{p,k}$ in L-space homology sphere with $2g(K_{p,k}) - 1 = p$ (see [14] for this conjecture).

Definition 1.2. *Let K be a knot in an L-space homology sphere Y. We call lens surgery $Y_p(K) = L(p, q)$ with $2g(K) \leq p$ an admissible lens surgery. Such a knot K is called an admissible lens space knot.*

Throughout this paper, we only treat any admissible lens surgery on an L-space homology sphere or $\{K_{p,k}\}$ in homology spheres, where $K_{p,k}$ is the knot whose dual knot is a simple 1-bridge knot in a lens space. The Seifert genus $g(K)$ of those knots coincides with the degree d of the Alexander polynomial. Rasmussen proved the following:

Theorem 1.2 (Rasmussen [11])**.** *Let $K \subset Z$ be a knot in an L-space, and suppose that some integral surgery on K yields a homology sphere Y. If $2g(K) < p + 1$, then Y is an L-space, while if $2g(K) > p + 1$, then Y is not an L-space.*

Thus, if p is a lens surgery slope of a double-primitive knot K in a non-L-space homology sphere, then $2g(K) \geq p + 1$ holds.

1.3 Results

Let K be a lens space knot with a parameter (p, k). We set the Alexander coefficients $\mathfrak{A} = \{a_{-d}, a_{-d+1}, \cdots, a_d\}$ on $\mathbb{Z}^2$ along the line $j = j_0 \in \mathbb{Z}$. On the other vertical lines we also set $\mathfrak{A}$ with the shift by k or $|k_2|$. Theorem 2.3 (one of the main theorems in this paper) says the non-zero coefficients are *weakly-increasing (or weakly-decreasing) and traversable* in some ways. We call the traversable curve *non-zero curve*. This curve is connected and simple.

Results in this section are obtained by using a connectivity of the non-zero curve in the plane $\mathbb{R}^2$ and the property (2).

We mainly deal with the following types of non-trivial lens space knots:

(A) An admissible lens space knot in an L-space homology sphere Y.

(B) A lens space knot whose dual is a simple 1-bridge knot in genus one Heegaard splitting of the lens space.

Any knot of (A), or (B) is called *type-(A)*, or *type-(B)* knot. The type-(B) knot is parametrized by coprime positive integers (p, k) and denote by $K_{p,k}$.

For example, all the lens space knots in S^3 are contained in the knots of type (A). Conjecturely, all $K_{p,k}$ in the Poincaré homology sphere would be in (A).

In the several main theorems of the present paper the same assertion will be proven for both types (A) and (B) separately, however knots of both types have a common traversable property (Theorem 2.3) and we prove these theorems by using the same method.

As a corollary, we can determine the second term as an application of the non-zero curve. This result was also proven independently in [8] and [6].

Theorem 1.3. *Let $K \subset Y$ be a non-trivial knot of (A) or (B), then we have*

$$n_2 = d - 1. \tag{5}$$

In particular, any non-trivial lens space knot in S^3 satisfies $n_2 = d - 1$.

This property (5) gives an interesting strict constraint. For example, if $\Delta_K(t)$ is of form $f(t^m)$ (e.g., $(m, 1)$-cable knot of any knot), then m must be 1. Hence, the $(m, 1)$-cable of any knot for $m > 1$ is not a lens space knot, which this was already proven in [2].

In the following, we will prove a characterization of lens space knots with $(2, n)$-torus knot polynomial.

Theorem 1.4. *Let $K \subset Y$ be a non-trivial knot of (A) or (B) with surgery parameter (p, k, k_2) and $k \leq |k_2|$. Then the following conditions are equivalent:*

1. $\Delta_K(t) = \Delta_{T(2,2d+1)}(t)$
2. *The lens surgery parameter of $Y_p(K) = L(p, q)$ is $(p, 2, 2d + 1)$.*
3. *The lens surgery can be realized by the surgery of $(2, 2d + 1)$-torus knot*
4. $|k_2| = 2g(K)$ *or* $|k_2| = 2g + 1$.

This theorem says that if $K_{p,k}$ in a homology sphere Y satisfies $\Delta_{K_{p,k}}(t) = \Delta_{T(2,2d+1)}(t)$, then Y is homeomorphic to S^3 and K is isotopic to $T(2, 2d+1)$. Hence, we obtain the following corollary:

Corollary 1.1. *Let $K_{p,k}$ be a type-(B) knot in a non-L-space homology sphere Y. Then $\Delta_{K_{p,k}}(t) \neq \Delta_{T(2,2d+1)}(t)$ for any integer d.*

In general, it is unknown whether any double-primitive knot K in a non-L-space homology sphere satisfies $\Delta_K \neq \Delta_{T(2,2d+1)}(t)$.

Question 1.1. *Let $K \subset Y$ be a double-primitive knot in a homology sphere. If $\Delta_K(t) = \Delta_{T(r,s)}(t)$, then is Y homeomorphic to S^3 and is K isotopic to $T(r, s)$?*

Later, we give the some counterexamples for this question in Remark 2.1.

Let K be a knot whose Alexander polynomial satisfies with (2). For the non-zero sequence $\{n_i\}$ of the polynomial we define the following index:

$$\alpha(K) = \max\{n_1 - n_{2j+1} | n_{2i-1} - n_{2i} = 1,\ 1 \leq \forall i \leq j \leq r - 1\},$$

where $2r - 1$ is the number of the full non-zero sequence. The index α satisfies $2 \leq \alpha(K) \leq 2d$. The condition of the polynomial satisfying the equality $\alpha(K) = 2d$ is equivalent to $\Delta_K(t) = \Delta_{T(2,2d+1)}$. This index $\alpha(K)$ means the length of the maximal region which contains the top term and coefficients $1, -1$ are adjacent.

The non-zero sequence with $\alpha(K) = \alpha_0$ satisfies with

$$(n_1, n_2, \cdots, n_{2s-3}, n_{2s-2}, n_{2s-1}, \cdots) = (d_1, d_1 - 1, \cdots, d_{s-1}, d_{s-1} - 1, d_s, \cdots),$$

where $d_1 - d_s = \alpha_0$. We call the region $\{i \in \mathbb{Z} | n_{2s+1} \leq i \leq n_1\}$ *adjacent region.*

In this paper we use notations $(d_1, d_2, \cdots, d_s)$ for sequence of $+1$ coefficients in the adjacent region. It is called *the adjacent sequence* and denoted by $AS(K)$ and $d_{i-1} > d_i + 1$ for any $i \leq s$. Hence, s is the number of $+1$ in the adjacent region. For example, $\alpha(Pr(-2, 3, 7)) = 7$ holds from non-zero sequence (3) and $s = 4$.

We give the following lower estimate of the index α:

Theorem 1.5. *Let K be a lens surgery parameter with (p,k). Then we have*

$$\alpha+1\geq \max\{|k_2|,k\}. \quad (6)$$

Furthermore, if $d_1-d_{s_1}+1=|k_2|$ or k holds for some integer s_1, then the coefficient of $t^{d_{s_1}-1}$ is zero.

If the Alexander polynomial Δ_K can be expanded as follows

$$\Delta_K(t)=t^{d_1}-t^{d_1-1}+t^{d_2}-t^{d_2-1}+\cdots+t^{d_{\gamma-1}}-t^{d_{\gamma-1}-1}+t^{d_\gamma}-\cdots,$$

then $d_1-d_\gamma+1\geq\max\{|k_2|,k\}$.

From the inequality, we classify lens space knots with $\alpha(K)=2$.

Corollary 1.2. *Let K be an admissible lens space knot. If $\alpha(K)=2$, then the surgery can be realized by the trefoil knot.*

Proof. The inequality implies $\max\{k,|k_2|\}\leq 3$. The surgery parameters with this condition are $(5,2),(7,2)$, $(8,3)$ or $(10,3)$. The parameters $(5,2)$ and $(7,2)$ can be realized by the trefoil. The non-zero sequence of the parameters $(8,3)$ and $(10,3)$ are $(4,3,1,0)$ and $(6,5,3,2,0)$ respectively. These sequences do not satisfy $\alpha=2$. □

Let $AS=(d_1,d_2,\cdots,d_s)$ be an adjacent sequence. We will prove the following relationship between the adjacent sequence and the surgery parameter.

Proposition 1.1. *Let (p,k) and $(d_1,\cdots,d_s)$ be a surgery parameter of (A) or (B) and the adjacent sequence respectively. Then there exist integers $1\leq s_1,s_2\leq s$ such that*

$$d_{s_1}=\begin{cases}d_1-k\\ d_1-k+1\end{cases}\quad \text{and } d_{s_2}=\begin{cases}d_1-|k_2|\\ d_1-|k_2|+1.\end{cases}$$

The connectivity of the non-zero curve gives further determination of non-zero coefficients.

Proposition 1.2. *If $d_{s_1}=d_1-k+1$ or $d_1-|k_2|+1$, then $\alpha(K)=d_1-d_{s_1}$ holds. Furthermore one of the following cases holds:*

- *(a) The case of $n_{2s-1}-n_{2s}>3$. Then we have $n_3=n_2-1$.*
- *(b) The case of $n_{2s-1}-n_{2s}=3$. Then we have $n_3<n_2-1$.*
- *(c) The case of $n_{2s-1}-n_{2s}=2$.*

If $n_{2s-1}-n_{2s}=2$ and $n_2-n_3=1$, then $n_{2s}-n_{2s+1}=1$ holds.

We immediately get the two corollaries:

Corollary 1.3. *Suppose that K a knot in (A) or (B) with parameter $(p,k,|k_2|)$ satisfies $|k_2|=k+1$. Then, $\alpha(K)=k$ holds.*

The equality holds when $K=T(u,u+1)$ where we have $u\geq 2$. In this case, we have $k=u$ and $k_2=u+1$. The expansion of $\Delta_{T(u,u+1)}(t)$ is as follows:

$$\Delta_{T(u,u+1)}(t)=t^d-t^{d-1}+t^{d-u}-t^{d-u-2}+\cdots,$$

where $d=\frac{u(u-1)}{2}$. The non-zero sequence is $AS(T(u,u+1))=(d,d-u)$ and $\alpha(T(u,u+1))=u$.

Theorem 1.3 and 1.5 were proven by the author before in [13], although, here we reprove by using the global coefficient relationship of Alexander polynomials.

Corollary 1.4. *The number r of non-zero exponents of $\Delta_K(t)$ is bounded as follows:*

$$\max\{k, |k_2|\} \leq 2r + 1.$$

By using the following results, we determine the third and fourth non-zero coefficients of any lens space knot.

Theorem 1.6. *Let K be a lens space knot of type (A) or (B) with at least 4 non-zero coefficients of the Alexander polynomial. Then, the third coefficient n_3 and fourth coefficient n_4 are one of the following:*

$$(n_1, n_2, n_3, n_4) = (d, d-1, d_2, d_2-1), (d, d-1, d_2, d_2-2), \text{ or } (d, d-1, d_2, d_2-3),$$

where $d > d_2 + 1$. If (p, k) be a surgery parameter of lens space knot of type (A), then $(n_1, n_2, n_3.n_4) = (d, d-1, d_2, d_2-3)$ does not occur.

In particular, the Alexander polynomial of lens space knot K in S^3 can be expanded as follows:

$$\Delta_K(t) = t^d - t^{d-1} + t^{d_2} - t^{d_2-1} + \cdots$$

or

$$\Delta_K(t) = t^d - t^{d-1} + t^{d_2} - t^{d_2-2} + \cdots .$$

In Section 3.2, we give all lens surgeries with $2g-4 \leq |k_2| \leq 2g-1$, in Section 3.3, we give all lens surgeries with $g \leq 5$ or with 7 non-zero coefficients

Theorem 1.7. *Let (p, k, k_2) be a type-(A) lens space knot with $2g(K) - 4 \leq |k_2| \leq 2g(K) - 1$. Then K can be realized by the following knots:*

$$T(4,3), \text{ or } Pr(-2,3,7)$$

These lens surgeries just correspond to the ones with the half non-zero sequence

$$(d, d-1, d-3, d-4, \cdots, 2, 1, 0).$$

Any lens surgery of type-(B) has $g \geq 6$ as proven in Proposition 2.2. We list surgery parameters with $g \leq 5$ and the realizations.

