THE THIRD TERM IN LENS SURGERY POLYNOMIALS

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Abstract. It is well-known that the second coefficient of the Alexander polynomial of any lens space knot is -1. We show that the non-zero third coefficient condition of the Alexander polynomial of a lens space knot K in S^3 confines the surgery to the one realized by the $(2, 2g+1)$-torus knot, where g is the genus of K. In particular, such a lens surgery polynomial coincides with $\Delta_{T(2,2g+1)}(t)$.

1. Introduction

1.1. **Lens space knots.** If a knot K in a homology sphere Y yields a lens space by an integral Dehn surgery, then we call K a *lens space knot* in Y. The result obtained by a Dehn surgery is written by $Y_p(K)$. Hence, the lens space surgery is presented as $Y_p(K) = L(p, q)$. The homology class represented by the dual knot of the surgery is identified with an element k in $(\mathbb{Z}/p\mathbb{Z})^\times$. Precisely it is explained in Section 2. The pair (p, k) is called a *lens surgery parameter*.

We call a polynomial $\Delta(t)$ *lens surgery polynomial (in Y)* if there exists a lens space knot K in Y such that $\Delta(t) = \Delta_K(t)$. It is well-known that any lens surgery polynomials have interesting properties.

In [5], Ozsváth and Szabó proved that any lens surgery polynomials in S^3 are flat and alternating. If the absolute values of all coefficients of a polynomial are smaller than or equal to 1, we call the polynomial *flat*. If the non-zero coefficients of a polynomial are alternating sign in order, then we call the polynomial *alternating*. We call a polynomial Δ *trivial*, if $\Delta = 1$.

Any lens space knot with trivial Alexander polynomial in S^3 is isotopic to the unknot due to [4]. Generally, if K is a lens space knot, then the degree of the Alexander polynomial coincides with the Seifert genus g.

In this paper we use the following notations for coefficients of any lens surgery polynomial:

$$\Delta(t) = t^{-g} \sum_{i=0}^{2g} \alpha_i t^i = \sum_{i=-g}^{g} a_i t^i.$$

In other words, this equality implies $\alpha_i = a_{i-g}$. By the symmetry of Alexander polynomial we obtain $a_i = a_{-i}$ and $\alpha_i = \alpha_{2g-i}$.

We consider non-trivial lens surgery polynomials from now. Then due to the author [7] and Hedden and Watson [3], any lens surgery polynomial in S^3 becomes the following form around the top coefficient t^g:

$$\Delta = t^g - t^{g-1} + \cdots .$$

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In [3], it is shown that any L-space knot has the same form. Namely, the second coefficient from the top has -1.

In [7], it is proven that the second top coefficient of any lens space knot in any L-space homology sphere is -1.

1.2. **Third top term of lens space polynomial.** Let $T(p,q)$ be the right-handed (p,q)-torus knot. Teragaito asked an interesting question about the next coefficient:

Question 1.1. *If a non-trivial lens surgery polynomial in S^3 has the following form:*

$$\Delta = t^g - t^{g-1} + t^{g-2} + \cdots ,$$

then does Δ coincide with $\Delta_{T(2,2g+1)}$?

In other words, if a lens surgery polynomial is not a $(2,2g+1)$-torus knot polynomial for some integer g, then $\alpha_2 = 0$?.

Here we give an affirmative answer for this question.

Theorem 1.2. *Teragaito's question (Question 1.1) is true.*

This theorem is true even if the lens space knot K lies in an L-space homology sphere and satisfies $2g(K) \leq p$ by applying the same method.

Theorem 1.15 in [7] gave a criterion for a lens space knot K to satisfy $\Delta_K(t) = \Delta_{T(2,2g+1)}(t)$ for some positive integer g. On the other hand, we can also say that Theorem 1.2 gives a new criterion for a lens space knot to have the same Alexander polynomial as that of $T(2,2g+1)$.

