EXOTIC 4-MANIFOLDS OBTAINED BY AN INFINTE ORDER PLUG

MOTOO TANGE

Abstract. In the previous paper the author defined an infinite order plug (P, φ) which gives rise to infinite Fintushel-Stern's knot-surgeries. Here, we give two 4-dimensional infinitely many exotic families Y_n, Z_n of exotic enlargements of the plug. The families Y_n, Z_n have $b_2 = 3$, 4 and the boundaries are 3-manifolds with $b_1 = 1$, 0 respectively. We give a plug (or g-cork) twist $(P, \varphi_{p,q})$ producing the 2-bridge knot or link surgery by combining the plug (P, φ). As a further example, we describe a 4-dimensional twist (M, μ) between knot-surgeries for two mutant knots. The twisted double concerning (M, μ) gives a candidate of exotic $\#^2 S^2 \times S^2$.

1. Introduction

1.1. **Corks and plugs.** If two smooth manifolds X, X' are homeomorphic but non-diffeomorphic, then we say that X and X' are *exotic (or exotic pair)*.

A cut-and-paste is a performance removing a submanifold Z from X and regluing Y via $\phi : \partial Y \to \partial Z$. We use the notation $(X - Z) \cup_\phi Y$ for the cut-and-paste. We call a cut-and-paste *a local move* in this paper. Let Y be a (codimension 0) submanifold of a 4-manifold X. Let ϕ be a diffeomorphism $\partial Y \to \partial Y$. We denote the local move along (Y, ϕ) by

$$X(Y, \phi) := [X - Y] \cup_\phi Y,$$

and call such a local move *a twist*.

For a pair of exotic 4-manifolds X, X', we call a compact contractible Stein manifold Cr *a cork*, if Cr is smoothly embedded in X, and X' is obtained by a cut-and-paste of $Cr \subset X$ according to a diffeomorphism $\tau : \partial Cr \to \partial Cr$. Hence, the boundary diffeomorphism τ cannot extend to inside Cr as a diffeomorphism. We also call the deformation a *cork twist*. Suppose that X and X' are two exotic simply-connected closed oriented 4-manifolds. Then they are changed to each other by a cork twist (Cr, τ) with

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an order 2 boundary diffeomorphism τ ([3], [7], [13]). Namely, this means

$$X' = X(Cr, \tau).$$

Hence, in some sense, the existence of such a cork (Cr, τ) causes 4-dimensional differential structures. Akbulut and Yasui in [4] defined another kind of twists, which are called *plug twists*, and which change smooth structures. The definition is given in the later section. A study of cork and plug should play a key role in understanding differential structures of 4-manifolds.

In this paper, we produce infinitely many exotic enlargements of P. The meaning of studying enlargements is to investigate to what extent the 'exotic producer' like cork or plug can extend to a larger 4-manifold. Combining a plug (P, φ) defined in [16] and isotopy, we give many enlargements. We also apply 2-component links, rational tangles, and the knot mutation to 4-dimensional twists.

1.2. **Infinite order cork and plug.** Let P denote a 4-manifold described in FIGURE 1. A diffeomorphism $\varphi : \partial P \to \partial P$ is defined to be FIGURE 2.

Remark 1. *Throughout this paper, any unlabeled component in any diagrams of 4-manifolds or of 3-manifolds stands for a* 0*-framed 2-handle or a* 0*-surgery respectively.*

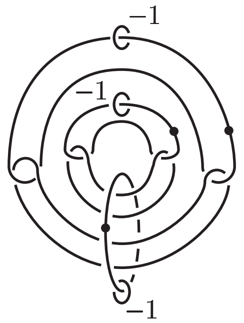

FIGURE 1. A handle decomposition of P.

The paper [16] shows that the twist (P, φ) is an infinite order plug, and the square twist (P, φ^2) is a generalized cork (a g-cork), as defined later. Namely, (P, φ) and (P, φ^2) satisfy the following:

Theorem 1 ([16])**.** *P is a Stein 4-manifold. The map $\varphi : \partial P \to \partial P$ has infinite order and φ cannot extend to a self-homeomorphism on inside P. There exists a 4-manifold X such that $\{X(P, \varphi^k)\}$ is a family of mutually exotic 4-manifolds.*

The map φ^2 can extend to a self-homeomorphism, but cannot extend to any self-diffeomorphism on P.

In general, we define an infinite order plug, cork and g-cork.

Definition 1 (Infinite order plug)**.** $(\mathcal{P}, \phi)$ *is an infinite order plug if it satisfies the following conditions:*

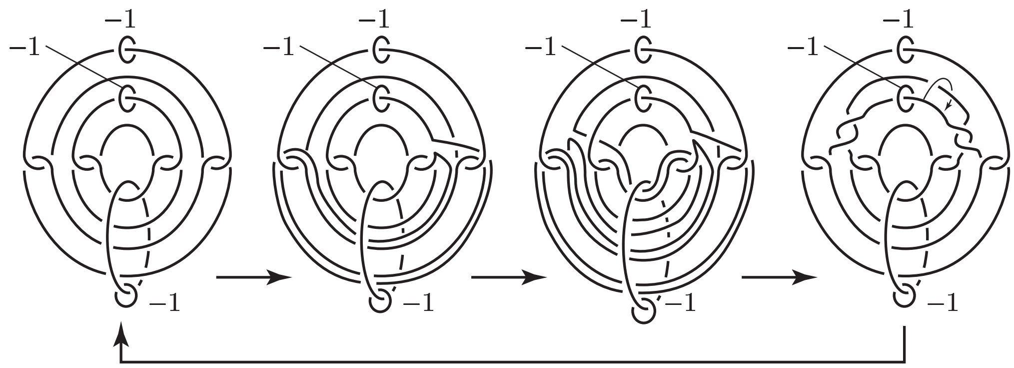

FIGURE 2. A diffeomorphism $\varphi : \partial P \to \partial P$.

(1) *$\mathcal{P}$ is a compact Stein 4-manifold.*
(2) *ϕ cannot extend to a self-homeomorphism on $\mathcal{P}$.*
(3) *There exists a 4-manifold X and embedding $\mathcal{P} \subset X$ such that $\{X(\mathcal{P}, \varphi^k)\}$ is a family of mutually exotic 4-manifolds.*

Definition 2 (Infinite order cork)**.** *$(\mathcal{C}, \phi)$ is an infinite order cork if it satisfies the following conditions:*

(1) *$\mathcal{C}$ is a compact contractible Stein 4-manifold.*
(2) *ϕ^k cannot extend to any self-diffeomorphism on $\mathcal{C}$ for any positive integer k.*

If $X(\mathcal{C}, \phi^k)$ are mutually exotic 4-manifolds, then $(\mathcal{C}, \phi)$ is an infinite order cork for $\{X(\mathcal{C}, \phi^k)\}$. In the case where $\mathcal{C}$ is not contractible, in place of being contractible in the (1) condition, we call $(\mathcal{C}, \phi)$ a generalized cork (or g-cork).

The order of each ϕ in Definition 1 and 2 as a mapping class on the boundary 3-manifold is infinite. The twist (P, φ) defined above is an infinite order plug and (P, φ^2) is an infinite order g-cork. Furthermore, the plug twist (P, φ) can make a Fintushel-Stern's knot-surgery. Let V and C denote the neighborhoods of Kodaira's singularity III and II. See FIGURE 3 for the diagrams. C is called a *cusp neighborhood*. These diagrams can be also seen in [11].

Theorem 2 ([16])**.** *Let X be a 4-manifold containing V and let K be a knot. Let X_K be a knot-surgery of X along the general fiber of V. For a knot K' obtained by changing a crossing of a diagram of K, there exists an embedding $i : P \hookrightarrow X_K$ such that for the embedding i we have*

$$X_{K'} = X_K(P, \varphi).$$

The n-th power (P, φ^n) makes an n times full-twist

$$X_{K_n} = X_K(P, \varphi^n),$$

where K and K_n are the two knots whose local diagrams are FIGURE *4 and whose remaining diagrams are the same thing.*

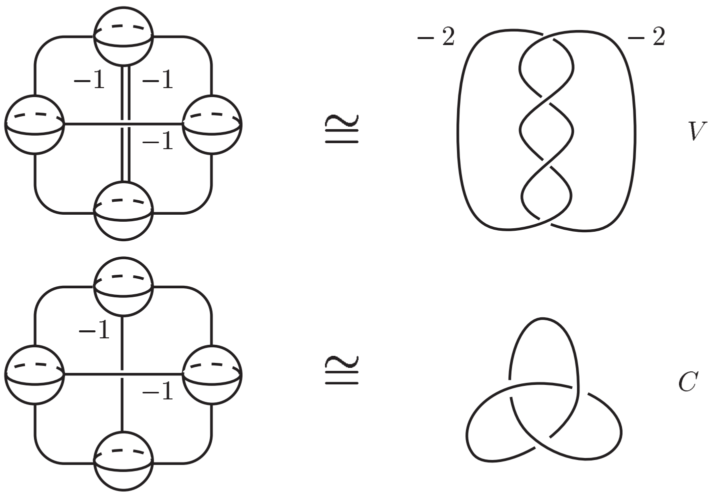

FIGURE 3. The neighborhoods of Kodaira's singularity III and II (cusp).