Theorem 1.8. *Let K be a lens space knot with $g(K) \leq 5$ of type-(A). Then K can be realized by one of the following:*

$$T(2,3),\ T(2,5),\ T(2,7),\ T(3,4),\ T(2,9),\ T(3,5),\ T(2,11), \text{ or } Pr(-2,3,7)$$

Theorem 1.9. *Let K be a lens space knot with at most 7 non-zero coefficients of type-(A) or -(B). Then K can be realized by one of the following:*

$$T(2,3),\ T(2,5),\ T(4,3),\ T(2,7),\ T(3,5), \text{ or } T(4,5)$$

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2 Preliminaries

In this section we give several basics of lens space surgery. To prove the main theorems we define the non-zero curve in the plane and prove that this curve is traversable in a non-zero region, which is called a *traversable theorem.*

2.1 Lens surgery parameter and Alexander polynomial

Here we review some definitions, notations and formulae used in this paper. Let Y be a homology sphere. Suppose that the p-surgery of $K \subset Y$ yields lens space $L(p,q)$ i.e., $Y_p(K) = L(p,q)$. The dual knot $\tilde{K}$, which is the surgery core of the Dehn surgery, gives a homology class $[\tilde{K}] \in H_1(L(p,q))$. The class can be represented as an integer k as follows: $[\tilde{K}] = k[C]$, where C is the core of the genus one Heegaard splitting of $L(p,q)$. The integer k is a multiplicative generator in $(\mathbb{Z}/p\mathbb{Z})^\times$. The choice of the core of genus one Heegaard splitting of $L(p,q)$ gives an ambiguity $k \to k^{-1} \bmod p$. The change of orientation gives the involution $k \to -k$. We change the integer $k \to -k, k^{-1}, -k^{-1} \bmod p$, if necessary, to get $k^2 = -q \bmod p$ and $0 < k < \frac{p}{2}$.

We define the integer k_2 satisfying $kk_2 = 1 \bmod p$ and $-\frac{p}{2} < k_2 < \frac{p}{2}$.

Definition 2.1. *We call the pair $(p, k, |k_2|)$ (lens) surgery parameter of $Y_p(K) = L(p,q)$. We omit the third parameter $|k_2|$ in some cases. In the case where $(p,k,|k_2|)$ gives an admissible lens surgery, we call it an admissible parameter.*

Theorem 2.1 ([12],[7]). *Let Y be a homology sphere and K a lens space knot of type-(A) with surgery parameter (p,k). Then, for an integer l with $kl = 1 \bmod p$ and $\gcd(k,l) = 1$, we have*

$$\Delta_K(t) \equiv \Delta_{T(k,l)}(t) \mod t^p - 1.$$

Furthermore, if the surgery is an admissible lens surgery, then $\Delta_K(t)$ is the smallest symmetric representative of $\Delta_{T(k,l)}$ in $\mathbb{Z}[t^{\pm 1}]/(t^p - 1)$.

For a Laurent polynomial $f(t) = \sum_i \beta_i t^i \in \mathbb{Q}[t^{\pm 1}]$, the *smallest symmetric representative* of $f(t)$ in $\bmod t^p - 1$ means

$$\bar{f}(t) = \begin{cases} \sum_{|i|<\frac{p}{2}} \alpha_i t^i & p \equiv 1 \bmod 2 \\ \sum_{|i|<\frac{p}{2}} \alpha_i t^i + \frac{\alpha_{\frac{p}{2}}}{2}(t^{\frac{p}{2}} + t^{-\frac{p}{2}}) & p \equiv 0 \bmod 2 \end{cases},$$

where $\alpha_i = \sum_{j \equiv i \bmod p} \beta_j$. The polynomial $\bar{f}(t)$ is equivalent to $f(t)$ in $\mathbb{Q}[t^{\pm 1}]/(t^p - 1)$. Hence, the last statement is equivalent to $\Delta_K(t) = \overline{\Delta_{T(k,l)}}(t)$.

2.2 The coefficient formula of Alexander polynomial for admissible lens surgery.

Here we give the coefficient formula of the Alexander polynomials of admissible lens space knots. This formula has been proven in [12], and we will reprove it as a formula with a bit different form.

We put $e = \mathrm{sgn}(k_2)$, $c = \frac{(k+1-p)(k-1)}{2}$, $m = \frac{kk_2 - 1}{p}$, and

$$I_\alpha = \begin{cases} \{1, 2, \cdots, \alpha\} & \alpha > 0 \\ \{\alpha+1, \alpha+2, \cdots, -1, 0\} & \alpha < 0 \end{cases}$$

The bracket $[\cdot]_p$ stands for the least absolute remainder with respect to p. Namely, the remainder satisfies $-\frac{p}{2} < [y]_p \leq \frac{p}{2}$ for integer y. In [12] the coefficient a_i of the symmetrized Alexander polynomial $\Delta_K(t)$ is computed as follows:

Proposition 2.1 ([12])**.** *Let K be a lens space knot of type-(A). Suppose that $Y_p(K) = L(p,q)$ and $2g(K) < p$, where Y is an L-space homology sphere. Then we have*

$$a_i(K) = -m + e \cdot \#\{j \in I_k | [-q(j + ki + c)]_p \in I_{k_2}\}. \tag{7}$$

If $2g(K) = p$, then

$$a_i(K) = \begin{cases} -m + e \cdot \#\{j \in I_k | [-q(j + ki + c)]_p \in I_{k_2}\} & |i| < \frac{p}{2} \\ 1 & i = \pm \frac{p}{2} \end{cases}.$$

From the upper bound of $g(K)$ by Greene [4] in the case of $Y = S^3$, for the formula of a_i we use (7). Here we prove this proposition. Essentially, this equality (7) was proven in [12]

Proof. We set the right hand side of (7) as $\alpha_i(K)$. Then

$$t^{lc+1}(t^k-1)(t^l-1) \sum_{0\le i<p} \alpha_i t^i \equiv \sum_{0\le i<p} (\alpha_{i-k-l-lc-1} - \alpha_{i-k-lc-1} - \alpha_{i-l-lc-1} + \alpha_{i-lc-1}) t^i \bmod t^p - 1. \tag{8}$$

Since we have

$$\alpha_{i-l} - \alpha_i = eE_{k_2}(-q(ki + c)) - eE_{k_2}(-q(k(i + 1) + c)),$$

$$\begin{aligned} & \alpha_{i-k-l-lc-1} - \alpha_{i-k-lc-1} - \alpha_{i-l-lc-1} + \alpha_{i-lc-1} \\ = \ & eE_{k_2}(k_2(i-1)-1) - eE_{k_2}(k_2(i-1)) - eE_{k_2}(k_2 i - 1) + eE_{k_2}(k_2 i) \\ = \ & \begin{cases} 1 & i = k+2, k \\ -2 & i = k+1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, (8) is $t^k(t-1)^2$. Thus we have

$$t^{lc+1}(t^k - 1)(t^l - 1)\Delta_K(t) \equiv t^k(t^{kl} - 1)(t - 1) \bmod t^p - 1.$$

Thus, we have

$$\Delta_K(t) \equiv t^{-lc-1+k} \frac{(t^{kl} - 1)(t - 1)}{(t^k - 1)(t^l - 1)} \tag{9}$$

in $\mathbb{Q}[t^{\pm 1}]/\sum_{i=0}^{p-1} t^i$. Here $lc + 1 - k = \frac{1}{2}(k-1)(l-1) \bmod p$ holds. In fact, $2lc + 2 - 2k - (k-1)(l-1) = l(k-1)(k+1-p) + 2 - k + l + kl - 1 \bmod 2p$. Further, when $t = 1$, the right hand side of (9) is 1. $\Delta_K(1) = -mp + e\sum_{i=0}^{p-1} \#\{j \in I_k | [-q(j + ki + c)]_p \in I_{k_2}\} = -mp + kk_2 = 1$. Thus, (9) lifts as the equality in $\mathbb{Z}[t^{\pm 1}]/t^p - 1$. □

For coefficients a_i of the symmetrized Alexander polynomial we define the coefficient $\bar{a}_i \in \mathbb{Z}$ to be $\sum_{j \equiv i \bmod p} a_j$. The coefficients have the period p namely, $\bar{a}_{i+p} = \bar{a}_i$. We define *A-function* $A(x)$ and *A-matrix* $(A_{i,j})$ to be

$$A : \mathbb{Z} \to \mathbb{Z}, \quad A(x) = \bar{a}_{-k_2(x+c)}$$

and

$$A_{i,j} = \bar{a}_{-j-k_2(i+c)}, \quad (i,j) \in \mathbb{Z}^2$$

respectively. A-function and A-matrix define essentially the same function by the following relation $A(i + jk) = A_{i,j}$. Further, we denote the difference $A(x+1) - A(x)$ by $dA(x)$, and $A_{i+1,j} - A_{i,j} = dA_{i,j}$. We call these *dA-function* and *dA-matrix*.

We define *A′-function* and *A′-matrix* to be $A'(x) = \bar{a}_{-k(x+c')}$, where $c = (k_2 + 1 - p)(k_2 - 1)/2$ and $A'(x) = \bar{a}_{-k_2(x+c')}$ and $A'_{i,j} = \bar{a}_{-j-k_2(i+c')}$.

Definition 2.2. *Let β be a non-zero integer. We define a function E_β and $A_{i,j}$ as follows:*

$$E_\beta(\alpha) = \begin{cases} e & [\alpha]_p \in I_\beta \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 2.1. *The difference $dA(x)$ is computed by*

$$dA(x) = E_{k_2}(xq + k_2) - E_{k_2}(xq) = \begin{cases} -1 & [xq]_p \in I_{|k_2|} \\ 1 & [xq]_p \in I_{-|k_2|} \\ 0 & \textit{otherwise} \end{cases} \tag{10}$$

and

$$dA(x) = -1 \Leftrightarrow dA(x + ek) = 1.$$

Proof. $dA(x) = -m + e\{j \in I_k | [-q(j-x)]_p \in I_{k_2}\}$

$$\begin{aligned} dA(x) &= A(x) - A(x+1) \\ &= e\{j \in I_k | [-q(j-x)]_p \in I_{k_2}\} - e\{j \in I_k | [-q(j-x-1)]_p \in I_{k_2}\} \\ &= E_{k_2}(xq + k_2) - E_{k_2}(xq) \end{aligned}$$

If $dA(x) = -1$ and $e = 1$, then $E_{k_2}(xq) = 1$ and $dA(x+k) = E_{k_2}(xq) - E_{k_2}(xq - k_2) = 1$. If $dA(x) = -1$ and $e = -1$, then $E_{k_2}(xq + k_2) = 1$ and $dA(x-k) = E_{k_2}(xq + 2k_2) - E_{k_2}(xq + k_2) = 1$ holds. Hence, $dA(x) = -1$ implies $dA(x + ek) = -1$. The converse is also true. □

In the same way as the case of dA, we have $dA'(x) = E_{ek}(xq' + ek) - E_{ek}(xq')$ and

$$dA'(x) = -1 \Leftrightarrow dA'(x + k_2) = 1.$$

2.3 The knot $K_{p,k}$ in $Y_{p,k}$.

In this and next section we deal lens space surgery of the type-(B), which the dual of the knot is a simple 1-bridge knot in the lens space.

Definition 2.3. *We call $K \subset Y$ a double-primitive knot if it satisfies the following properties:*

(i) *Y has a genus 2 Heegaard splitting $H_0 \cup_{\Sigma_2} H_1$, where H_i is the genus 2 handlebody and Σ_2 is the Heegaard surface.*

(ii) $K \subset \Sigma_2$

(iii) *Fo $i = 1, 2$, the induced element $[K] \in \pi_1(H_i)$ is a primitive one.*

Due to [1], any double-primitive knot $K \subset Y$ produces a lens space by an integral Dehn surgery. A simple class of double-primitive knots is the set of knots whose dual knots in the lens spaces have 1-bridge position in the genus one Heegaard splitting of the lens space as in Figure 1. The α-curve and β-curve are the circles compressing in the two handlebodies H_0 and H_1 respectively. Each of broken lines in Figure 1 presents an arc in the handlebody H_i. Joining the arcs, we produce a knot $\tilde{K}$ in the lens space. The knot $\tilde{K}$ in the lens space gives the homology class $[\tilde{K}] = k[C] \in H_1(L(p,q))$, where C is the core of the genus one Heegaard decomposition of $L(p,q)$. By doing a surgery along $\tilde{K}$, we get the homology sphere and a double-primitive knot K on the genus two Heegaard surface induced by the surgery of the lens space and the genus one Heegaard splitting of the lens space.

Definition 2.4. *We denote a double-primitive knot contructed in the this way by $K_{p,k}$, and the homology sphere obtained by the surgery of the lens space by $Y_{p,k}$.*

The formula of $\Delta_{K_{p,k}}(t)$ will be given in Theorem 2.2.

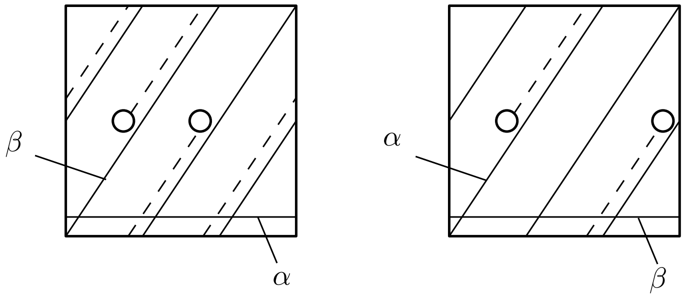


Figure 1: The knot $K_{3,1}$ in the Heegaard splitting of $L(3,1)$.