1.3. **Realization of lens surgery.** We define the following terminology.

Definition 1.3. *Let p,k be relatively prime positive integers. If a lens surgery $Y_p(K) = L(p,q)$ in a homology sphere Y has the lens surgery parameter (p,k), then we say that the parameter (p,k) is realized by a lens space knot K.*

Corollary 1.4. *Let K be a lens space knot in S^3 with the surgery parameter (p,k). The Alexander polynomial $\Delta_K(t)$ has the following form:*

$$\Delta_K = t^g - t^{g-1} + t^{g-2} + \cdots ,$$

if and only if (p,k) is realized by $T(2,2g+1)$.

This condition in this corollary is equivalent to the condition of $k=2$.

1.4. **The cases of lens space knots $K_{p,k}$ in $Y_{p,k}$.** Consider a simple $(1,1)$-knot in a lens space yielding a homology sphere by some integer slope. The 'simple' is defined in [6] and [9]. If such a $(1,1)$-simple knot generates the 1st homology of the lens space, we can always find such a slope. Hence any simple $(1,1)$-knot is parameterized by a relatively prime integers (p,k). The dual knot is a lens space knot in the homology sphere. The dual knot is denoted by $K_{p,k}$ and the homology sphere by $Y_{p,k}$. The reader should probably understand these facts by reading [6] and [9]. The main result in [2] gave a formula of the Alexander polynomial of $K_{p,k}$ by using p,k. Here we give the following conjecture:

Conjecture 1.5. *If the third top term of the symmetrized Alexander polynomial $\Delta_{K_{p,k}}(t)$ is non-zero, then $\Delta_{K_{p,k}}(t)$ coincides with $\Delta_{T(2,2g+1)}(t)$ for some integer g, in other words, $k=2$ holds.*

This conjecture can be easily checked by a computer program ($p \leq 600$) based on the formula in [2]. Conjecture 1.5 is true under a little strong condition that $Y_{p,k}$ is homeomorphic to S^3, because of Theorem 1.2. The essential point is what if the third top term of $\Delta_{K_{p,k}}$ is non-zero, then $Y_{p,k}$ is homeomorphic to S^3. Notice that in [7] the author proved that $k = 2$ holds if and only if $Y_{p,k}$ is homeomorphic to S^3 and $K_{p,k}$ is isotopic to $T(2, 2g+1)$ for some integer g. This condition is also equivalent to the equality $\Delta_{K_{p,k}}(t) = \Delta_{T(2,2g+1)}(t)$.

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2. Preliminaries and Proofs

2.1. **Brief preliminaries.** Here we define the lens surgery parameter (p, k).

Definition 2.1. *Let K be a knot in a homology sphere Y. Suppose that $Y_p(K) = L(p, q)$ and the dual knot $\tilde{K}$ has $[\tilde{K}] = k[c] \in H_1(L(p, q), \mathbb{Z})$ for some orientation of $\tilde{K}$. Here the dual knot is the core knot in the solid torus obtained by the Dehn surgery. Furthermore, c is either of core circles of genus one Heegaard decomposition of $L(p, q)$. Then we call (p, k) lens surgery parameter. The integer k is called a dual class.*

If $L(p, q)$ is a Dehn surgery of a homology sphere, the surgery parameter (p, k) is relatively prime and $q = k^2 \bmod p$. Note that we adopt the orientation of $L(p, q)$ as the p/q-surgery of the unknot in S^3.

The ambiguity of the orientation of $\tilde{K}$ and the choices of the core circles of genus one Heegaard decomposition give (at most) four possibilities of the dual class $k_0, -k_0, k_0^{-1}, -k_0^{-1}$ (in $\mathbb{Z}/p\mathbb{Z}$), for some integer k_0. We always take the minimal integer k as a representative satisfying $0 < k < p/2$.