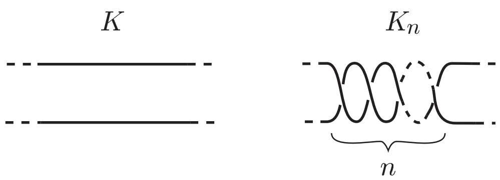

FIGURE 4. K_n is the n-full twist of K.

Remark 2. *The crossing change is a local move for knots or links. Theorem 2 means that the plug twist* (P, φ) *plays a role in 'the crossing change of 4-manifolds' obtained by knot-surgery in some sense. Similarly, for many other local moves of knots or links, one can construct a local move over 4-manifolds. We will give an example of a 4-dimensional local move (twist) coming from a local move (knot mutation) of knots and links at the later section.*

Here we define the knot-surgery and (2-component) link-surgery according to [8]. Let $T \subset X$ be an embedded torus with trivial neighborhood and K a knot in S^3. The 0-surgery M_K cross S^1 naturally contains an embedded torus $T_m = \{\text{meridian}\} \times S^1$ with the trivial neighborhood. Then, *(Fintushel-Stern's) knot-surgery* X_K is defined to be the fiber-sum

$$X_K = (M_K \times S^1)\#_{T_m=T} X.$$

Let U_1, U_2 be two 4-manifolds containing an embedded tori $T_i \subset U_i$ with the trivial neighborhoods. Let $L = K_1 \cup K_2$ be a 2-component link. Let

$$\alpha_L : \pi_1(S^3 - L) \to \mathbb{Z}$$

be a homomorphism satisfying $\alpha_L(m_i) = 1$, where m_i is the meridian curve of K_i. Let M_L be the $\alpha(\ell_i)$-surgery of L, where ℓ_i is the longitude of K_i.

Let T_{m_i} be a torus $m_i \times S^1 \subset M_L \times S^1$. Then, we denote by $(U_1, U_2)_L$ the following double fiber-sum operation:

$$(U_1, U_2)_L = U_1 \#_{T_1 = T_{m_1}} (M_L \times S^1) \#_{T_{m_2} = T_2} U_2.$$

In the case of $U = U_1 = U_2$, we write as $(U, U)_L = U_L$. We call $(U_1, U_2)_L$ *the link-surgery by the link* L.

1.3. **Two kinds of enlargements Y_n and Z_n.** Akbulut-Yasui's corks (W_n, f_n) and plugs $(W_{m,n}, f_{m,n})$ in [4] can give exotic enlargements by attaching 2-handles. In this paper we consider two kinds of enlargements $Y_0 = P \cup h_1$ and $Z_0 = P \cup h_1 \cup h_2$, where h_1, and h_2 are two 2-handles on P as indicated in FIGURE 5 and the framings are both -1. Hence, we have

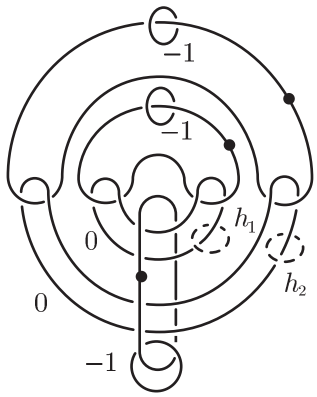

FIGURE 5. Two kinds of attachments $Y_0 = P \cup h_1$ and $Z_0 = P \cup h_1 \cup h_2$.

$$Y_0 = \tilde{Y} \# \overline{\mathbb{C}P^2}$$

and

$$Z_0 = \tilde{Z} \#^2 \overline{\mathbb{C}P^2}.$$

$\tilde{Y}$ (and $\tilde{Z}$) are 4-manifolds presented by the left (and right) diagrams in FIGURE 6.

Let Y_n and Z_n define to be other enlargements obtained by twists

$$Y_n = Y_0(P, \varphi^n) \quad (1)$$

and

$$Z_n = Z_0(P, \varphi^n) \quad (2)$$

with respect to the embeddings $P \hookrightarrow Y_0$ and Z_0. Since Y_n and Z_n are the 2-handle attachments of the simply-connected manifold P, they are also simply-connected and the Betti numbers b_2 of them are 3 and 4 respectively.

The g-cork (P, φ^2) in [16] gives the diffeomorphisms:

$$Y_{n+2} \simeq Y_n \quad (3)$$

and

$$Z_{n+2} \simeq Z_n. \tag{4}$$

In this paper we use notation $\cong$ and $\simeq$ as a diffeomorphism and a homeomorphism respectively. Hence, Y_{2n} (or Z_{2n}) is homeomorphic to Y_0 (or Z_0) and Y_{2n+1} (or Z_{2n+1}) is homeomorphic to Y_1 (or Z_1). Actually Y_n and Z_n give the four homeomorphism types.

Proposition 1. *Let X be Y or Z. In $\{X_n\}$ there exist two homeomorphism types X_0 and X_1 and we have*

$$X_n \simeq \begin{cases} X_0 & n \equiv 0 \bmod 2 \\ X_1 & n \equiv 1 \bmod 2. \end{cases}$$

This proposition is proven by seeing intersection forms in later section. The boundary ∂Y_n is diffeomorphic to the 3-manifold described by the left diagram in FIGURE 6. This is a 0-surgery on $-\Sigma(2,3,5)$ as the left diagram in FIGURE 6. The boundary ∂Z_n is 1-surgery of the granny knot. The proof is in FIGURE 7.

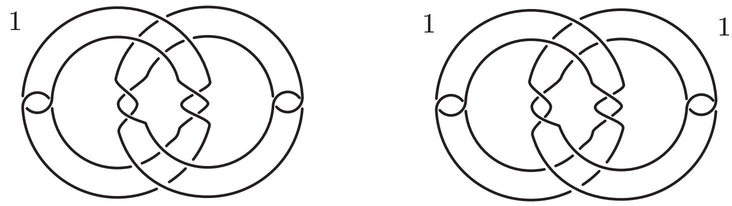

FIGURE 6. Diagrams of $\tilde{Y}$ and $\tilde{Z}$ (as 4-manifolds) and $\partial\tilde{Y}$ and $\partial\tilde{Z}$ (as 3-manifolds).

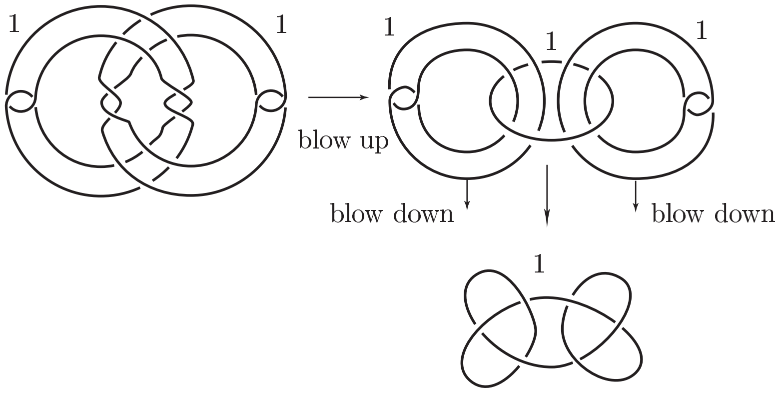

FIGURE 7. ∂Z_n is homeomorphic to 1-surgery of the granny knot.

From the view point of geometry, Y_n and Z_n have the following property.

Theorem 3. *Let n be a positive integer. Y_n and Z_n are submanifolds of irreducible symplectic manifolds.*

For the differential structures, we get the following theorem.

Theorem 4. *Let n be any positive integer n. Then Y_{2n} and Y_0 are exotic.*

Whether $\{Y_n\}$ are mutually non-diffeomorphic manifolds is unknown, however, we can prove the following.

Theorem 5. *Each of $\{Y_{2n}|n \in \mathbb{N}\}$ and $\{Y_{2n+1}|n \in \mathbb{N}\}$ contains infinitely many differential structures.*

We will prove this theorem in Section 2.1. The differential structures $\{Z_n\}$ satisfy the following.

Theorem 6. *$\{Z_{2n}|n \geq 0\}$ and $\{Z_{2n+1}|n \geq 0\}$ are two families of mutually exotic 4-manifolds.*

1.4. **A twist for a rational tangle replacement.** Let K_i be a knot or link for $i = 1, 2$. K_2 is a *tangle replacement* of K_1, if the local move $K_1 \rightsquigarrow K_2$ satisfies the following:

- K_2 is a local move of K_1 with respect to a closed 3-ball B^3 that K_i and ∂B^3 transversely intersects at $K_i \cap \partial B^3$.
- $K_1 \cap B^3$ and $K_2 \cap B^3$ are proper embeddings of several arcs in B^3.
- the arcs are homotopic to each other by a homotopy that fixes the boundary.