2.4 Alexander polynomial of $K_{p,k}$.

Ichihara, Saito, and Teragaito gave a formula of the Alexander polynomial of any type-(B) knot $K_{p,k}$ in S^3. This formula works for any 1-bridge knot $K_{p,k}$ in any homology sphere as well as S^3. Here a symbol $[[\cdot]]_p$ presents the remainder between 1 and p when divided by p.

Theorem 2.2 ([9]). *Let $K_{p,k}$ be a type-(B) knot in a homology sphere Y. Then $\Delta_{K_{p,k}}(t)$ is computed by the following formula:*

$$\Delta_{K_{p,k}}(t) \doteq \frac{\sum_i t^{\Phi(i)\cdot p-[[q'i]]_p\cdot k}}{\sum_{i=0}^{k-1} t^k}, \tag{11}$$

where q' is the inverse of q in $(\mathbb{Z}/p\mathbb{Z})^\times$ and $\Phi(i) = \#\{j \in I_{k-1} | [[q'j]]_p < [[q'i]]_p\}$.

Here notice that the right hand side of the formula (11) is not symmetrized. In this paper we use this formula in many times to compute the Alexander polynomials and the genus of $K_{p,k}$. We denote the coefficient of the right hand side of (11) by c_i.

We define a *B-function* and *B-matrix* to be

$$B : \mathbb{Z} \to \mathbb{Z} \quad B(x) = b_{-kx}$$

and

$$B_{i,j} = b_{i-jk_2} \in \mathbb{Z}, \quad (i,j) \in \mathbb{Z}^2.$$

Similarly, we define the difference dB-function dB-matrix as follows: $dB : \mathbb{Z} \to \mathbb{Z}$, and $dB_{i,j} \in \mathbb{Z}$ to be

$$dB(x) := B(x) - B(x+1)$$

and

$$dB_{i,j} := B_{i,j} - B_{i+1,j}.$$

Lemma 2.2. *Let b_l be the expanding coefficients of the right hand side of (11). Then we have*

$$dB_{i,j} = b_{j-ik} - b_{j-(i+1)k} = \begin{cases} 1 & \Phi(l)\cdot p - [[q'l]]_p \cdot k = j - ik \text{ for some } l \in I_{k-1} \\ -1 & \Phi(l)\cdot p - [[q'l]]_p \cdot k = j - 1 - ik \text{ for some } l \in I_{k-1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $dB(i,j) = 1 \Leftrightarrow dB(i,j+1) = -1$.

Proof. Let $F(t)$ denote the right hand side of (11).

$$\begin{aligned}(t^k-1)F(t) &= \sum_i (b_{i-k}-b_i)t^i \\ &= (t-1)\sum_{i=1}^{k-1} t^{\Phi(i)\cdot p-[[q'i]]_p\cdot k} = \sum_{i=1}^{k-1}(t^{\Phi(i)\cdot p-[[q'i]]_p\cdot k+1}-t^{\Phi(i)\cdot p-[[q'i]]_p\cdot k}) \\ &= \sum_l \big(\#\{i\in I_{k-1}|\Phi(i)\cdot p-[[q'i]]_p\cdot k=l-1\}\cdot t^l \\ &\quad -\#\{i\in I_{k-1}|\Phi(i)\cdot p-[[q'i]]_p\cdot k=l\}\cdot t^l\big)\end{aligned}$$

We have, therefore,

$$\begin{aligned}b_l-b_{l-k} &= \#\{m\in I_{k-1}|\Phi(m)\cdot p-[[q'm]]_p\cdot k=l\}-\#\{m\in I_{k-1}|\Phi(m)\cdot p-[[q'm]]_p\cdot k=l-1\} \\ &= \begin{cases}1 & \Phi(i)\cdot p-[[q'i]]_p\cdot k=l \text{ for integer } i\in I_{k-1} \\ -1 & \Phi(i)\cdot p-[[q'i]]_p\cdot k=l-1 \text{ for integer } i\in I_{k-1} \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

From this formula, the last assertion follows easily. □

Here we prove the following:

Proposition 2.2. *Let $K_{p,k}$ be a type-(B) knot in a non-L-space homology sphere. Then $g(K)\geq 6$ holds. If $g(K)=6$, then it is a double-primitive knot in $\Sigma(2,3,7)$ with the surgery parameter $(10,3)$.*

The knot $K_{10,3}$ lies in $\Sigma(2,3,7)$ and $\Sigma(2,3,7)_{10}(K_{10,3})=L(10,1)$.

Proof. When the slope is $p\leq 9$, any double-primitive knot $K_{p,k}$ lies in S^3 or $\Sigma(2,3,5)$ (see the list in [1] and [15]). Hence, any double-primitive knot in a non-L-space homology sphere satisfies $p\geq 10$. From Theorem 1.2, those knots satisfy $g(K_{p,k})\geq \frac{p+1}{2}>5$. From the formula (11), the surgery parameter with $g(K_{p,k})=6$ is $(p,k)=(10,3)$. This example is all the double-primitive knots in a non-L-space homology sphere $10\leq p\leq 11$. Other double-primitive knots with $p\geq 12$ in a non-L-space homology sphere have $g(K_{p,k})\geq\frac{p+1}{2}>6$. □

By [16], the homology sphere $Y_{12,5}$ is homeomorphic to $\Sigma(3,5,7)$, and we have $\Sigma(3,5,7)_{12}(K_{12,5})=L(12,11)$. By the formula (11), we have $g(K_{12,5})=12$. Here we put the list of $K_{p,k}$'s in non-L-space homology spheres with the slope $p\leq 23$.

Remark 2.1 (Counterexample of Question 1.1 [16])**.** *From Table 1 we get the equalities:*

$$NS(K_{10,3})=NS(T(3,7)),\ NS(K_{12,5})=NS(K_{17,5})=NS(T(5,7))$$

$$NS(K_{13,5})=NS(T(5,8)),\ NS(K_{15,4})=NS(T(4,11))$$

$$NS(K_{17,3})=NS(T(3,11)),\ NS(K_{19,3})=NS(T(3,13))$$

$$NS(K_{17,4})=NS(T(4,13)),\ NS(K_{20,9})=NS(T(9,11))$$

$$NS(K_{21,8})=NS(T(8,13)),\ NS(K_{23,5})=NS(T(5,9)).$$

Namely, there exist several knots $K_{p,k}$ in non-L-space homology spheres with the same Alexander polynomials as the ones of torus knots. On the other hand, $K_{23,7}$ is a type-(B) knot in $\Sigma(2,3,11)$ and the polynomial $\Delta_{K_{23,7}}$ is not a cyclotomic polynomial and furthermore, it is not the Alexander polynomial of any admissible lens space knot.

p	k	$Y_{p,k}$	$g(K_{p,k})$	NS_h
10	3	$\Sigma(2,3,7)$	6	$(6,5,3,2,0)$
12	5	$\Sigma(3,5,7)$	12	$(12,11,7,6,5,4,2,1,0)$
13	5	$\Sigma(3,5,8)$	14	$(14,13,9,8,6,5,4,3,1,0)$
15	4	$\Sigma(3,4,11)$	15	$(15,14,11,10,7,6,4,2,0)$
16	7	$\Sigma(4,7,9)$	24	$(24,23,17,16,15,14,10,9,8,7,6,5,3,2,1,0)$
17	3	$\Sigma(2,3,11)$	10	$(10,9,7,6,4,2,1)$
17	4	$\Sigma(3,4,13)$	18	$(18,17,14,13,10,9,6,4,2,0)$
17	5	$\Sigma(2,5,7)$	12	$(12,11,7,6,5,4,2,1,0)$
19	3	$\Sigma(2,3,13)$	12	$(12,11,9,8,6,5,3,2,0)$
20	9	$\Sigma(5,9,11)$	40	$(40,39,31,30,29,28,22,21,20,19,18,17,$ $13,12,11,10,9,8,7,6,4,3,2,1,0)$
21	8	$\Sigma(5,8,13)$	42	$(42,41,34,33,29,28,26,25,21,20,18,17,$ $16,15,13,12,10,9,8,7,5,4,3,1,0)$
23	5	$\Sigma(2,5,9)$	16	$(16,15,11,10,7,5,2,0)$
23	7	$\Sigma(2,3,11)$	13	$(13,12,10,9,6,5,3,2,0)$

Table 1: Type-(B) knots in non-L-space homology spheres up to $p \leq 23$.

2.5 Non-zero curves and Alexander region.

We give the way to visualize all the non-zero coefficients in the A-matrix $(A_{i,j})$ and B-matrix $(B_{i,j})$ from the values of $dA_{i,j}$ or $dB_{i,j}$ in Lemma 2.1 and 2.2.

Lemma 2.3. *Let K and X be a knot of type-(A) or -(B) and A or B respectively. Let $(X_{i,j})$ be the X-matrix of the lens space surgery for a knot $K \subset Y$ with $2g(K) \neq p$. If $dX_{i,j} \neq 0$ and $dX_{i,j+1} \neq 0$, then the values of X-matrix around (i,j) have one of the following local behaviors:*

($e = 1$ and $X = A$)

$$\begin{array}{c|cc} j+1 & 1 & 0 \\ j & -1 & 0 \\ \hline & i & i+1 \end{array} \qquad \begin{array}{cc} 0 & -1 \\ 0 & 1 \end{array} \qquad \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \qquad \begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array}$$

($e = -1$ and $X = A$ or $X = B$),

$$\begin{array}{cc} -1 & 0 \\ 1 & 0 \end{array} \qquad \begin{array}{cc} 0 & 1 \\ 0 & -1 \end{array} \qquad \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \qquad \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}$$

Definition 2.5 (Non-zero curves). *Let K and X be a lens space knot of type-(A) or -(B) with $g(K) \neq p/2$ and $X = A$ or B respectively. Then we describe curves on the 2-plane by regarding the matrix as a function on the lattice points $\mathbb{Z}^2$ in $\mathbb{R}^2$.*

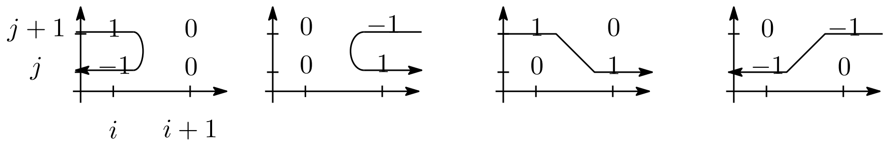
(a) Draw a horizontal arrow on any lattice point (i,j) with $A_{i,j} \neq 0$. The direction is the right when $A_{i,j} = 1$ and the left when $A_{i,j} = -1$ as below. Draw nothing on the point (i,j) with $A_{i,j} = 0$.

$$\xrightarrow{\ 1\ } \qquad \xleftarrow{\ -1\ } \qquad 0$$

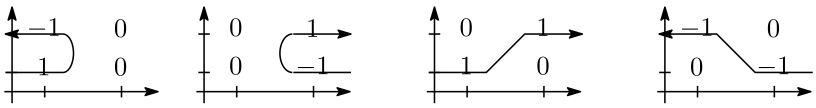
(b) Connect the horizontally adjacent arrows with the same direction. Namely, if there exist two arrows on (i,j) and $(i+1,j)$ with the same direction, then we connect them as below:

(c) For any point (i,j) satisfying $dA(i,j+1) = -dA(i,j) = e$ or $dB(i,j) = -dB(i,j+1) = 1$, connect the corresponding two non-empty arrows around the point (i,j) as figure below. The four patterns are the four possibilities in Lemma 2.3:

($X = A$ and $e = 1$)



($X = A$ and $e = -1$, or $X = B$)



Then we can make curves with direction on $\mathbb{R}^2$ and call the curves non-zero curves.

Notice that on no two points (i,j) and $(i+1,j)$ opposite arrows are drawn, because the absolute values of $dA(x)$ are less than 0 or ± 1. Furthermore, in the case of $X = A$, we assume that the curves are periodic.

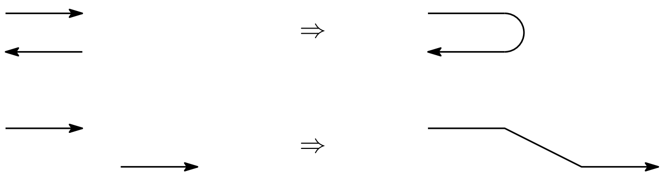
The curves are weakly-monotonicity about j. If $X = A$ and $e = 1$, then each of non-zero curves are weakly-decreasing about j, and conversely $X = A$ and $e = -1$ or $X = B$, then any non-zero curves are weakly-increasing or weakly-decreasing function about j.