For any integer i we define the integer $[i]_p$ to be the integer with $i \equiv [i]_p \bmod p$ and $-\frac{p}{2} < [i]_p \leq \frac{p}{2}$. Let k_2 be the absolute value of the integer $[k']_p$ satisfying $kk' \equiv 1 \bmod p$. We call k_2 *the second dual class*. We set $kk_2 \equiv e \bmod p$, $e = \pm 1$, $m = \frac{kk_2 - e}{p}$, $q = [k^2]_p$, $q_2 = [(k_2)^2]_p$. $c = \frac{(k-1)(k+1-p)}{2}$ and for some non-zero integer ℓ

$$I_\ell := \begin{cases} \{1, 2, \cdots, \ell\} & \ell > 0 \\ \{\ell + 1, \cdots, -1, 0\} & \ell < 0. \end{cases}$$

From these data, we can compute the coefficient a_i due to [8].

Proposition 2.2 (Proposition 2.3 in [7]). *Let K be a lens space knot in S^3. For any integer i with $|i| \leq p/2$, the i-th coefficient of the Alexander polynomial*

$$a_i = -e(m - \#\{j \in I_k | [q_2(j + ki + c)]_p \in I_{ek_2}\}).$$

To prove Theorem 1.2, we use the *non-zero curve* defined in [7]. First, we extend the coefficients a_i of the Alexander polynomial periodically as $\bar{a}_i = a_{[i]_p}$. By the estimate in [1] proven in [4], $\bar{a}_i$ is determined, where $g(K)$ is the Seifert genus of K.

We define *A-matrix* and *dA-matrix* as follows:

$$A_{i,j} = \bar{a}_{k_2(i+jek-c)}, \; dA_{i,j} = A_{i,j} - A_{i-1,j},$$

where $c=(k-1)(k+1-p)/2$. Due to the formula (9) in Lemma 2.6 in [7], we have

$$dA_{i,j}=E_{ek_2}(q_2i+k_2(j+e))-E_{ek_2}(q_2i+k_2j)=\begin{cases}1 & [q_2i+k_2j]_p\in I_{-k_2}\\ -1 & [q_2i+k_2j]_p\in I_{k_2}\\ 0 & \text{otherwise.}\end{cases}\tag{1}$$

We put $A_{i,j}$ on each lattice point (i,j) in $\mathbb{Z}^2\subset\mathbb{R}^2$. For a non-zero coefficient $A_{i,j}$ we draw a horizontal positive or negative arrow on (i,j) according to $A_{i,j}=1$ or -1 respectively, where a positive (or negative) arrow means a horizontal arrow with positive (or negative) in the i-direction. After that, we connect the horizontally adjacent arrows with the same orientation and compatibly connect arrows around the non-zero $dA_{i,j}$ as in [7]. Then we can obtain an infinite family of simple curves on $\mathbb{R}^2$ with no finite ends (i.e., they are properly embedded curves in $\mathbb{R}^2$). The arrows are not-increasing with respect to the j-coordinate. We call the curves *non-zero curves*.

Proposition 2.3 ([7])**.** *Any non-zero curve for any lens space knot in S^3 is included in a non-zero region $\mathcal{N}$. In each non-zero region there is a single component non-zero curve.*

Here a non-zero region $\mathcal{N}$ (introduced in [7]) is defined as follows. First, we consider the union of $2g+1$ box-shaped neighborhoods of a vertical sequent lattice points corresponding to $\alpha_0,\alpha_1\cdots,\alpha_{2g}$. Two adjacent box neighborhoods are overlapped with a horizontal unit segment. Next, we take the infinite parallel copies moved by $n\cdot\mathbf{v}$ where $\mathbf{v}$ is the vector $(1,-k_2)$ and n is any integer. We denote the union of the infinite parallel copies of $\mathcal{N}$ and call it *non-zero region*. Moving a non-zero region $\mathcal{N}$ by $n\cdot(0,p)$ for any integer n, we obtain infinite non-zero regions on $\mathbb{R}^2$.