The usual crossing change of knots and links is one example of tangle replacements. Figure 8 is a picture of tangle replacement which is described schematically.

In this paper we treat the tangle replacements satisfying the following conditions. Let T_i denote $K_i \cap B^3$.

- $\partial T_i \subset \partial B^3$ are four points
- $B^3 \setminus K_1$ is homeomorphic to $B^3 \setminus K_2$.

The first example is the case where $B^3 \setminus K_i$ is homeomorphic to the genus two handlebody. We call the replacement *rational tangle replacement.*

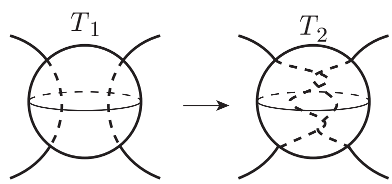

Figure 8. The tangle replacement $T_1 \to T_2$.

Let (p, q) be relatively prime integers with p even. We define a diffeomorphism $\varphi_{p,q} : \partial P \to \partial P$ in Section 3. The pair $(P, \varphi_{p,q})$ satisfies the following:

Proposition 2. *Let p be an even integer with $p \neq 0$. The twist $(P, \varphi_{p,q})$ is an infinite order*

$$\begin{cases} \textit{plug} & p \equiv 2 \bmod 4 \quad \textit{or} \\ \textit{g-cork} & p \equiv 0 \bmod 4. \end{cases}$$

Let O_n denote the n-component unlink.

Theorem 7. *Let X be a 4-manifold containing V and let $K_{p,q}$ be a non-trivial 2-bridge knot. Then there exists an embedding $i : P \hookrightarrow V \subset X$ such that the twist $(P, \varphi_{p-1,q})$ with respect to i gives the knot-surgery*

$$X := X_{O_1} \rightsquigarrow X(P, \varphi_{p-1,q}) = X_{K_{p,q}}.$$

Let X_i be a 4-manifold containing C and let $K_{p,q}$ be a non-trivial 2-bridge link. Let X be $X_1 \# X_2 \# S^2 \times S^2$. Then there exists an embedding $j : P \hookrightarrow X$ such that the twist $(P, \varphi_{p,q})$ with respect to j gives the link-surgery

$$X = (X_1, X_2)_{O_2} \rightsquigarrow X(P, \varphi_{p,q}) = (X_1, X_2)_{K_{p,q}}.$$

This is a generalization of the result (Theorem 1) that (P, φ) is a plug and (P, φ^2) is a g-cork. Namely, the case of $(p, q) = (2n, 1)$ corresponds to the equality $\varphi_{2n,1} = \varphi^n$.

By combining the twist and the inverse in Theorem 7 we also obtain a general rational tangle replacement

$$X_K \overset{\varphi_{p,q}^{-1}}{\rightsquigarrow} X_{O_1} \overset{\varphi_{r,s}}{\rightsquigarrow} X_{K'}.$$

1.5. **A twist for mutant knots.** We call an involutive tangle replacement as in Figure 9 *knot mutation* and we call two knots K, K' which are obtained by the knot mutation *mutant knots*. It is well-known that mutant

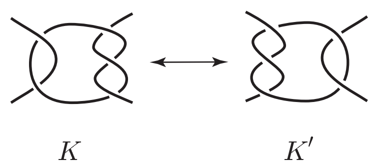

Figure 9. A knot mutation.

knots have similar topological properties. Any two mutant knots have the same hyperbolic volume and HOMFLY polynomial, in particular, the same Alexander polynomial.

The next variation of (P, φ) is a twist between knot-surgeries for any two mutant knots. The knot mutation is not a rational tangle replacement, because the local tangle complement is not homeomorphic to a handlebody. Indeed, compute the fundamental group of the local tangle complement. We found a twist (M, μ) of 4-manifold between the knot-surgeries for two mutant knots. Let M be a 4-manifold described by Figure 10. A map $\mu : \partial M \to \partial M$ is defined in Section 3.3. From the diagram in Figure 10, we can prove that M is an oriented, simply-connected 4-manifold with $\partial M =$

$\partial P\#S^2 \times S^1$, $H_*(M) \cong H_*(\vee^3 S^2)$, $b_3(M) = 0$. $\partial M \cong \partial P\#S^2 \times S^1$ is described in Figure 11.

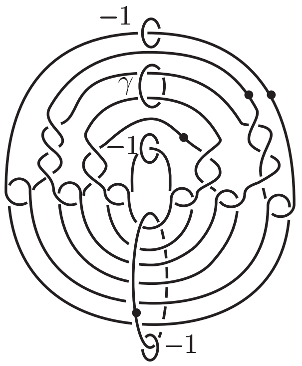

Figure 10. The manifold M.

Theorem 8. *Let X be a 4-manifold containing V. Let K, K' be any mutant knots. Then there exist a twist (M, μ) and an inclusion $i : M \hookrightarrow X_K$ such that the square of the gluing map $\mu : \partial M \to \partial M$ is homotopic to the trivial map on ∂M and changes the knot-surgeries as follows:*

$$X_{K'} = X_K(M, \mu).$$

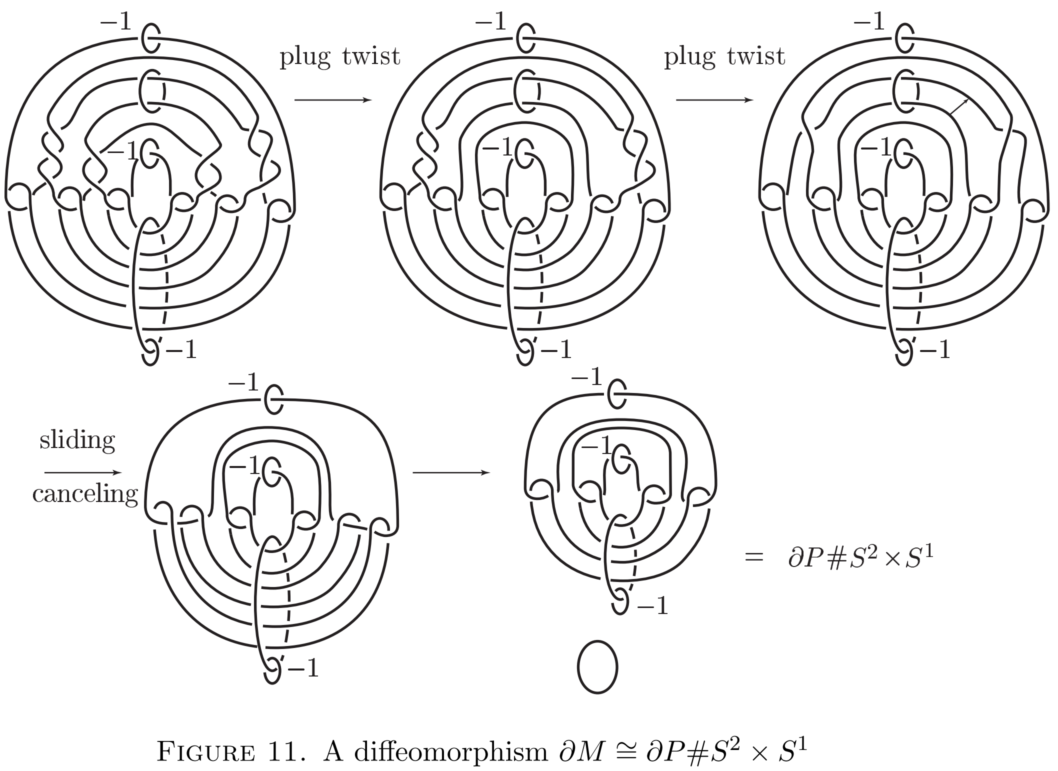

Figure 11. A diffeomorphism $\partial M \cong \partial P\#S^2 \times S^1$

Proposition 3. *The map $\mu : \partial M \to \partial M$ extends to a self-homeomorphism on M.*

It is a subtle problem whether μ extends to a self-diffeomorphism on M. One reason is what M *may not* be a Stein manifold. Another reason is what for mutant knots K and K', the Seiberg-Witten invariants are the same by Fintushel-Stern's formula in [8].

Remark 3. *If μ can extend to M as a diffeomorphism, then two knot-surgeries of all pairs of mutant knots are diffeomorphic to each other. Unlike the examples by Akbulut* [2]*, and Akaho* [1]*, this diffeomorphism suggests a meaningful map coming from knot mutation.*

If μ cannot extend to inside M as any diffeomorphism, then (M, μ) would be a new (possibly not-Stein) g-cork giving a subtle effect.

The twisted double $D_\mu(M) := M \cup_\mu (-M)$ is homeomorphic to $\#^3 S^2 \times S^2$. Its diffeomorphism type is not-known. $D_\mu(M)$ has one connected-sum component of $S^2 \times S^2$, i.e. it is not irreducible.