Lemma 2.4. *Let γ be a non-zero curve of type-(A) or -(B) and $p > 1$. Each non-zero curve is simple and does not have end points. In particular, a component of a non-zero curve is the image of an embedding of $\mathbb{R}$ in $\mathbb{R}^2$.*

The curve is monotone and unbounded about j.

Here an end point means a lattice point that does not connect two lattice points with respect to (b) and (c) in Lemma 2.5.

Proof. What the curve is simple is clear from the construction. If the curve has the end, which the point is (i,j) or $(i+1,j)$, then $dA_{i,j}$ implies non-zero. Thus, by using Lemma 2.1, $dA_{i+1,j} = -dA_{i,j}$ or $dA_{i-1,j} = -dA_{i,j}$ holds. Thus $(i+1,j)$ or $(i-1,j)$ is also end point with opposite direction as (i,j) or $(i+1,j)$. From the definition of non-zero curve, connecting the two end points smoothly, we can cancel the two end point. By using this cancelation, we can vanish all the possible end points as in the figure below. The figures below are examples when $e = 1$.



The second assertion for lens parameter of type-(B) is clear. We assume γ is of type-(A). The monotonicity about j is true due to the construction of the non-zero curve. If the curve is bounded about j, then from the monotonicity, j-value of the curve is convergent to the an integer. However, since the curve is periodic, the curve is constant. On the line $j = j_0 \in \mathbb{Z}$, there exists all the coefficients of the Alexander polynomial. This implies $p = 1$. We, therefore, figure out that the γ is unbounded. □

The definition of non-zero curve immediately implies the following:

Proposition 2.3. *A non-zero curve of type-(A) or -(B) has symmetry about a point in $\mathbb{R}^2$.*

In the end, we can get some symmetric infinite curve with arrow and no end points on $\mathbb{R}^2$. Furthermore, in the case of type-(A), any two non-zero curves are congruent each other by some parallel translation. This will be proven later.

2.6 The case of $2g(K) = p$ case.

In the case of type-(A) and $2g(K) = p$ we have to consider the behavior of A-function around non-trivial dA-function with $\bar{a}_g = 2$. Since any other coefficients are ± 1 or 0, the behaviors of those non-zero coefficients are the same as the case of $2g(K) < p$.

Lemma 2.5. *Let (p, k) and $A_{i,j}$ be a lens space knot K of type-(A) with $2g(K) = p$ and the A-matrix. If $dA_{i,j} \neq 0$ and $dA_{i,j+1} \neq 0$, then the values of A-matrix around (i, j) have one of the following local behaviors:*

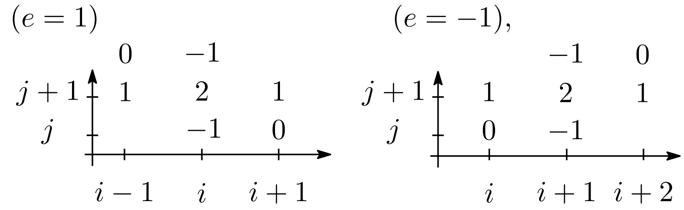
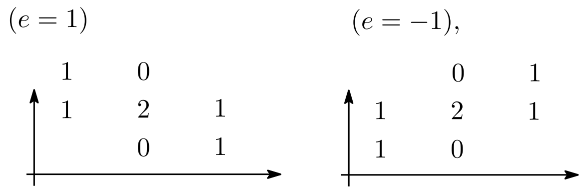


Figure 2:

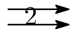
Proof. Suppose $dA_{i,j} = -dA_{i,j+1} = -e$ holds, and one of (i, j), $(i + 1, j)$, $(i, j + 1)$ and $(i + 1, j + 1)$ contains 2. Since the A-function has the symmetry about $a_{\frac{p}{2}}$ the behavior of the coefficient 2 is one of the two possibilities in Figure 2. Because the other possibilities as below can find an adjacent coefficient pair $1, 1$ against the condition (2).



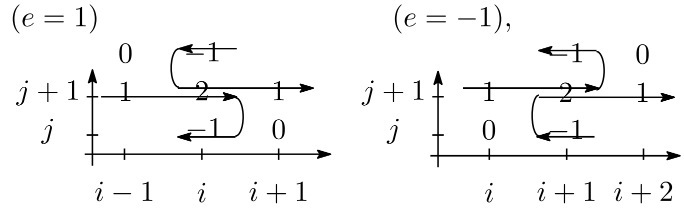
□

Definition 2.6. *Let (p, k) and $A_{i,j}$ be a lens space knot K of type-(A) with $2g(K) = p$ and the A-matrix. We do (a)' and (c)' in addition to (a) and (c).*

(a)' Draw a horizontal double arrow at the values 2 as follows:



(c)' Connect the arrows around (i,j) with $dA_{i,j} \neq 0$ and $dA_{i,j+1} \neq 0$.



Notice that since no two 2's are adjacent in A-function, the length of any double arrow is one.

2.7 Regions containing non-zero curves.

In the following, we investigate some domain in which the arrow lies.

Definition 2.7 (Non-zero region)**.** *Let (p,k) be the lens surgery parameter of type-(A) or -(B). We define a subset $\mathcal{N}$ in $\mathbb{R}^2$ as follows.*

Let (p,k) and d be a parameter of type-(A) and the degree of $\Delta_K(t)$ for the lens space knot K respectively. Suppose that $(i,j) \in \mathbb{Z}^2$ is a point satisfying $-j-k_2(i+c) \equiv 0 \bmod p$. We denote by $\mathcal{N}_0$ the union of closed $\frac{1}{2}$-neighborhood of points of

$$\{(i,j-d),(i,j-d+1),\cdots,(i,j+d)\}. \tag{12}$$

Let (p,k) and d be a parameter of type-(B) and the top degree of $\Delta_{K_{p,k}}$ respectively. For a fixed integer $j \in \mathbb{Z}$ there exists $(i,j) \in \mathbb{Z}^2$ such that

$$\{(i,j-d),(i,j-d+1),\cdots,(i,j+d)\}$$

contain all the non-zero coefficients. We denote by $\mathcal{N}_0$ the union of closed $\frac{1}{2}$-neighborhood of these points.

In both cases, let $\mathcal{N}_l$ denote $\{(x+l,y-k_2l) \in \mathbb{R}^2 | (x,y) \in \mathcal{N}_0\}$, which is the parallel translation of $\mathcal{N}_0$ by $(l,-k_2l)$ on $\mathbb{R}^2$. We call the region

$$\mathcal{N} = \cup_{l\in\mathbb{Z}}\mathcal{N}_l$$

a non-zero region.

Here we define closed ϵ-neighborhood of (i',j') to be $\{(x,y) \in \mathbb{R}^2 |\ |x-i'|+|y-j'| \leq \epsilon\}$. In the case of type-(A), changing the choice of (i,j) satisfying $-j-k_2(i+c) \equiv 0 \bmod p$, we get another non-zero region. A non-zero region $\mathcal{N}$ is connected due to $-p/2 < k_2 < p/2$.

Lemma 2.6. *Let K be a lens space knot of type-(A) and -(B). Then any connected component of non-zero curves is contained in a non-zero region,*

Proof. Suppose that (p,k) be a lens space surgery of type-(B). Since all the non-zero coefficients are contained in the non-zero region, the non-zero curve is also contained in the non-zero region.

Let (p,k) be a lens space surgery of type-(A) with $e=-1$. Suppose that a non-zero curve γ is passing on two non-zero regions $\mathcal{N}$ and $\mathcal{N}'$ as in Figure 3. Let s be the vertical segment in which γ traverses $\partial\mathcal{N} \cap \partial\mathcal{N}'$. We assume that the curve γ meets at the highest meeting point on s and at the meeting point the direction is from the

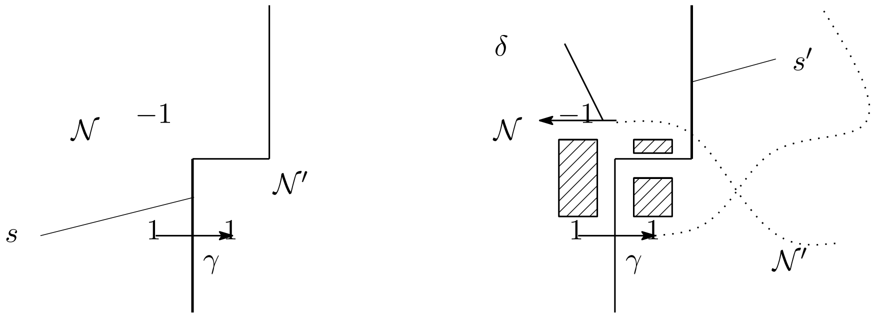


Figure 3: A non-zero curves passing two non-zero regions.

left to the right. See Figure 3. If the left 1 is not the top coefficient of $\Delta_K(t)$, then from the alternating condition of Δ_K, the next above non-zero coefficient is -1. Let δ be the non-zero curve on that -1. Actually, the curve δ agrees with γ. In fact, if the curve δ is not γ, then from the monotonicity and unboundedness about j (Lemma 2.4), δ and γ have to meet at least a point as in Figure 3. Hence, we have $\delta = \gamma$.

Suppose that the curve γ meets on the next segment s'. The left of the meeting point is -1. In the shaded region of the right in Figure 3 there is no non-zero coefficient. This contradicts the fact that the top coefficient of the Alexander polynomial is 1.

Suppose that the left 1 is the top coefficient of $\Delta_K(t)$. Let $\mathcal{N}_1'$ be a non-zero part $\{(x+1, y-k_2-p) \in \mathbb{R} | (x,y) \in \mathcal{N}_0\}$, which is the right below non-zero part of $\mathcal{N}_1$. For the assumption to be true, we have to have $-k_2 - p \geq 0$. This case does not occur.

Suppose that γ is passing from the right to the left as in Figure 4. We may assume the meeting point is the highest one among such meeting points. Then the next above non-zero coefficient is 1, however, we cannot draw a curve passing the point. Because the curve passing the coefficient 1 has to agree with γ, which is passing from the left to the right at a point in the segment. From what we proved above, such curve does not exist.

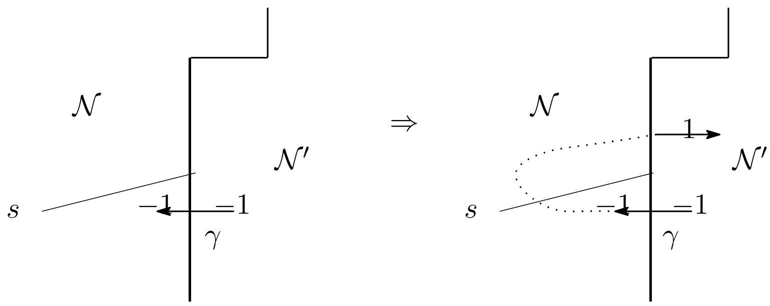


Figure 4: A part of non-zero curves passing two non-zero regions from the right to the left.

In the case of $e = 1$, since the argument is parallel by the mirror image about the j-axis, then the same assertion holds.

Hence, any non-zero curve is contained in the non-zero region. □

Lemma 2.7. *Let K be a lens space knot of type-(A). Then, the i-coordinate of any non-zero curve is unbounded both above and below.*

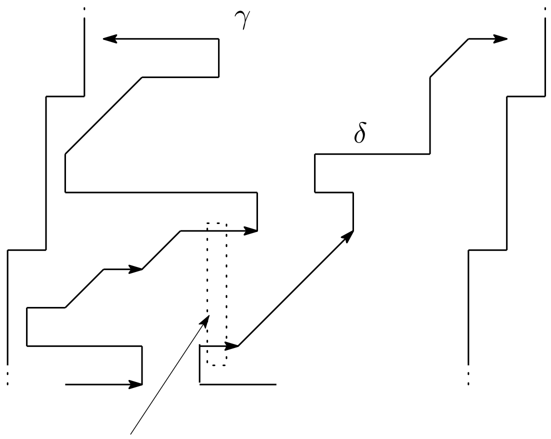
Proof. Suppose that $e = -1$. Let γ be a non-zero curve in a non-zero region $\mathcal{N}$. If γ is bounded above by $i = i_0$. Let j_0 be the minimal integer in $\mathcal{A} \cap \{i = i_0\}$. The lattice points on $R := \mathcal{A} \cap \{i \leq i_0 + \frac{1}{2}\} \cap \{j \geq j_0 - \frac{1}{2}\}$ are finite. Once the curve γ enters in R, the curve does not go out from R. This contradicts the fact that the lattice points in R are finite. Thus the curve γ is unbounded above about i.

In the case of $e = 1$, the argument of the proof is the mirror image or the case of $e = -1$ about the j-axis. □

The unboundedness about i of the non-zero curve of type-(B) is clear from the construction of the non-zero region $\mathcal{N}$.