The following lemma is important to prove the main theorem. This is also the case of $m=0$ in Lemma 4.4 in [7].

Lemma 2.4. *If there exist integers i_0,j_0 such that $dA_{i_0,j_0}=-dA_{i_0,j_0+1}=-1$, then for any integer i, there are no two adjacent zeros in the sequence $\{dA_{i,s}|s\in\mathbb{Z}\}$.*

Proof. We assume the existence of integers i_0,j_0. Let x be $i_0+(j_0-1)ek$. Using the formula (1), we have $[q_2x]_p\in I_{-k_2}$, $[q_2x+k_2]_p\in I_{k_2}$, and $[q_2x+2k_2]_p\in I_{-k_2}$. Hence, the sequence $[q_2x+sk_2]_p$ starts at $[q_2x]_p$ and returns in I_{-k_2} at $s=2$. Therefore, we have $p-k_2<(q_2x+2k_2)-q_2x<p+k_2$ and this means $p<3k_2$.

We suppose $dA_{i,j}=dA_{i,j+1}=0$ for some integers i,j. Then $[q_2(i+jek)]_p,[q_2(i+jek)+k_2]_p\notin I_{-k_2}\cup I_{k_2}$. This implies $p-k_2-k_2\geq k_2$. This contradicts the inequality above.

If for an integer I, the sequence $\{dA_{I,s}|s\in\mathbb{Z}\}$ has no adjacent zeros, for any integer i the same thing holds because the $\{dA_{i,s}|s\in\mathbb{Z}\}$ is a parallel copies of $\{dA_{I,s}|s\in\mathbb{Z}\}$. Hence, the desired condition is satisfied. □

Note that this lemma holds for any relatively prime positive integers (p,k). Actually, to prove this lemma we do not require that the matrices A and dA come from a lens space knot in S^3. In particular, if any $K_{p,k}$ in $Y_{p,k}$ (defined in Section 1.4) has non-zero third term in the Alexander polynomial, then $p<3k_2$ holds. To prove Conjecture 1.5, first we should probably classify $(Y_{p,k},K_{p,k})$ in the case of $3k_2<p$.

2.2. **Proof of Theorem 1.2.** Let K be a lens space knot with lens surgery parameter (p,k) and with $g=g(K)$. Suppose that $\alpha_0=1$, $\alpha_1=-1$, and $\alpha_2=1$. Now we assume that $2g-k_2\geq 3$. Since any non-zero curve has no finite ends, $\alpha_4=-1$ holds naturally.

Hence, we can assume that

$$\begin{cases} \alpha_0 = 1 \\ \alpha_1 = -1 \\ \alpha_3 = 1 \\ \alpha_4 = -1. \end{cases} \qquad (*)$$

Let i, j be fixed integers with $k_2(i + jek - c) = -g \bmod p$. Then $A_{i,j} = \alpha_0 = 1$. $A_{i-1,j} = A_{i-1,j+1} = A_{i-1,j+2} = A_{i-1,j+3} = 0$, because any non-zero curve is included in a non-zero region $\mathcal{N}$ due to Proposition 2.3.

We notice that the assumption of Lemma 2.4 is satisfied. Thus, we have $dA_{i,j} = 1$, $dA_{i,j+1} = -1$, $dA_{i,j+2} = 1$, and $dA_{i,j+3} = -1$. The local values for matrices A and dA are drawn in the top pictures in Figure 1.

Our situation falls into the following two cases (I), and (II) as in Figure 2.

(I): $A_{i+1,j+1} = -1$, and $A_{i+1,j+2} = 1$

(II): $A_{i+1,j+1} = 0$, and $A_{i+1,j+2} = 0$.

In the case of (I), we obtain $dA_{i+1,j+1} = dA_{i+1,j+2} = 0$. This contradicts Lemma 2.4.