Proposition 4. *M_0 be a 4-manifold M with a 2-handle deleted and let μ_0 be a boundary diffeomorphism $\partial M_0 \to \partial M_0$ naturally induced from μ. Then we have $D_\mu(M) = D_{\mu_0}(M_0) \# S^2 \times S^2$.*

Here we summarize several questions.

Question 1. *Does M admit a Stein structure?*

Question 2. *Can the map μ extend to a self-diffeomorphism on M?*

Question 3. *Is $D_\mu(M)$ (or $D_{\mu_0}(M_0)$) an exotic $\#^3 S^2 \times S^2$ (or $\#^2 S^2 \times S^2$)?*

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2. Exotic enlargements.

2.1. The homeomorphism types.

We consider four homeomorphism types Y_0, Y_1, Z_0 and Z_1 of the enlargement of P.

Lemma 1. *The intersection forms of Y_n and Z_n are as follows:*

$$Q_{Y_n} \cong \begin{cases} \langle 0 \rangle \oplus H & n : odd \\ \langle 0 \rangle \oplus \langle 1 \rangle \oplus \langle -1 \rangle & n : even, \end{cases}$$

$$Q_{Z_n} = \begin{cases} \oplus^2 H & n{:}odd \\ \oplus^2 \langle 1 \rangle \oplus^2 \langle -1 \rangle & n{:}even, \end{cases}$$

where H is the quadratic form presented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof of Proposition 1. Lemma 1, (3), and (4) imply the required assertion. □

Proof of Lemma 1. From the homeomorphisms (3) and (4), we may consider homeomorphism types Y_0, Y_1 and Z_0, Z_1 respectively. From the picture in FIGURE 5 together with 2-handles, the intersection forms of Y_0 and Z_0 can be immediately seen $\langle 0\rangle \oplus \langle 1\rangle \oplus \langle -1\rangle$ and $\oplus^2\langle 1\rangle \oplus^2 \langle -1\rangle$ respectively. The diagram of Z_1 is the left of FIGURE 12 and the diagram of Y_1 is FIGURE 12 with the -2-framed component erased. Hence, the intersection forms of Y_1 and Z_1 are isomorphic to $\langle 0\rangle \oplus H$ and $\oplus^2 H$ respectively. □

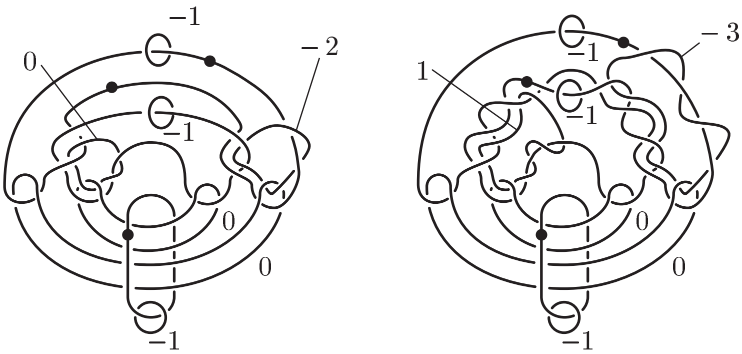

FIGURE 12. Z_1 and Z_2.

2.2. **Infinitely many exotic structures on Y_0 and Y_1.** Fintushel and Stern in [8] computed the Seiberg-Witten invariant of the link-surgery. Let L_n be the $(2, 2n)$-torus link, in particular, the $(2, 2)$-torus link is the Hopf link. Then the Seiberg-Witten invariant is as follows:

$$SW_{E(1)_{L_n}} = \Delta_{L_n}(t_1, t_2) = (t_1t_2)^{n-1} + (t_1t_2)^{n-3} + \cdots + (t_1t_2)^{-n+1}. \tag{5}$$

Thus, the basic classes are the following:

$$\mathcal{B}_{E(1)_{L_n}} = \{i(t_1 + t_2)|i = -n + 1, n + 3, \cdots , n - 1\}. \tag{6}$$

Each variable $t_i \in H^2(E(1)_{L_n})$ is the Poincaré dual $PD(2[T_i])$. The submanifolds T_1, and T_2 are general fibers of the two copies of $E(1)$. This implies that $E(1)_{L_n}$ are mutually non-diffeomorphic manifolds. We prove the following lemma about $E(1)_{L_n}$:

Lemma 2. *For any positive integer n, $E(1)_{L_n}$ is an irreducible symplectic manifold.*

Proof. We assume that $E(1)_{L_n}$ has an embedded sphere C with $[C]^2 = -1$. Since the intersection form is odd, n is even. We may assume C is a symplectic sphere. Let E' be the blow-downed manifold along C. Then the Seiberg-Witten basic classes $\mathcal{B}_{E(1)_{L_n}}$ are of form $\{k \pm PD(C)|k \in \mathcal{B}_{E'}\}$. The basic classes $k_\pm = k \pm PD(C)$ satisfy $(PD(k_+) - PD(k_-))^2 = 4C^2 = -4$. However, from the basic classes (6), the self-intersection number of the difference of any two of the basic classes is zero. This is contradiction.

Since $E(1)_{L_n}(n \neq 0)$ is a simply-connected, minimal symplectic manifold with $b_2^+ > 1$, it is irreducible due to [12]. Thus Z_n is also an irreducible symplectic manifold. □

We prove Theorem 3.

Proof of Theorem 3. We will prove $Y_n \subset E(1)_{L_n}$. The manifold $P \subset Y_0$ is embedded in $E(1)_{L_0}$ by the definition. See [15] for the embedding. The twist of (P, φ^n) via the embedding $P \hookrightarrow E(1)_{L_0}$ gets $E(1)_{L_n}$. Then Y_0 changes to Y_n in $E(1)_{L_n}$. This result is due to Theorem 2 or [16]. From Lemma 2, Y_n and Z_n are submanifolds of an irreducible symplectic 4-manifold. For $n = 1$, see Figure 13.

Applying the same twist for $Z_0 \subset E(1)_{L_0}$, we can obtain an embedding $Z_n \hookrightarrow E(1)_{L_n}$. □

Notice that each of 2-handles h_1 or h_2 in Y_n and Z_n corresponds to the sections in $E(1) - \nu(T^2)$.

Proof of Theorem 4. From Theorem 3, if n is positive, then Y_{2n} is irreducible, however Y_0 has a (-1)-sphere. Thus Y_{2n} is not diffeomorphic to Y_0. From Proposition 1, Y_0 and Y_{2n} are homeomorphic. □

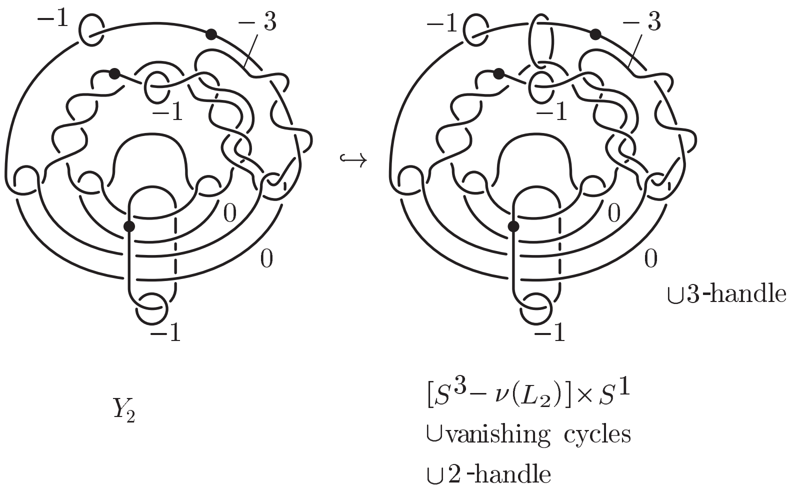

Figure 13. $Y_2 \hookrightarrow [S^3 - \nu(L_n)] \times S^1 \cup 3$ vanishing cycles$\cup$2-handle.

Next, we will prove the existence of infinitely many mutually exotic differential structures in $\{Y_n\}$. First, we prove the following lemmas:

Lemma 3. *Let Q be a quadratic form $\langle 0\rangle \oplus \langle -1\rangle \oplus \langle 1\rangle$ on $\mathbb{Z}^3$. Any isomorphism $(\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q)$ preserving Q is presented by*

$$\begin{pmatrix} \epsilon_1 & a & b \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix},$$

where each ϵ_i is ± 1 and a, b are any integers.