Theorem 2.3 (Traversable Theorem)**.** *Let (p, k) be a lens surgery parameter of type-(A) or -(B). The non-zero curves which are contained in each non-zero region have a single connected component.*

Proof. Suppose that two connected components γ and δ of non-zero curves are contained in a non-zero region. Since γ and δ are disjoint each other, we may assume that one of three components in $\mathbb{R}^2 - \gamma - \delta$ does not have any non-zero component and γ is upper of δ. We can find a not-alternating pair 1 and 1 in $\Delta_K(t)$ as seen in Figure 5 (the case of $e = -1$). If you cannot find not-alternating coefficients, then γ



Two non-zero coefficients which are not allowed.
Figure 5: The case of $e = -1$.

and δ must be separated by a vertical line $i = x_0$ for a real number x_0. However, from Lemma 2.7 any non-zero curve is unbounded about the i-coordinate. □

Thus, we can get the inequality.

Corollary 2.1. *Let (p, k) be a lens space surgery of type-(A). Then an inequality $\max\{k, |k_2|\} \leq 2g(K) + 1$ holds.*

Proof. By exchanging k and $|k_2|$, we assume that $k \leq |k_2|$. Let $\mathcal{N} = \cup_{i \in \mathbb{Z}} \mathcal{N}_i$ be a non-zero region. If $|k_2| \geq 2g(K) + 2$, then by the translation $(1, -k_2)$, $\mathcal{N}_i$ and $\mathcal{N}_j$ $(i \neq j)$ are disconnected. Thus, the non-zero region is infinite disjoint union of $\mathcal{N}_0$. Since $\mathcal{N}_i$ has at most finite lattice points, we cannot embed any non-zero curve passing infinite non-zero coefficients. This contradicts Theorem 2.3. □

Corollary 2.2. *Let (p,k) be a lens surgery parameter of type-(A). If $k \le |k_2|$ and $2g(K) \le |k_2| \le 2g(K)+1$, then we have $e=-1$, $\Delta_K(t) = \Delta_{T(2,2d+1)}$ and $(p,k,|k_2|) = (4d+1,2,2d)$ or $(4d+3,2,2d+1)$.*

There exists no lens surgery with $|k_2| = 2g(K)-1$.

Proof. Let (p,k) be a lens surgery parameter of type-(A). If $|k_2| = 2d+1$, then by the translation $(1,-k_2) = (1,-e(2d+1))$ two adjacent non-zero regions $\mathcal{N}_0$ and $\mathcal{N}_1$ meet one corner point. Then, we can embed a non-zero curve as in Figure 6. Hence, naturally all the lattice points in the non-zero region $\mathcal{N}_i$ have non-zero coefficients. Thus we have

$$\Delta_K(t) = t^d - t^{d-1} + t^{d-2} - \cdots + t^{-d}.$$

Suppose that $d > 1$. Then if $p > 2(2d+1)+1$, then the dA-function has adjacent two 0s on the line $i = i_0$. This contradicts that there exists an adjacent pair $(1,-1)$ and $(-1,1)$ of dA-function in order on the line with respect to $e=1$ or $e=-1$ respectively. This is equivalent to $p < 3|k_2|$. Thus we have $p = 2(2d+1)+1$ or $2(2d+1)$. Since $\gcd(p,k)=1$, $p = 4d+3$ and $e=-1$.

Suppose that $d=1$. Then $\Delta_K(t) = t-1+t^{-1}$ holds. The only lens surgery is the trefoil knot surgery with $(p,k) = (7,2)$, $k_2 = -3$ and $e=-1$ (see [12]).

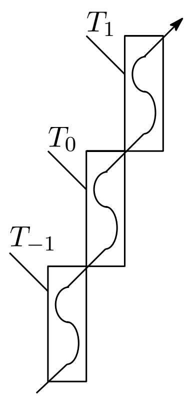


Figure 6: The case of $e=-1$ with $|k_2| = 2d+1$.

If $|k_2| = 2d$, then the non-zero region becomes Figure 7. Since the non-zero curve is connected in the region, other lattice points in the region are all non-zero. This means that $\Delta_K(t) = \Delta_{T(2,2d+1)}(t)$. Furthermore, in the same way as above, we have $p < 3|k_2|$. Thus we have $p = 4d+1, 4d+2$. Since $\gcd(p,|k_2|)=1$, we have $p = 4d+1$.

If $|k_2| = 2g(K)-1$, then since the coefficients a_g and a_{g-1} are adjacent in the plane, there exist values ± 2 in the dA-function. This is contradiction. □

Corollary 2.3. *Let K be a lens space knot in Y of type-(A) with surgery parameter (p,k). If $\Delta_K(t) \ne \Delta_{T(2,2m+1)}(t)$, then we have $\max\{k,|k_2|\} \le 2g(K)-2$.*

Proof. From Corollary 2.2, the required assertion holds. □

3 The coefficients of $X_{i,j}$.

3.1 The second term n_2 and adjacent region.

The aim of this section is to prove the main theorem. First we prove the following

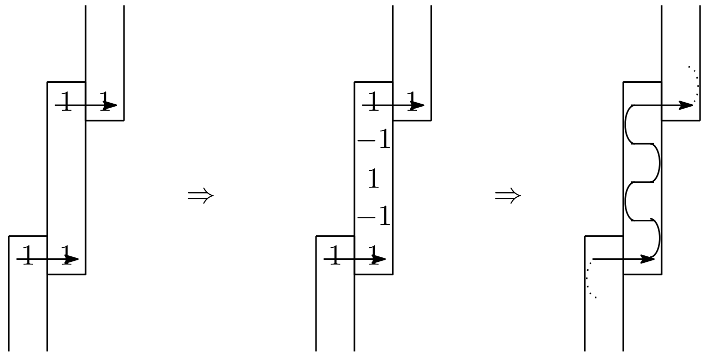


Figure 7: The case of $e = -1$.

Proposition 3.1. *Let K be a nontrivial lens space knot of type-(A) or -(B). Suppose that (p, k) be a surgery parameter of K. Then $n_2 = d - 1$ holds.*

Proof. Let K be a lens space knot of type-(A) or -(B) and (p, k) the surgery parameter of K. Suppose that $e = -1$. Let $\mathcal{N}$ be the non-zero region and $(i, j) \in \mathcal{N}$ the bottom point with the fixed $i = i_0$. The point (i_0, j) has the right arrow. Since $|k_2| > 1$ and the traversable theorem, the lattice point next to (i_0, j) must be $(i_0, j + 1)$. Thus $(i_0, j + 1)$ has left arrow, that is, $a_{g-1} = -1$. In the case of $e = 1$, the same argument follows $\bar{a}_{g-1} = -1$. □

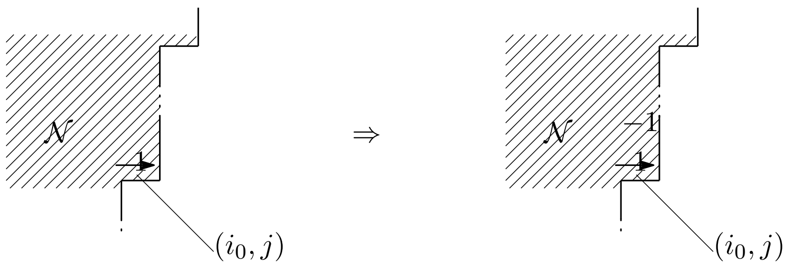
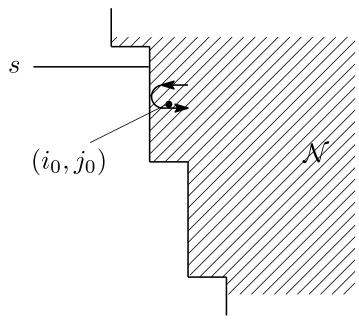


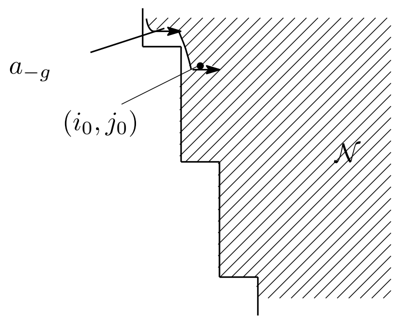
Figure 8: The case of $e = -1$.

This theorem implies Theorem 1.3. Next, we prove Theorem 1.5

Proof of Theorem 1.5. Suppose that $e = 1$. In the case of $e = -1$ the proof is the mirror image with orientation reversing of the non-zero curve about the j-axis. Let s be a vertical left segment in $\partial\mathcal{N}$. Let (i_0, j_0) be a lattice point satisfying $(i_0, j_0 - \frac{e}{2}) \in s$ and $X(i_0, j_0) = 1$. Then, the non-zero curve passing (i_0, j_0) connects $(i_0 - e, j_0 + 1)$ or $(i_0, j_0 + 1)$ since it is contained in the non-zero region. If $(i_0 - e, j_0 + 1) \notin \mathcal{N}$, then the former case holds. If $(i_0 - e, j_0 + 1) \in \mathcal{N}$, then the latter case holds and $(i_0 - e, j_0 + 1)$ corresponds to the bottom term t^{-g}. See the figure below.

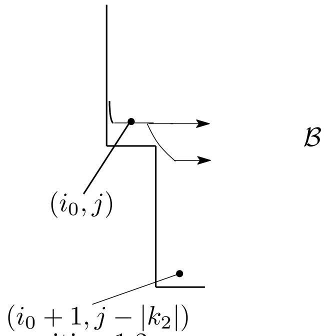
Thus, on the right lattice points of the segment s are contained in the adjacent region. Hence, we have $\alpha \geq |k_2| - 1$. Exchanging the k and $|k_2|$, we have $\alpha + 1 \geq \max\{k, |k_2|\}$.





□

Proof of Proposition 1.1. Suppose $e = 1$. Let (i_0, j) be a bottom coefficient of the Alexander polynomial on $i = i_0$. The next point is $(i_0 + 1, j)$ or $(i_0 + 1, j - 1)$. Thus, there exists an integer s_2 such that $d_1 - d_{s_2} = |k_2|$ or $|k_2| - 1$. In the same way there exists s_1 such that $d_1 - d_{s_1} = k$ or $k - 1$.



In the following, we prove Proposition 1.2.

Proof of Proposition 1.2. Suppose that $e = 1$. Let s be an integer satisfying $d - d_s = |k_2| - 1$ and d the degree of the Alexander polynomial. Let (i_0, j_0) be the lattice point of the bottom of non-zero region as in the picture below. From Proposition 1.1, we have $A(i_0 + 1, j_0) = 0$. In the case of (a) in the proposition, the non-zero curve is as

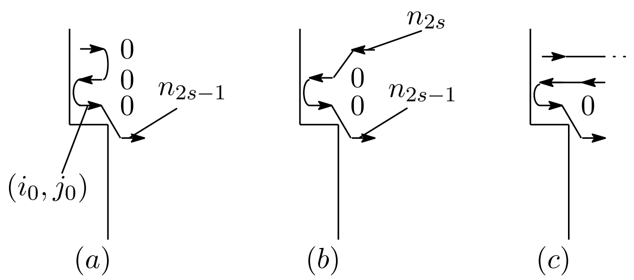


Figure 9:

the leftmost picture of Figure 9. Since the non-zero curve is connected, $n_3 = d - 2$ holds.

In the case of (b) in the proposition, the non-zero curve is the central picture of Figure 9. Hence, $n_3 - n_2 > 2$ holds.

If $n_{2s-1} - n_{2s} = 2$ and $n_3 = d - 2$, then the connectedness of the non-zero curve means $n_{2s} - n_{2s+1} = 1$. □

We prove Corollary 1.3.

Proof of Corollary 1.3. We use the X-function and X'-function, which is obtained by exchanging k and k_2. Suppose that $e = 1$. If for some integer s_1, we have $d_1 - d_{s_1} = k - 1$, then the coefficient of $n_{2s_1-1} - 1$ is 0. Exchanging k and k_2, we consider the X-function. The point corresponding to n_{2s_1-1} has coefficient 1 in the X-function, see Figure 10. Then, the point would make an end point on the curve. This contradicts Lemma 2.4. Thus $d_1 - d_{s_1} = k = |k_2| - 1 = \alpha(K)$ holds. □

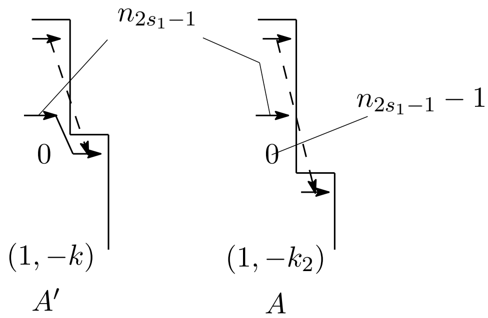


Figure 10: The case of $|k_2| = k + 1$.