Next, consider the case of (II). We claim $A_{i+2,j+1} = A_{i+2,j+2} = 0$. If $A_{i+2,j+1}$ or $A_{i+2,j+2}$ is non-zero, then the non-zero term is included in the non-zero region right next to $\mathcal{N}$, because there is only one non-zero curves in any non-zero region (Proposition 2.3). This implies that by seeing the vertical coordinate in $\mathbb{R}^2$, we have

$$p - 2k_2 \leq 2.$$

Since $2k_2 < p$, we have $p = 2k_2 + 1$ or $2k_2 + 2$. The equality $p = 2k_2 + 1$ means it gives a $(2, 2g + 1)$-torus knot surgery. We consider the case of $p = 2k_2 + 2$. Since p, k_2 are relatively prime, k_2 is an odd number. The equality $p = 2k_2 + 2$ can be deformed into $k_2^2 - 1 = \frac{k_2-1}{2}p \equiv 0 \bmod p$. It is a lens space surgery yielding $L(p, 1)$. This is the $k_2 = 1$ case only due to [4]. Thus the claim above is true.

The remaining cases are $-2 \leq -2g + k_2 \leq 1$. By using Theorem 1.15 and Theorem 4.20 in [7], the cases are realized by the $(2, 2g + 1)$-torus knot surgeries or the lens surgery on $T(3, 4)$ or $Pr(-2, 3, 7)$. The knots $T(3, 4)$ and $Pr(-2, 3, 7)$ both do not satisfy $(*)$. Thus, our remaining cases satisfying this condition $(*)$ give the equality $\Delta_K(t) = \Delta_{T(2,2g+1)}$. □

2.3. Proof of Corollary 1.4.

We give a proof of Corollary 1.4. Let K be a lens space knot in S^3 with parameter (p, k). If $\Delta_K(t)$ has $\alpha_0 = -\alpha_1 = \alpha_2 = 1$, then $\Delta_K(t) = \Delta_{T(2,2g+1)}$ holds by Theorem 1.2. Using Theorem 1.15, the lens surgery parameter is $(p, 2)$. The parameter is realized by a $(2, 2g + 1)$-torus knot. □

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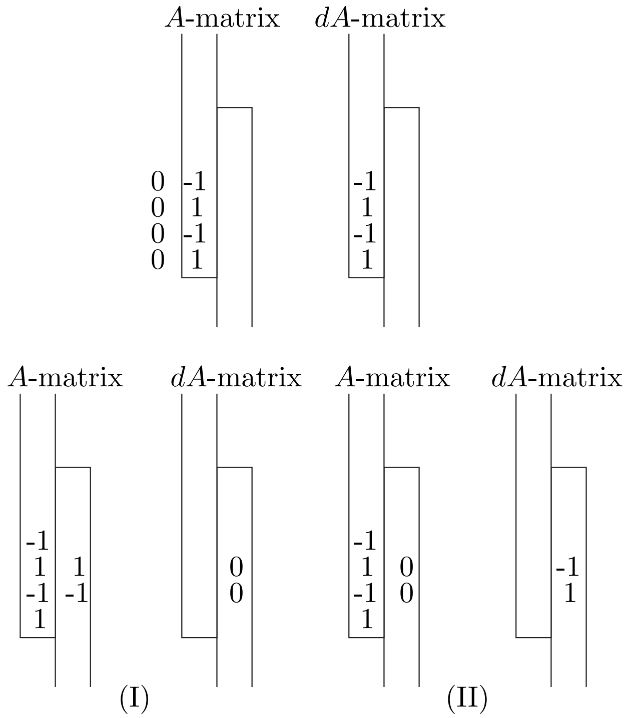

FIGURE 1. Case (I) and (II)

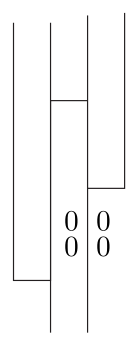

FIGURE 2. An A-matrix of (II)

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