Proof. Let ϕ be any isomorphism $\phi : (\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q)$ preserving the quadratic form $Q = \langle 0\rangle \oplus \langle -1\rangle \oplus \langle 1\rangle$. For the standard generator $\{\mathbf{e}_i\}$ in

$\mathbb{Z}^3$, we denote the images by $\phi(\mathbf{e}_i) = a_i\mathbf{e}_1 + b_i\mathbf{e}_2 + c_i\mathbf{e}_3$. Since ϕ preserves Q, we have

$$\begin{cases} -b_1^2 + c_1^2 = 0, \ -b_2^2 + c_2^2 = -1, \\ -b_3^2 + c_3^2 = 1, \ -b_1b_2 + c_1c_2 = 0, \\ -b_1b_3 + c_1c_3 = 0, \ -b_2b_3 + c_2c_3 = 0. \end{cases}$$

Solving these equations, we have $c_2 = 0$, $b_3 = 0$, $b_2 = \pm 1$, and $c_3 = \pm 1$. Furthermore, we have $b_1 = c_1 = 0$. Here we put $b_2 =: \epsilon_2$, and $c_3 =: \epsilon_3$. Since the map ϕ is an automorphism on $\mathbb{Z}^3$, we have $a_1 =: \epsilon_1$, where $\epsilon_1 = \pm 1$. Hence, defining as $a = a_2$ and $b = a_3$, we get the presentation matrix of ϕ. □

Lemma 4. *Let Q be a quadratic form $\langle 0\rangle \oplus H$ on $\mathbb{Z}^3$. Any isomorphism $(\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q)$ preserving Q is presented by*

$$\begin{pmatrix} \epsilon_1 & a & b \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}, \ or \ \begin{pmatrix} \epsilon_1 & a & b \\ 0 & 0 & \epsilon_2 \\ 0 & \epsilon_2 & 0 \end{pmatrix}$$

where each ϵ_i is ± 1 and a, b are any integers.

Proof. Let ϕ be any isomorphism $(\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q)$. For the standard generator $\{\mathbf{e}_i\}$ in $\mathbb{Z}^3$, we denote the images by $\phi(\mathbf{e}_i) = a_i\mathbf{e}_1 + b_i\mathbf{e}_2 + c_i\mathbf{e}_3$. Since ϕ preserves Q, we have

$$\begin{cases} b_1c_1 = 0, \ b_2c_2 = 0, \ b_3c_3 = 0, \\ b_1c_2 + c_1b_2 = 0, \ b_1c_3 + c_1b_3 = 0, \ b_2c_3 + c_2b_3 = 1. \end{cases}$$

If $b_1 \neq 0$, then $c_1 = 0$ holds from the first equation. Then, from $c_2 = -\frac{c_1b_2}{b_1}$ and $c_3 = -\frac{c_1b_3}{b_1}$, we obtain $c_2 = c_3 = 0$. This is contradiction for the last equation. Thus $b_1 = 0$ holds. In the same way $c_1 = 0$ holds.

Since $b_2b_3c_2c_3 = 0$, we have $b_2c_3 = 0$ or $c_2b_3 = 0$. If $b_2c_3 = 0$, then $c_2b_3 = 1$, hence, $c_2 = b_3 = \pm 1$ and $b_2 = c_3 = 0$ (because $b_2c_2 = 0$ and $c_3b_3 = 0$). If $c_2b_3 = 0$, then $b_2c_3 = 1$, hence $c_2 = b_3 = \pm 1$ and $b_3 = c_2 = 0$.

Since the map ϕ is an isomorphism, we get $a_1 = \pm 1$. Therefore, we get the presented matrix of ϕ as above. □

Here we introduce the following result in [14]:

Proposition 5 ([14])**.** *Suppose that Σ is a smooth, embedded, closed 2-dimensional submanifold in a smooth 4-manifold X with $b_2^+(X) > 1$ and for a basic class K we have $\chi(\Sigma) - [\Sigma]^2 - K([\Sigma]) = 2n < 0$. Let ϵ denote the sign of $K([\Sigma])$. Then the cohomology class $K + 2\epsilon PD([\Sigma])$ is also a basic class.*

Lemma 5. *Let m be a positive integer. There exists a generator $\{T_1, T_2, S_m\}$ in $H_2(P_m)$ such that T_i $(i = 1, 2)$ are realized by tori and the genus of the surface realizing S_m is $m(m-1)$. The presentation matrix with respect to this generator is*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2m^2 - m - 1 \end{pmatrix}.$$

Proof. Recall that $Y_0 = P \cup h_1$ and $Y_m = Y_0(P, \varphi^m)$. Here h_1 is the

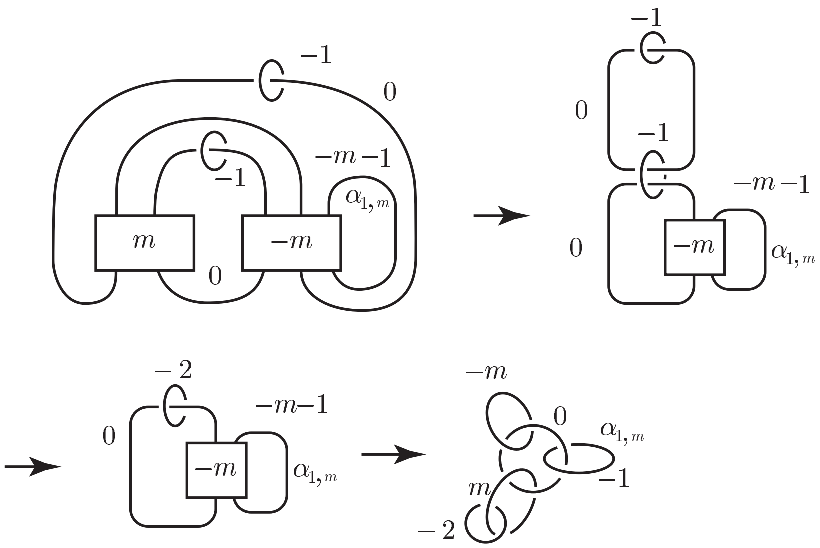

FIGURE 14. The torus knot for S.

2-handle in FIGURE 5. We denote by α_1 the attaching sphere of h_1 and by $\alpha_{1,m}$ the image $\varphi^m(\alpha_1)$. The attaching sphere $\alpha_{1,m}$ is the $(m, 2m+1)$-torus knot on the boundary of the 0-handle (see the fourth pictures in FIGURE 14).

Let m_1, m_2 be the meridians for the link L_m. Let T_i be the embedded torus $T_{m_i} = m_i \times S^1$ in Y_m corresponding to m_i. T_i can be seen in Figure 15.

Let S_m be an embedded surface made from the union of a slice surface in P of $\alpha_{1,m}$ and the core disk of h_1. Hence, the pair $\{T_1, T_2, S_m\}$ is embedded surfaces generating $H_2(Y_m)$, because Y_m consists of $\alpha_{1,m}$ and 0-framed 2-handles by canceling two 1-/2-handle canceling pairs. The latter 0-framed 2-handles correspond to the handle decomposition of P.

The genus is $g(S_m) = \frac{(m-1)2m}{2} = m(m-1)$ since $\alpha_{1,m}$ is the $(m, 2m+1)$-torus knot. The self-intersection number of S_m is $-2m^2 - m - 1$ by

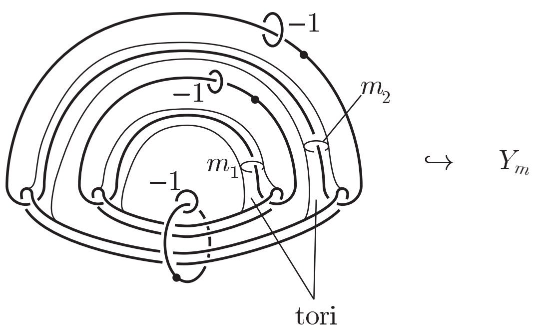

FIGURE 15. These tori are embedded in Y_m as T_1, T_2.

canceling other components by handle calculus. The intersection of T_2 and S_m can be understood from what attaching sphere of T_2 is a meridian of S_m homologically in the same way as FIGURE 13 in [16].

Thus, the presentation matrix for the generators $\{T_1, T_2, S_m\}$ becomes the claimed one.

Proof of Theorem 5. Suppose that there exists a diffeomorphism $\delta : Y_m \cong Y_n$ for some m, n with $0 \leq m < n$ and $n \equiv m(\mathrm{mod} 2)$. We denote by $\{T_1', T_2', S_n\}$ such a pair corresponding to Y_n. We get a smooth inclusion:

$$S_m \subset Y_m \stackrel{\delta}{\to} Y_n \hookrightarrow E(1)_{L_n}.$$

We denote $\delta(S_m)$ simply by S_m in $E(1)_{L_n}$.