For example the lens surgery parameter of $Pr(-2, 3, 7)$ is $(19, 7, -8)$. The coefficient is expanded as follows:

$$\Delta_{Pr(-2,3,7)}(t) = t^5 - t^4 + \cdots + t^{5-7} - t^{5-9} + \cdots .$$

Here $\alpha(Pr(-2, 3, 7)) = 7$ holds. The double-primitive knot $K_{61,13}$ in S^3 is expanded as follows:

$$\Delta_{K_{61,13}}(t) = t^{22} - t^{21} + \cdots + t^{22-13} - t^{22-15} + \cdots .$$

Thus $\alpha(K_{61,13}) = 13$ holds.

Proof of Corollary 1.4. If the non-zero curve starts with a fixed lattice point $\mathbf{p}_0$ and goes to the point $\mathbf{p}_0 + (1, -k_2)$, then the curve tracks all the non-zero coefficients of the Alexander polynomial. The number is at least $\max\{|k_2|, k\}$ by considering the shifting length of non-zero region about the j-coordinate. □

3.2 Lens surgeries with $\Delta_K(t) = \Delta_{T(2,n)}(t)$ or $2g-4 \leq |k_2| \leq 2g-1$.

In this section we give a characterization of lens space surgery with $\Delta_K(t) = \Delta_{T(2,n)}(t)$ (Theorem 1.4). This case corresponds to $|k_2| = 2g$ or $|k_2| = 2g + 1$. In this section we give the realization of lens surgeries with $2g - 4 \leq |k_2| \leq 2g - 1$. Furthermore, we classify lens surgery $NS_h = (d, d - 1, d - 3, d - 4, \cdots, 2, 1, 0)$ of type-(A).

Lemma 3.1. *We have $dA(0) = dA_{0,0} = e$ and $dA(k) = dA_{0,1} = -e$.*

Proof. By using the equality (10), $dA(0) = E_{k_2}(k_2) - E_{k_2}(0) = e$ holds. We have $dA(k) = E_{k_2}(kq + k_2) - E_{k_2}(kq) = E_{k_2}(0) - E_{k_2}(-k_2) = -e$. □

Lemma 3.2. *Let (p, k) be a surgery parameter of type-(A). Then there exists integer m such that the number of 0s between the two pairs of the adjacent $\{1, -1\}$ is m or $m + 1$.*

Proof. From the computation of dA-function we have the following:

$$dA_{i_0,j} = dA(i_0 + jk) = \begin{cases} -1 & [i_0q + jk_2]_p \in I_{|k_2|} \\ 1 & [i_0q + jk_2]_p \in I_{-|k_2|} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the number of sequent 0s in dA-function in a vertical line is determined by the sequence $[nk_2]_p \notin I_{|k_2|} \cup I_{-|k_2|}$. □

Proof of Theorem 1.4. The non-trivial part is from the condition 1 to the condition 2 and the equivalence of the conditions 1,2,3 and the condition 4. Let (p, k) be a lens surgery parameter of an admissible lens surgery with $\Delta_K(t) = \Delta_{T(2,n)}$. We will show that the surgery parameter is $(p, 2)$ or $(p, \frac{p-1}{2})$. Suppose that $p - |k_2| - 2g - 1 < 0$. Then a non-zero curve passes on the different non-zero regions, see the left picture in Figure 11. This contradicts the traversable theorem.

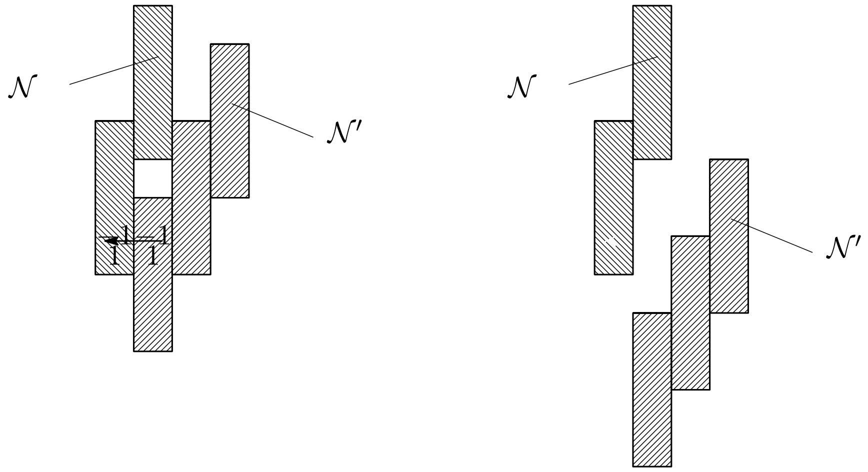


Figure 11: The case of $p - |k_2| - 2g - 1 < 0$ and $p - |k_2| - 2g - 1 \geq 0$.

Hence we have $p-|k_2|-2g-1 \geq 0$ (see the right picture in Figure 11). Then there exists $x \in \mathbb{Z}$ such that $dA(x) = e, dA(x+k) = -e, dA(x+2k) = e, \cdots, dA(x+(|k_2|-1)k) = -e$, in particular k_2 is an even integer or $|k_2| = 2g+1$. See Figure 12. If $4 \leq |k_2|$, then the number of 0s between two vertical pairs of $\{1, -1\}$ in the dA-function must be zero or one. Namely, we have $p < 3|k_2|$ and $p = |k_2| + 2g + 1$ or $p = |k_2| + 2g + 2$. Hence $|k_2| = 2g$ or $2g + 1$ holds. This implies the proof from the condition 1 to the condition 4. The only possibilities are $(p, |k_2|) = (4g + 1, 2g), (4g + 3, 2g + 1)$ in Figure 13. Hence, we have $e = -1$ and $|k_2| = \frac{p-1}{2}$.

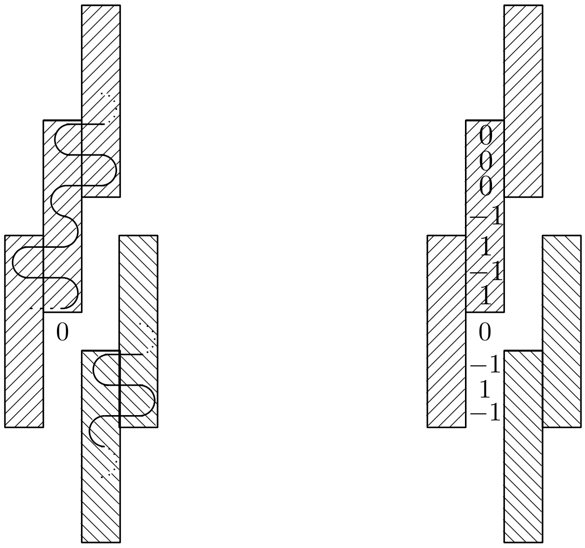
If $|k_2| < 4$, then $|k_2| = 2$ holds. In this case, we have $2k = |k_2|k < p$ and $k|k_2| = p - 1$ holds. Thus, we have $k = \frac{p-1}{2}$.

If (p, k) is a lens surgery parameter with the condition 4, then the Alexander polynomial is the same as the one of $T(2, 2d+1)$. This proves 'from 4 to from 1'. □

Corollary 3.1. *If a lens space knot K with $|k_2| = 2g(K)$, then $(p, k) = (5, 2)$. In particular K is the trefoil knot.*

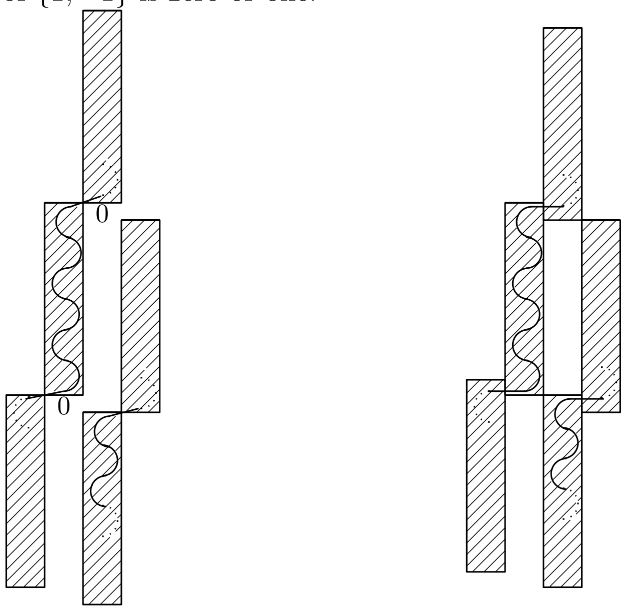
Proof. Let (p, k) be a lens surgery parameter with $|k_2| = 2g(K)$. Since there exists no case of $|k_2| \geq 4$ from the proof of the previous theorem, we have $|k_2| = 2 = 2g(K)$. Thus $g(K) = 1$ holds. The genus one case is classified in [3]. The knot is the trefoil. □

Let (p, k) be a surgery parameter and $k_2 = [k^{-1}]_p$. We classified lens surgeries with $2g(K) \leq |k_2|$. Here we discuss further classification.



The A-function. The dA-function.

Figure 12: The case of $p = |k_2| + 2g + 2$. The three 0s contradict the fact that the number of 0s between the pairs of $\{1, -1\}$ is zero or one.



The $(p, |k_2|) = (4g + 3, 2g + 1)$ case. The $p = |k_2| + 2g(K) + 1$ case.

Figure 13: The case of $p = |k_2| + 2g + |k_2| + 1$.

Proposition 3.2. *There exists no lens space knot with $2g(K) - 1 = |k_2|$.*

If a lens space knot K satisfies $2g(K) - 2 = |k_2|$ and $g(K) < \frac{p-1}{2}$, then the only surgery parameters are $(19, 7)$ or $(11, 3)$. The Alexander polynomials are

$$\Delta_K(t) = \Delta_{Pr(-2,3,7)}(t), \text{ or } \Delta_K(t) = \Delta_{T(3,4)}(t)$$

respectively.

Proof. In the case of $2g(K) - 1 = |k_2|$, the coefficients $a_g = 1$ and $a_{g-1} = -1$ are adjacent on the horizontal line $j = j_0$.

In the case of $2g(K)-2 = |k_2|$, the non-zero sequence is $NS_h = (d, d-1, d-2, \cdots, 0)$ or $(d, d-1, d-3, d-4, \cdots, 2, 1, 0)$.

The former case corresponds to the surgery parameter $(p, 2)$ due to Theorem 1.4. We consider the latter case. Let (p, k) be a surgery parameter with $|k_2| = 2g - 2$. In the case of $g \geq 5$, the number of sequent 0s on a line of dA is 0 or 1 (Lemma 3.2).

By Lemma 3.2, we have $p - |k_2| = 2g + 1$ and $p - |k_2| = 2g + 2$. Thus, we have $(p, |k_2|) = (4g - 1, 2g - 2)$. The case $(4g, 2g - 2)$ is ruled out since $\gcd(p, k) = 1$. See Figure 14 (for $e = -1$) and Figure 15 (for $e = 1$).

The lengths between the 3 0-values in a vertical line in dA-function are 5, $2g - 3$ and $2g - 3$, see Figure 14. Thus, $k = 2g - 3$ holds. Therefore, we have $k|k_2| = (2g - 3)(2g - 2) = 4g^2 - 10g + 6 \equiv 3g + 3 \equiv \pm 1 \bmod p$. This means $3g + 3 = 4g - 2$, that is, $(p, k) = (19, 7)$ and $g = 5$. This case is realized by the $(-2, 3, 7)$-pretzel knot.