Suppose that m is even. The isomorphism $f_\delta : (\mathbb{Z}^3, Q_{Y_m}) \to (\mathbb{Z}^3, Q_{Y_n})$ can be decomposed as follows:

$$(\mathbb{Z}^3, Q_{Y_m}) \to (\mathbb{Z}^3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \to (\mathbb{Z}^3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \to (\mathbb{Z}^3, Q_{Y_n}).$$

Using Lemma 3, we obtain the following presentation for f_δ:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -n^2 - \frac{n}{2} & n^2 + \frac{n}{2} + 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 & a & b \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m^2 - \frac{m}{2} - 1 \\ 0 & 1 & -m^2 - \frac{m}{2} \end{pmatrix}.$$

Hence, the class of S_m in Y_n via δ is presented as follows:

$$\begin{aligned} [S_m] &= \left(-(a+b)\left(m^2 + \frac{m}{2}\right) - a\right)[T_1'] \\ &\quad + \left\{(\epsilon_2 - \epsilon_3)\left(m^2 + \frac{m}{2}\right)\left(n^2 + \frac{n}{2}\right) + \left(n^2 + \frac{n}{2}\right)\epsilon_2 \right. \\ &\quad \left. - \left(m^2 + \frac{m}{2}\right)\epsilon_3\right\}[T_2'] + \left((\epsilon_2 - \epsilon_3)\left(m^2 + \frac{m}{2}\right) + \epsilon_2\right)[S_n]. \end{aligned} \tag{7}$$

Thus, we have the following intersection number

$$[S_m] \cdot ([T_1'] + [T_2']) = (\epsilon_2 - \epsilon_3)\left(m^2 + \frac{m}{2}\right) + \epsilon_2.$$

Here putting $k = PD(\epsilon_2(n-1)([T_1'] + [T_2']))$ and $\eta = \frac{1 - \epsilon_2\epsilon_3}{2}$, we have $k([S_m]) = (n-1)((2m^2 + m)\eta + 1) > 0$.

Here we have

$$\begin{aligned} \chi(S_m) - [S_m]^2 - k([S_m]) &= 2 - 2m(m-1) + (2m^2 + m + 1) \\ &\quad -(n-1)((2m^2 + m)\eta + 1) \\ &= 3m + 3 - (n-1)((2m^2 + m)\eta + 1) \\ &= 3m - n + 4 - (n-1)(2m^2 + m)\eta \\ &\leq 3m - n + 4. \end{aligned}$$

If n satisfies $3m + 4 < n$, then $\chi(S_m) - [S_m]^2 - k([S_m]) = 2\ell < 0$ holds. Using Proposition 5, we have a basic class $k + 2PD([S_m])$.

Here S_n represents a section in $E(1)_{L_n}$ thus $[S_n]$ is a non-vanishing class in $H_2(E(1)_{L_n})$. From the basic classes (6) of $E(1)_{L_n}$, the coefficient of $[S_n]$ in $[S_m]$ must be 0. The coefficient of $[S_n]$ is an odd number. See the coefficient in (7). Thus, this has some contradiction. Therefore, if $3m + 4 < n$ is satisfied, then Y_n is not diffeomorphic to Y_m.

Suppose that m is odd. Any isomorphism $(\mathbb{Z}^3, Q_{Y_m}) \to (\mathbb{Z}^3, Q_{Y_n})$ can be decomposed as follows:

$$Q_{Y_m} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to Q_{Y_n}.$$

Using Lemma 4, we obtain the following presentation for φ:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n^2 + \frac{n+1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 & a & b \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m^2 - \frac{m+1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n^2 + \frac{n+1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 & a & b \\ 0 & 0 & \epsilon_2 \\ 0 & \epsilon_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m^2 - \frac{m+1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

In fact, any automorphism preserving $\langle 0 \rangle \oplus H$ is the solution of

$$[S_m] = \left(b - a\left(m^2 + \frac{m+1}{2}\right)\right)[T_1'] - \epsilon_2(m-n)\left(m + n + \frac{1}{2}\right)[T_2'] + \epsilon_2[S_n]$$

or

$$\begin{aligned} [S_m] &= \left(b - a\left(m^2 + \frac{m+1}{2}\right)\right)[T_1'] + \epsilon_2\left(1 - \left(m^2 + \frac{m+1}{2}\right)\left(n^2 + \frac{n+1}{2}\right)\right)[T_2'] \\ &\quad -\epsilon_2\left(m^2 + \frac{m+1}{2}\right)[S_n]. \end{aligned}$$

Thus, we have

$$[S_m] \cdot ([T_1'] + [T_2']) = \epsilon_2 \text{ or } -\epsilon_2\left(m^2 + \frac{m+1}{2}\right).$$

Here putting $k = PD(\epsilon_2(n-1)([T_1'] + [T_2']))$ or $PD(-\epsilon_2(n-1)([T_1'] + [T_2']))$, we have $k([S_m]) = n - 1 > 0$ or $(n-1)\left(m^2 + \frac{m+1}{2}\right) > 0$ respectively. Thus, we have

$$\begin{aligned} \chi(S_m) - [S_m]^2 - k([S_m]) &= 2 - 2m(m-1) + 2m^2 + m + 1 - \begin{cases} n - 1 \\ (n-1)(m^2 + \frac{m+1}{2}) \end{cases} \\ &= \begin{cases} 3m + 4 - n \\ 3m + 3 - (n-1)(m^2 + \frac{m+1}{2}) \end{cases} \leq 3m + 4 - n. \end{aligned}$$

If n satisfies $3m+4 < n$, then $\chi(S_m) - [S_m]^2 - k([S_m]) = 2\ell < 0$ holds. Using Proposition 5, $k + 2PD([S_m])$ is also a basic class. In the same reason as the case where m is even, the coefficient of $[S_n]$ in $[S_m]$ must be 0, namely, we have

$$2m^2 + m + 1 = 0.$$

Since this equation does not have any integer solution, Y_m is non-diffeomorphic to Y_n.

In both parities of m and n, we can get an infinite subsequence $\{m_i\}$ in $\mathbb{N}$ such that Y_{m_i} are mutually non-diffeomorphic to each other. □

2.3. **4-manifolds $\{Z_{2n}\}$ and $\{Z_{2n+1}\}$ obtained by a g-cork (P, φ^{2n}).** In this section we show infinitely many non-diffeomorphic exotic enlargements Z_n of P.

$$Z_0 = P \cup h_1 \cup h_2 = \tilde{Z}_0 \#^2 \overline{\mathbb{C}P^2},$$

Proof of Theorem 6. Let $E_{D,i} \to D^2$ $(i = 1, 2)$ be two copies of the

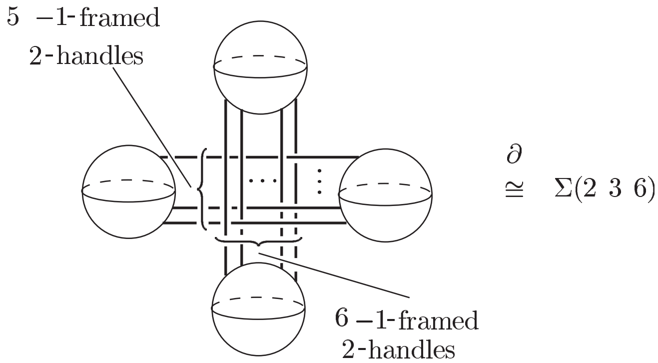

FIGURE 16. Milnor fiber attached one 2-handle ($\tilde{M}_c(2,3,5)$). The boundary is $\Sigma(2,3,6)$.

fibration of the complement $E(1) - \nu(T^2)$ of the neighborhood of a fiber T^2. The definition of the link-surgery gives $E(1)_{L_n} = ([S^3 - \nu(L_n)] \times S^1) \cup_{\omega_1} E_{D,1} \cup_{\omega_2} E_{D,2}$ (see the first picture in FIGURE 18). Each gluing map ω_i is a map from ∂E_{D_i} to one component of $\partial\nu(L_n) \times S^1$.

Here, Dv_1, and Ds_1 in $E_{D,1}$ are the neighborhoods of the compressing disk for the vanishing cycle and a section of $E_{D,1} \to D^2$. Dv_2, Dv_3, and Ds_2 in the other component $E_{D,2}$ are the neighborhoods of the compressing disks for the vanishing cycles and a section of $E_{D,2} \to D^2$. We use the same notation Dv_i, and Ds_j as the parts put on $[S^3 - \nu(L_n)] \times S^1$ via gluing maps ω_1 and ω_2 (see the second picture in FIGURE 18). Since the following holds:

$$E_{D,1} - Dv_1 - Ds_1 = M_c(2,3,6)$$

and

$$E_{D,2} - Dv_2 - Dv_3 - Ds_2 = M_c(2,3,5),$$

we get

$$E(1)_{L_n} = (([S^3 - \nu(L_n)] \times S^1) \cup_{i=1}^3 Dv_i \cup_{i=1}^2 Ds_i) \cup M_c(2,3,6) \cup M_c(2,3,5).$$

Here the Milnor fiber is defined to be the set

$$M_c(p,q,r) = \{(z_1,z_2,z_3) \in \mathbb{C}^3 | z_1^p + z_2^q + z_3^r = \epsilon \text{ and } |z_1|^2 + |z_2|^2 + |z_3|^2 \leq 1\},$$

for a non-zero complex number ϵ. The handle decomposition is seen in [11].