In the case of $g < 5$, we have $g = 3$ and $(p, k) = (11, 3)$ see Figure 16. □

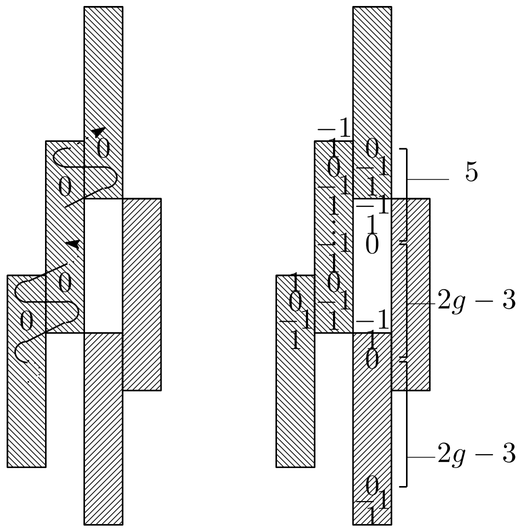
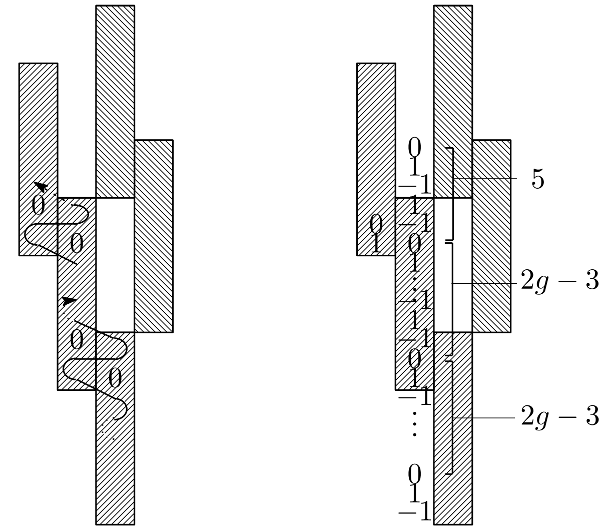


Figure 14:

Proposition 3.3. *If a lens space knot K satisfies $2g(K)-3 = |k_2|$ and type-(A), then the only surgery parameter is $(11, 4)$ and*

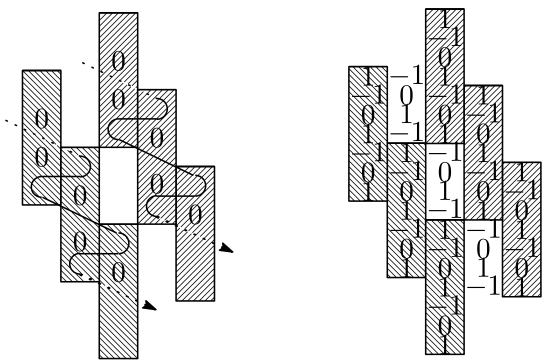
$$\Delta_K(t) = t^3 - t^2 + 1 - t^{-1} + t^{-2} + t^{-3}$$

This surgery can be realized by the $(3, 4)$-torus knot.



A-function. *dA*-function.

Figure 15:



A-function. *dA*-function.

Figure 16: The *A*-function or *dA*-function of the $(3, 4)$-torus knot.

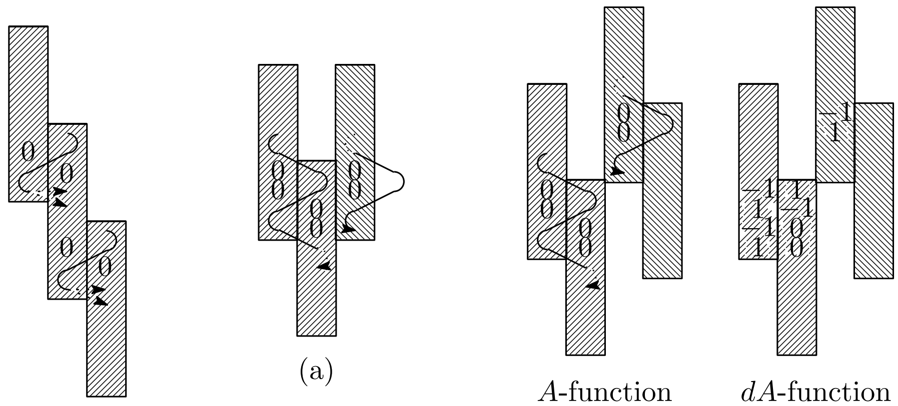
Proof. From the condition $2g(K) - 3 = |k_2|$, the Alexander polynomial is

$$NS_h(K) = (d, d-1, d-3, d-4, \cdots, 2, 1, 0) =: N_1 \text{ or } (d, d-1, d-4, d-5, \cdots, 2, 1, 0) =: N_2.$$

If $p - |k_2| < 2g + 1$holds, then the case of $NS_h = N_1$ holds and the A-function is the picture (a) in Figure 17. This picture has periodicity 2 for the i-direction. Thus $p = 2$ holds. This case does not occur. Hence $p - |k_2| \geq 2g + 1$ holds.

Suppose that $p - |k_2| \geq 2g + 1$. If $NS_h = N_1$, then the dA-function is the picture (b). This picture is inconsistent with Lemma 3.2. If $NS_h = N_2$ and $g \geq 5$, then the number m in Lemma 3.2 is 0. Thus $p - |k_2| = 2g + 1$ or $2g + 2$, therefore, $(p, k) = (4g-2, 2g-3), (4g-1, 2g-3)$. The number of 0-values in a vertical continuous p points in dA-function is 5 or 4 respectively. The lengths of the 5 0-values in a vertical line in dA-function are $2g - 6$, 3, 3, $2g - 5$, and 3. Here, the existence of the length 3 implies $3|k_2| - p = 2g + 8 \leq 4$. However this is contradiction of the condition $g \geq 5$. In the similar way, the case of $p = 4g - 1$ does not occur.

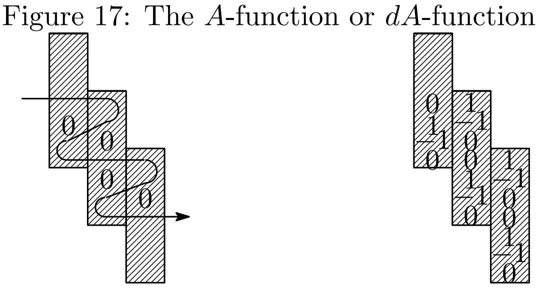
If $NS_h = N_2$ and $g < 5$, then $g = 3$ and the A-function is Figure 18. Since the case of $p - |k_2| < 2g + 1$ does not occur, $p - 3 = 8$, 9, or 10. Thus we have $(p, |k_2|) = (11, 3), (13, 3)$ due to $\gcd(p, |k_2|) = 1$. This case $NS_h = (3, 2, 0)$ is realized by the torus knot $T(3, 4)$. □



(a) A-function dA-function

The $2g(K) - 3 = |k_2|$ case. (b)

Figure 17: The A-function or dA-function.



The A-function. The dA-function.

Figure 18: The A-function or dA-function in the case of $NS_h = N_2$ and $g = 3$.

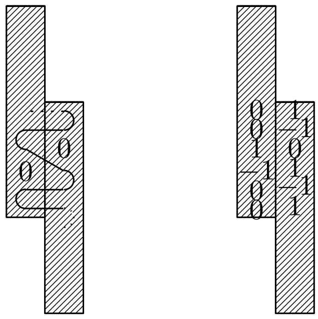
Proposition 3.4. *If a lens space knot K of type-(A) satisfies $2g(K) - 4 = |k_2|$, then the only surgery parameter is $(19, 8)$ and $\Delta_K(t) = \Delta_{Pr(-2,3,7)}(t)$.*

The proof can be done by the similar method to Proposition 3.3, hence, we omit it.

Proposition 3.5. *The only lens space knot K whose half non-zero exponent is* $(g, g-1, g-3, g-4, \cdots, 3, 2, 1, 0)$ *is* $g = 3, 5$. *This case can be realized by the* $(3,4)$*-torus knot and* $(-2,3,7)$*-pretzel knot respectively.*

Proof. We suppose the case of $2g-4 < |k_2|$. The only case is $|k_2| = 2g-5$, see Figure 19.

Suppose that $g \geq 5$. Since $p - |k_2| < 2g+1$ contradicts the traversable theorem, we may assume $p - |k_2| \geq 2g+1$. However, the values are inconsistent with Lemma 3.2 and, see the right of Figure 19.



$|k_2| = 2g-5$

Figure 19:

If $g \leq 4$, then $|k_2| < 2g-4$. This contradicts the assumption.

Thus, $|k_2| \leq 2g-4$ holds. The case is already classified in the previous proposition. Such lens surgeries are realized by $T(3,4)$ or $Pr(-2,3,7)$. □

Proof of Theorem 1.6. If $(n_1, n_2, n_3, n_4) \neq (d, d-1, d_2, d_2-1), (d, d-1, d_2, d_2-2), (d, d-1, d_2, d_2-3)$, then by using Proposition 1.2, $n_3 = n_2 + 1$ holds.

We suppose the type-(A). If $p - |k_2| \geq 2g+1$, then the number in Lemma 3.2 is 0. Thus $|k_2| = 3$ or 4 holds. The left two pictures in Figure 20 are non-zero curves of $|k_2| = 3$ or 4. However, each of pictures does not describe the non-zero curve of lens surgery. Because, in the case of $|k_2| = 3$, we cannot connect the curve as a connected curve and in the case of $|k_2| = 4$, the curve does not have a symmetry about a point (Proposition 2.3).

If $p - |k_2| < 2g+1$, then the right picture is the non-zero curve. Since the number m in Lemma 3.2 is 0, then $p = 2g+1$ or $2g+2$. These cases are not type-(A).

3.3 Lens surgeries with $g(K) \leq 5$ or with at most 7 non-zero coefficients.

From Proposition 2.2, if a knot in type-(A) or -(B) satisfies $g(K) \leq 5$, then the knot has type-(A). Before the classification of lens space surgery of $g(K) \leq 5$ or at most 7 non-zero coefficients, we characterize the lens surgery of torus knot surgery.

Proposition 3.6. *We consider any admissible lens surgery with surgery parameter* $(p, k, |k_2|)$ $(k \leq |k_2|)$. *Let γ be a non-zero curve and γ' the non-zero curve obtained by the* $(k, -1)$ *parallel translation. Then, the parameter* (p, k) *can be realized by a surgery parameter of a torus knot surgery in* S^3, *if and only if the interior of the strip between γ and γ' does not contain any other non-zero curve.*

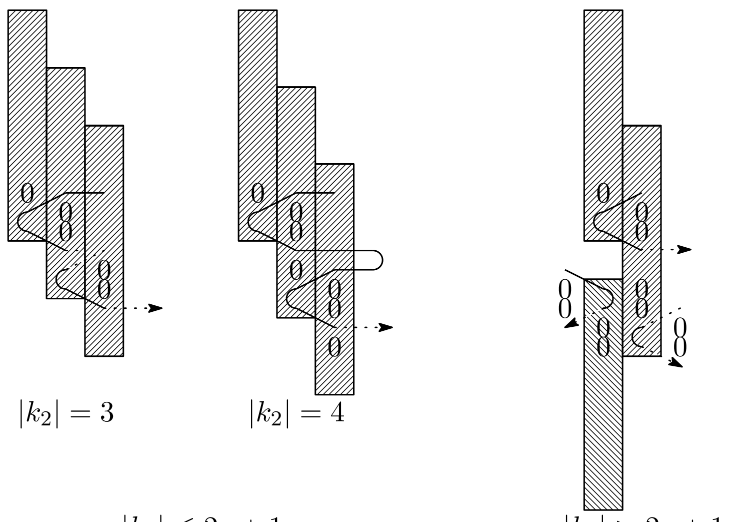


Figure 20: The case of $(n_1, n_2, n_3, n_4) = (d, d-1, d_2, d_2 - 3)$ and $|k_2| = 3$ or 4.

Proof. The parallel translation by a vector $\mathbf{v}_1 = (1, -k_2)$ gives a self-congruence map on non-zero regions, due to the definition of the non-zero region. The lattice points moved by this translation lie in the same non-zero region. The vector $\mathbf{v}_2 = (0, -p)$ gives a congruence map to the just right non-zero region. Thus $k\mathbf{v}_1 - m\mathbf{v}_2 = (k, -kk_2 + mp) = (k, -1)$ gives a congruence map to the $|m|$-th non-zero region on the right, where m is an integer defined by $m = \frac{kk_2 - 1}{p}$. If (p, k) is the lens surgery parameter of a torus knot surgery, then we have $p = kek_2 - e = emp$ and $m = e$. Hence the strip of two non-zero curves γ, γ' does not contain the other non-zero curves. □

Corollary 2.2, Corollary 2.3 and Proposition 3.6 can easily give a classification of non-torus polynomial with lens space surgeries with small Seifert genus.

Here we demonstrate the classification of $\Delta_K(t)$ of lens space knot when small k and $|k_2|$, or small $g(K) = d$. First, if $k = 1$, then $k_2 = 1$ holds, hence $\Delta_K(t) = 1$. The case of $k = 2$ is already classified in Theorem 1.4.

In the case of $g \leq 3$, all the lens surgery polynomials are torus polynomials. The torus knots with $g(K) = 4$ are $T(2, 9)$ and $T(3, 5)$. Other admissible lens space knots with $g = 4$ are the following:

Proposition 3.7. *Let $(p, k, |k_2|)$ be a lens surgery parameter of (A) with $k \leq |k_2|$. If $g = 4$, then any lens surgery polynomial is torus knot polynomial.*

Proof. We may assume $3 \leq k \leq |k_2|$. In the case of $g = 4$, the possible half non-zero sequence of non-torus polynomials is $(4, 3, 2, 0)$. The sequences $(4, 3, 1, 0)$ and $(4, 3, 2, 1, 0)$ present torus knot polynomials. The $(4, 3, 0)$ case fails to Theorem 1.6.