FIGURE 17 gives $Z_n \cup h_3 \cup h^3 = ([S^3 - \nu(L_n)] \times S^1) \cup_{i=1}^3 Dv_i \cup_{i=1}^2 Ds_i$. The

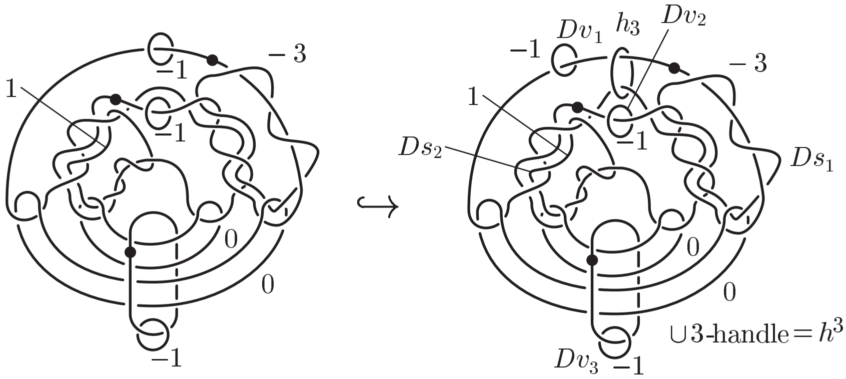

FIGURE 17. $Z_n \hookrightarrow ([S^3 - \nu(L_n)] \times S^1) \cup_{i=1}^3 Dv_i \cup_{i=1}^2 Ds_i$

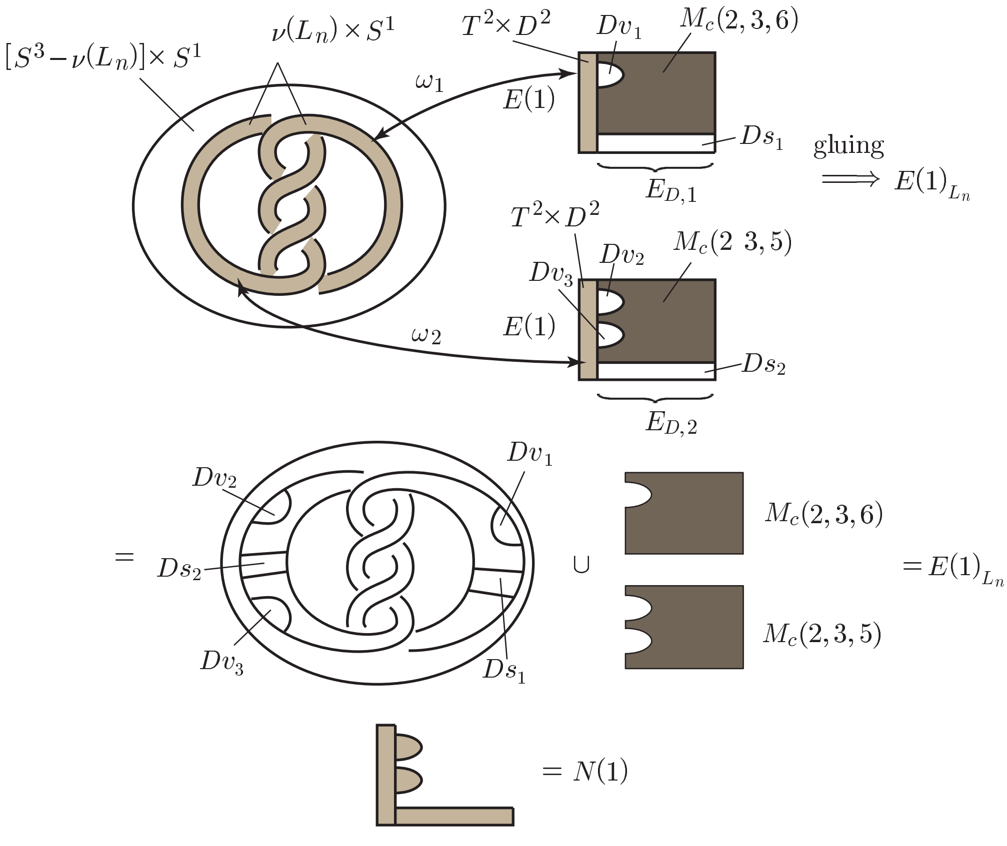

FIGURE 18. $E(1)_{L_n} = ([S^3 - \nu(L_n)] \times S^1) \cup E_D \cup E_D = ([S^3 - \nu(L_n)] \times S^1) \cup_{i=1}^3 Dv_i \cup_{i=1}^2 Ds_i \cup M_c(2,3,6) \cup M_c(2,3,5)$ and $N(1)$.

link-surgery is constructed as follows:

$$E(1)_{L_n} = Z_n \cup h_3 \cup h^3 \cup M_c(2,3,6) \cup M_c(2,3,5).$$

The handles h_3 and h^3 are the 2- and 3-handle indicated in FIGURE 17.

Let R denote the union $h_3 \cup h^3$. The attaching region of R is a thickened torus $T^2 \times D^1$ on ∂Z_n. The boundary $\partial(Z_n \cup R)$ is the disjoint union of $\Sigma(2,3,5)$ and $\Sigma(2,3,6)$. The isotopy class of the essential torus in ∂Z_n is uniquely determined from JSJ-theory. Thus the self-diffeomorphism on Z_n can extend to $Z_n \cup R$ uniquely.

Next we attach the Milnor fibers on the boundaries $\Sigma(2,3,5)$ and $\Sigma(2,3,6)$. Here we claim the following lemma:

Lemma 6 ([10],[15])**.** *Any diffeomorphism on* $\Sigma(2,3,5)$ *or* $\Sigma(2,3,6)$ *extends to* $M_c(2,3,5)$ *or* $M_c(2,3,6)$ *respectively.*

Proof. The proof is the same as Lemma 3.7 in [9]. We remark the case of $\Sigma(2,3,6)$ here. By the result in [6] the diffeotopy type of $\Sigma(2,3,6)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The non-trivial diffeomorphism on $\Sigma(2,3,6)$ is the restriction of rotating by the $180°$ about the horizontal line in FIGURE 16. Thus the diffeomorphism extends to $M_c(2,3,6)$. □

Thus, any self-diffeomorphism on $\Sigma(2,3,5)$ and $\Sigma(2,3,6)$ can extend to $M_c(2,3,5)$ or $M_c(2,3,6)$.

The diffeomorphism on Z_n can extend to $Z_n \cup R \cup M_c(2,3,5) \cup M_c(2,3,6) = E(1)_{L_n}$. This means that the diffeomorphism type of $E(1)_{L_n}$ is determined by that of Z_n. Conversely, if $m \neq n$, then Z_n and Z_m are non-diffeomorphic. □

Hence, we have the following corollary.

Corollary 1. *Any diffeomorphism* $Z_n \to Z_m$ *extends to a diffeomorphism* $E(1)_{L_n} \to E(1)_{L_m}$.

Hence, in this case Z has the same role as Gompf's nuclei N in [9].

3. Some variations of plug twists.

In this section, combining the plug twist (P, φ) and other twists, we show the 2-bridge knot-surgery and 2-bridge link-surgery (Theorem 7) are produced by the same P.

3.1. The 2-bridge knot-surgery.

For an irreducible fraction p/q, take the continued fraction

$$p/q = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_n}}} = [a_1, a_2, a_3, \cdots, a_n].$$

The continued fraction determines the 2-bridge knot or link diagram as FIGURE 19, where $\boxed{k}$ in the figure stands for the k-half twist. The isotopy type of $K_{p,q}$ depends only on the relatively prime integers (p,q) and does not depend on the way of the continued fraction.

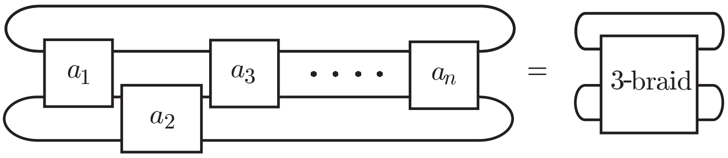

FIGURE 19. An example of the 2-bridge knot or link $K_{p,q}$.

The following deformations of coefficients do not change the isotopy class of $K_{p,q}$ and the rational number p/q:

$$(a_1, \cdots, a_i, a_{i+1}, \cdots, a_n) \leftrightarrow (a_1, \cdots, a_i \pm 1, \pm 1, a_{i+1} \pm 1, \cdots, a_n) \tag{8}$$

$$(a_1, \cdots, a_n) \leftrightarrow (\pm 1, a_1 \pm 1, \cdots, a_n), (a_1, \cdots, a_n \pm 1, \pm 1) \tag{9}$$

By using this deformation, for any irreducible fraction p/q we get the continued fraction

$$p/q = [b_1, b_2, \cdots, b_N]$$

such that N is an odd number and $b_3, b_5, \cdots, b_N$ are all even. If b_1 is odd or even, then $K_{p,q}$ is a knot or 2-component link respectively. We define the 3-braid indicating as in the right of FIGURE 19 with respect to $(b_1, b_2, \cdots, b_N)$ to be $B_{p,q}$.