From Corollary 2.3 $|k_2| \leq 6$ holds. If $|k_2| = 6$, then the lens surgery polynomial is $\Delta_{T(2,7)}$ or $\Delta_{T(3,5)}$ only. We assume $|k_2| \leq 5$. Since the surgery slope p satisfies $p \geq 2g(K)$ and is the divisor of $k|k_2| \pm 1$ less than $k|k_2| \pm 1$, the only possibility is $(p, k) = (8, 3)$. The half non-zero sequence is $(4, 3, 1, 0) = NS_h(T_{5,3})$. □

$(8, 3)$ can be realized by a double-primitive knot in the Poincare homology sphere as in [13] and has non-zero sequence $NS_h(K_{8,3}) = NS_h(K_{14,3})$. The sequence $(4, 3, 0)$ does not present any admissible lens space knot polynomial.

The author proved in [12] the following:

Theorem 3.1 (Theorem 16 in [12]). *If a knot K satisfying $\Delta_K(x) = x^n - 1 + x^{-n}$ admits lens surgery, then $n = 1$ and moreover K is the trefoil knot.*

This theorem was proved by a longer argument of coefficients, however, it is a corollary of Theorem 1.3. In the present paper, we can continue to discuss the existence of lens surgery in terms of the number of non-zero coefficients of the Alexander polynomial.

For example, the polynomial $t^n - t^{n-1} + 1 - t^{-n+1} + t^{-n}$ satisfies the condition $n_2 = d - 1$ in Theorem 1.3, however, there exists an upper bound of n for the polynomial to become a lens surgery polynomial for at least type-(A) or -(B) knots.

Corollary 3.2. *If the Alexander polynomial $\Delta_K(t)$ of lens space knot of type-(A) or -(B) has 5 non-zero coefficients, then $\Delta_K(t) = \Delta_{T(5,2)}(t)$, or $\Delta_{T(4,3)}(t)$.*

In other words, if the lens surgery polynomial is of form $t^n - t^{n-1} + 1 - t^{-n+1} + t^{-n}$, then $n = 2, 3$ holds.

This type of polynomial is not given from any type-(B) knot in any non-L-space homology sphere.

Proof. The case of type-(A) is a corollary of Theorem 1.6. In the case of type-(B) and $(n_1, n_2, n_3, n_4) = (d, d-1, d_2, d_2 - 3)$, we have $g(K) = 4$. However the genus of a type-(B) knot in a non-L-space homology sphere is $g(K) \geq 6$. This case does not occur. □

Corollary 3.3. *Admissible lens space knots with $g = 5$ are $T(2, 11)$ and $Pr(-2, 3, 7)$.*

Here $Pr(p, g, r)$ is the (p, q, r)-pretzel knot.

Proof. From Corollary 2.3, since $3 \leq k \leq |k_2| \leq 8$ holds, the parameters (p, k) satisfying $mp = kk_2 - 1$ with $|m| \geq 2$ are $(11, 3)$, $(13, 5)$, $(17, 5)$, $(18, 5)$, $(19, 7)$. The admissible parameters among these are $(11, 3)$, $(18, 5)$, $(19, 7)$, and $(22, 3)$. Thus, $K_{11,3} = T(3, 4)$, $T_{18,5} = T_{19,7} = Pr(-2, 3, 7)$ and $g(K_{22,3}) = 11$ holds. □

The non-zero sequences of these knots $T(2, 11)$ and $Pr(-2, 3, 7)$ are $(5, 4, 3, 2, 1, 0)$ and $(5, 4, 2, 1, 0)$.

The next classification is done for lens space knots with the 7 non-zero coefficients.

Corollary 3.4. *If the Alexander polynomial $\Delta_K(t)$ of lens space knot of type-(A) or -(B) has 7 non-zero coefficients, then $\Delta_K(t) = \Delta_{T(7,2)}(t)$, $\Delta_{T(5,3)}(t)$, or $\Delta_{T(4,5)}(t)$.*

If K is a lens space knot of type-(A) or -(B) with

$$\Delta_K(t) = t^n - t^{n-1} + t - 1 + t^{-1} - t^{-n+1} + t^{-n} \tag{13}$$

$$\Delta_K(t) = t^n - t^{n-1} + t^2 - 1 + t^{-2} - t^{-n+1} + t^{-n}, \tag{14}$$

then when (13), $n = 4, 5$ and when (14), $n = 6$.

Proof. If the lens surgery polynomial is (13), by describing the non-zero curve, we find $g(K) \leq 5$. Thus $n = 5$ or 4.

If the lens surgery polynomial is (14), then the possibilities of non-zero curves are the pictures in Figure 21. The next is the table of the 4 non-zero curves.

$\lvert k_2\rvert$	p	α
4	6	4
5	7	5
5	6	4
4	5	3

By using the inequality (6) in Theorem 1.5, considering all the cases $(p, k, |k_2|)$, we get $(p, |k_2|, g) = (19, 4, 6), (21, 4, 6)$ and $\Delta_K(t) = \Delta_{T_{4,5}}(t)$. These cases are torus knot surgeries.

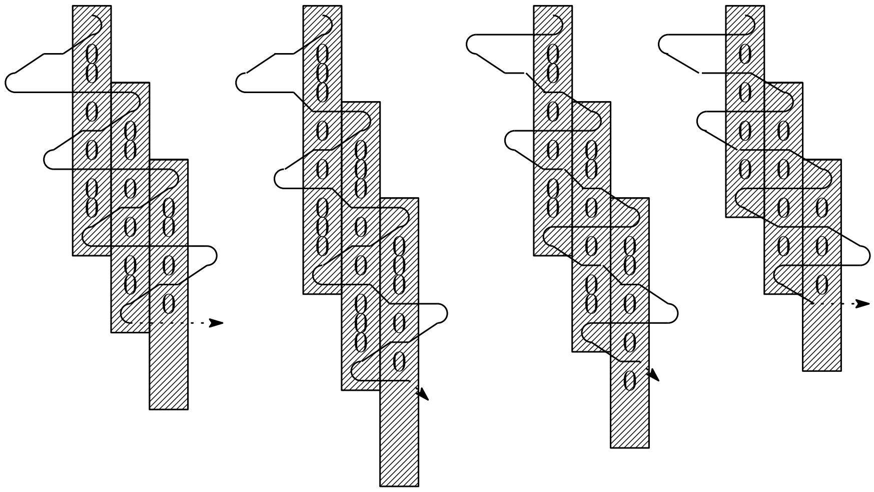


Figure 21: The cases of $(|k_2|, g) = (4, 6), (5, 7), (5, 6), (4, 5)$ respectively.

□

The next classification of admissible lens surgeries should be done for the surgeries with 9 non-zero coefficients. This is left for readers.

Problem 3.1. *Classify the Alexander polynomial with* 9 *non-zero coefficients. The polynomials are of the form:*

$$t^n - t^{n-1} + t^m - t^{m-1} + 1 + t^{-m} - t^{-m+1} - t^{-n} + t^n$$

or

$$t^n - t^{n-1} + t^m - t^{m-2} + 1 + t^{-m} - t^{-m+2} - t^{-n} + t^n.$$

3.4 Examples

Finally, in Table 2 and 3 we list type-(B) knots with $g(K_{p,k}) \leq 30$ in non-L-space homology spheres.

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g	p	k	$\lvert k_2 \rvert$	NS_h, AS	α
6	10	3	3	$(6,5,3,2,0)$ $(6,3,0)$	6
10	17	3	6	$(10,9,7,6,4,3,1,0)$ $(10,7,4,1,-1)$	11
12	12	5	5	$(12,11,7,6,5,4,2,1,0)$ $(12,7,5,2,0,-2)$	14
12	17	5	7	$(12,11,7,6,5,4,2,1,0)$ $(12,7,5,2,0,-2)$	14
12	19	3	6	$(12,11,9,8,6,5,3,2,0)$ $(12,9,6,3,0)$	12
13	23	7	10	$(13,12,10,9,6,5,3,2,0)$ $(13,10,6,5,3,0)$	13
14	13	5	5	$(14,13,9,8,6,5,4,3,1,0)$ $(14,9,6,4,1,-1)$	15
15	15	4	4	$(15,14,11,10,7,6,4,2,0)$ $(15,11,7,4)$	11
16	23	5	9	$(16,15,11,10,7,5,2,0)$ $(16,11,7)$	9
16	26	3	9	$(16,15,13,12,10,9,7,6,4,3,1,0)$ $(16,13,10,7,4,1,-1)$	17
16	26	7	11	$(16,15,12,11,9,8,5,4,2,0)$ $(16,12,9,5,2)$	14
16	29	8	11	$(16,15,13,12,8,7,5,4,2,1,0)$ $(16,13,8,5,2,0,-2)$	18
18	17	4	4	$(18,17,14,13,10,9,6,4,2,0)$ $(18,14,10,6)$	12
18	25	9	11	$(18,17,9,8,7,6,4,3,2,1,0)$ $(18,9,7,4,2,0,-2,-4)$	22
18	28	3	9	$(18,17,15,14,12,11,9,8,6,5,3,2,0)$ $(18,15,12,9,6,3,0)$	18
19	29	9	13	$(19,18,12,11,10,9,6,5,3,2,1,0)$ $(19,12,10,6,3,1,-1,-3)$	22
20	27	5	11	$(20,19,15,14,10,8,5,3,0)$ $(20,15,10)$	10
21	35	8	13	$(21,20,16,15,13,12,8,7,5,4,3,2,0)$ $(21,16,13,8,5,3,0)$	21
21	38	9	17	$(21,20,17,16,12,11,8,7,4,2,0)$ $(21,17,12,8,4)$	17

Table 2: The list of double-primitive knots $p \leq 21$

g	p	k	$\lvert k_2\rvert$	NS_h	α
22	35	3	12	$(22, 21, 19, 18, 16, 15, 13, 12, 10, 9, 7, 6, 4, 3, 1, 0)$	23
24	16	7	7	$(24, 23, 17, 16, 15, 14, 10, 9, 8, 7, 6, 5, 3, 2, 1, 0)$	27
24	32	7	9	$(24, 23, 17, 16, 15, 14, 10, 9, 8, 7, 6, 5, 3, 2, 1, 0)$	27
24	33	5	13	$(24, 23, 19, 18, 14, 13, 11, 10, 9, 8, 6, 5, 4, 3, 1, 0)$	25
24	35	11	16	$(24, 23, 13, 12, 11, 10, 8, 7, 5, 4, 2, 1, 0)$	26
24	37	3	12	$(24, 23, 21, 20, 18, 17, 15, 14, 12, 11, 9,$ $8, 6, 5, 3, 2, 0)$	24
25	37	13	17	$(25, 24, 14, 13, 12, 11, 8, 7, 5, 4, 3, 2, 1, 0)$	30
25	43	9	19	$(25, 24, 20, 19, 16, 15, 11, 10, 7, 5, 2, 0)$	18
26	42	11	19	$(26, 25, 22, 21, 18, 17, 15, 14, 11, 10, 7, 6, 4, 2, 0)$	22
26	47	5	19	$(26, 25, 21, 20, 16, 15, 11, 10, 7, 5, 2, 0)$	19
28	44	3	15	$(28, 27, 25, 24, 22, 21, 19, 18, 16, 15, 13, 12, 10, 9,$ $7, 6, 4, 3, 1, 0)$	29
28	44	7	19	$(28, 27, 21, 20, 16, 15, 14, 13, 9, 8,$ $7, 6, 3, 1, 0)$	25
28	44	13	17	$(28, 27, 18, 17, 15, 14, 11, 10, 8,$ $7, 5, 4, 2, 0)$	26
29	45	7	13	$(29, 28, 22, 21, 16, 14, 9, 7, 3, 0)$	13
29	55	16	24	$(29, 28, 22, 21, 15, 14, 13, 12, 8, 7, 6, 4, 1, 0)$	23
30	39	7	11	$(30, 29, 23, 22, 19, 18, 16, 15, 12, 11, 9, 7, 5, 4, 2, 0)$	21
30	43	15	20	$(30, 29, 15, 14, 13, 12, 10, 9, 7, 6, 4, 3, 2, 1, 0)$	34
30	46	3	15	$(30, 29, 27, 26, 24, 23, 21, 20, 18,$ $17, 15, 14, 12, 11, 9, 8, 6, 5, 3, 2, 0)$	30
30	53	3	18	$(30, 29, 25, 24, 20, 19, 15, 14, 10, 8, 5, 3, 0)$	20
30	58	7	25	$(30, 29, 23, 22, 16, 15, 14, 13, 9, 8, 7, 6, 5, 4, 2, 1, 0)$	32

Table 3: Non-zero sequences of double-primitive knots in non-L-space homology spheres up to $22 \leq p \leq 30$.

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