Let p be an even integer. Then we take a continued fraction $p/q = [b_1, \cdots, b_N]$ as above. Namely, N is an odd number and $b_1, b_3, \cdots, b_N$ are

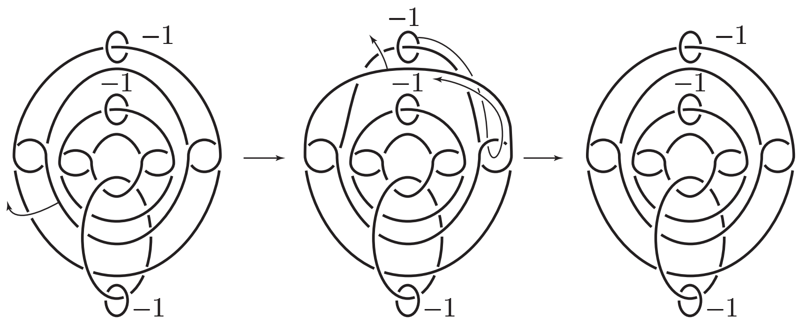

FIGURE 20. The definition of ψ.

even integers. We denote the map $\psi : \partial P \to \partial P$ as in FIGURE 20. We define $\varphi_{p,q} : \partial P \to \partial P$ as follows:

$$\varphi_{p,q} := \varphi^{\frac{b_N}{2}} \circ \psi^{b_{N-1}} \circ \cdots \circ \varphi^{\frac{b_3}{2}} \circ \psi^{b_2} \circ \varphi^{\frac{b_1}{2}}. \tag{10}$$

This definition may depend on the way of continued fraction of p/q. We choose such a continued fraction for the fraction p/q. Here we prove Proposition 2.

Proof. First, we show that $\varphi_{p,q}$ is not a torsion element. Let B_3 be the 3-braid group with the following presentation:

$$B_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$

and let B_3^0 be a subgroup generated by σ_1 and σ_2^2. The generators σ_1 and σ_2 are as in FIGURE 21. This group is a normal subgroup in B_3 and gives

FIGURE 21. The generators σ_1, σ_2 in B_3.

a homomorphism $\pi : B_3^0 \to MCG(\partial P)$ defined to be a map satisfying $\pi(\sigma_1) = \psi$ and $\pi(\sigma_2^2) = \varphi$. Hence, $\varphi_{p,q}$ lies in $\pi(B_3^0)$. Here $MCG(\partial P)$ is the mapping class group of ∂P.

Claim 1. *$B_3^0 \cong F_2 \rtimes \mathbb{Z}$, where F_2 is the rank 2 free group.*

Proof. We have the following short exact sequence:

$$1 \to F_2 \overset{f_1}{\to} B_3^0 \overset{f_2}{\to} \mathbb{Z} \to 0,$$

where f_2 is the number of half-twists between the first string and the second string, namely, it is the map $B_3^0 \to B_2 \cong \mathbb{Z}$ obtained by forgetting the third string. The subgroup in B_3^0 satisfying $f_2 = 0$ is considered as the homotopy class of a path on the 2 holed disk with a base point. Thus we have $\mathrm{Ker}(f_2) \cong F_2$. This exact sequence is splittable since $B_2 \cong \langle\sigma_1\rangle$ is the subgroup in B_3^0 as a lift of f_2. □

Since F_2 and $\mathbb{Z}$ are torsion-free, $F_2 \rtimes \mathbb{Z}$ is also torsion-free. This means that if $\varphi_{p,q}$ is torsion, then $\varphi_{p,q} = \mathrm{id}$ holds. Since the twist $(P, \varphi_{p,q})$ of $E(1)_{O_2} = 3\mathbb{C}P^2 \# 19\overline{\mathbb{C}P^2}$ is trivial, namely, $\Delta_{K_{p,q}}(t_1, t_2) = 0$. The 2-bridge knot with Alexander polynomial zero is the 2-component unlink only. Therefore if $p \neq 0$, then $\varphi_{p,q}$ is not torsion.

We compute the intersection form of $D_{\varphi_{p,q}}(P)$. The double is described in FIGURE 22 (the case of $N = 3$). The two (0-framed) fine curves are the attaching spheres of the upper manifolds of the double. The curve is parallel to the thick curve in each box with $\pm b_{2k+1}$-half twist and is twisted in each box with $\pm b_{2k}$-half twist. The parallel and twisted diagram is described in FIGURE 23. The first deformation (homeomorphism) in FIGURE 23 is also seen in [16] and the second and fourth deformations (diffeomorphisms) are also seen in [16]. The third deformation in FIGURE 23 is an isotopy of the diagram. Hence, the intersection form is $\oplus^2 \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{2}(b_1 + b_3 + \cdots + b_N) \end{pmatrix}$.

We claim the following:

Lemma 7. *Let $[b_1, \cdots, b_N]$ be a continued fraction of p/q with N an odd natural number. If $b_1, b_3, \cdots b_N$ are all even integers, then $p \equiv (-1)^{\frac{N-1}{2}}(b_1 + b_3 + \cdots + b_N) \bmod 4$.*

Proof. The integer p is equal to the (1,1)-component in the following matrix.

$$\begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_2 & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} b_N & -1 \\ 1 & 0 \end{pmatrix}.$$

Since we have

$$\begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_3 & -1 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} -b_1-b_3 & 1-b_1b_2 \\ 1-b_2b_3 & -b_2 \end{pmatrix} \bmod 4.$$

Suppose that

(11)
$$\prod_{l=1}^{2k+1}\begin{pmatrix} b_l & -1 \\ 1 & 0 \end{pmatrix} \equiv (-1)^k\begin{pmatrix} \sum_{l=0}^k b_{2l+1} & -1+\sum_{s=1}^k c_s b_{2s-1} \\ 1+\sum_{s=1}^k d_s b_{2s+1} & e \end{pmatrix} \bmod 4,$$

where c_i, d_i, e are some integers. Then we have

$$\begin{aligned}
&\begin{pmatrix} \sum_{s=0}^k b_{2s+1} & -1+\sum_{s=1}^k c_s b_{2s-1} \\ 1+\sum_{s=1}^k d_s b_{2s+1} & e \end{pmatrix}\begin{pmatrix} b_{2k+2} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_{2k+3} & -1 \\ 1 & 0 \end{pmatrix} \\
= &\begin{pmatrix} \sum_{s=0}^k b_{2s+1} & -1+\sum_{s=1}^k c_s b_{2s-1} \\ 1+\sum_{s=1}^k d_s b_{2s+1} & e \end{pmatrix}\begin{pmatrix} b_{2k+2}b_{2k+3}-1 & -b_{2k+2} \\ b_{2k+3} & -1 \end{pmatrix} \\
\equiv &\begin{pmatrix} -\sum_{s=0}^{k+1} b_{2s+1} & b_{2k+2}\sum_{s=0}^k b_{2s+1}+1+\sum_{s=1}^k c_s b_{2s-1} \\ -1-b_{2k+2}b_{2k+3}-\sum_{s=1}^k d_s b_{2s+1}+eb_{2k+3} & -e' \end{pmatrix} \bmod 4 \\
= &-\begin{pmatrix} \sum_{s=0}^{k+1} b_{2s+1} & -1+\sum_{s=1}^{k+1} c'_s b_{2s-1} \\ 1+\sum_{s=1}^{k+1} d'_s b_{2s+1} & e' \end{pmatrix},
\end{aligned}$$

where c'_i, d'_i, e' are some integers. Thus (11) holds for $k+1$ instead of k. The induction implies $p \equiv (-1)^{\frac{N-1}{2}}(b_1+b_3+\cdots+b_N) \bmod 4$. □

We go back to the proof of Proposition 2. The intersection form of $D_{\varphi_{p,q}}(P)$ is

$$\oplus^2\begin{pmatrix} 0 & 1 \\ 1 & (-1)^{\frac{N+1}{2}}\frac{p}{2} \end{pmatrix} \cong \begin{cases} \oplus^2\langle 1\rangle \oplus^2 \langle -1\rangle & p\equiv 2 \bmod 4 \\ \oplus^2 H & p \equiv 0 \bmod 4 \end{cases}$$

The Boyer's result means that if $p \equiv 2 \bmod 4$, then $(P, \varphi_{p,q})$ is a plug and if $p \equiv 0 \bmod 4$, then $(P, \varphi_{p,q})$ is a g-cork. □

We decompose Theorem 7 into two propositions (Proposition 6 and 7).

Proposition 6. *Let X be a 4-manifold containing V and $K_{p,q}$ be a non-trivial 2-bridge knot (i.e. p is an odd number). Then there exists an embedding $i : P \hookrightarrow V \subset X$ such that the twist $(P, \varphi_{p-1,q})$ gives the deformation:*

$$X_{K_{p,q}} = X(P, \varphi_{p-1,q}, i),$$

where the embedding i is defined in Figure *24 and independent of $K_{p,q}$.*

Proof. The embedding $i : P \hookrightarrow V$ is constructed in Figure 24. The twist $(P, \varphi^{\frac{b_1-1}{2}})$ is described in the first deformation in Figure 25. Consecutively, we do the twist (P, ψ^{b_2}) (the second deformation in Figure 25). Continuing the twists along (10), we totally obtain the twist $(P, \varphi_{p,q})$ in the

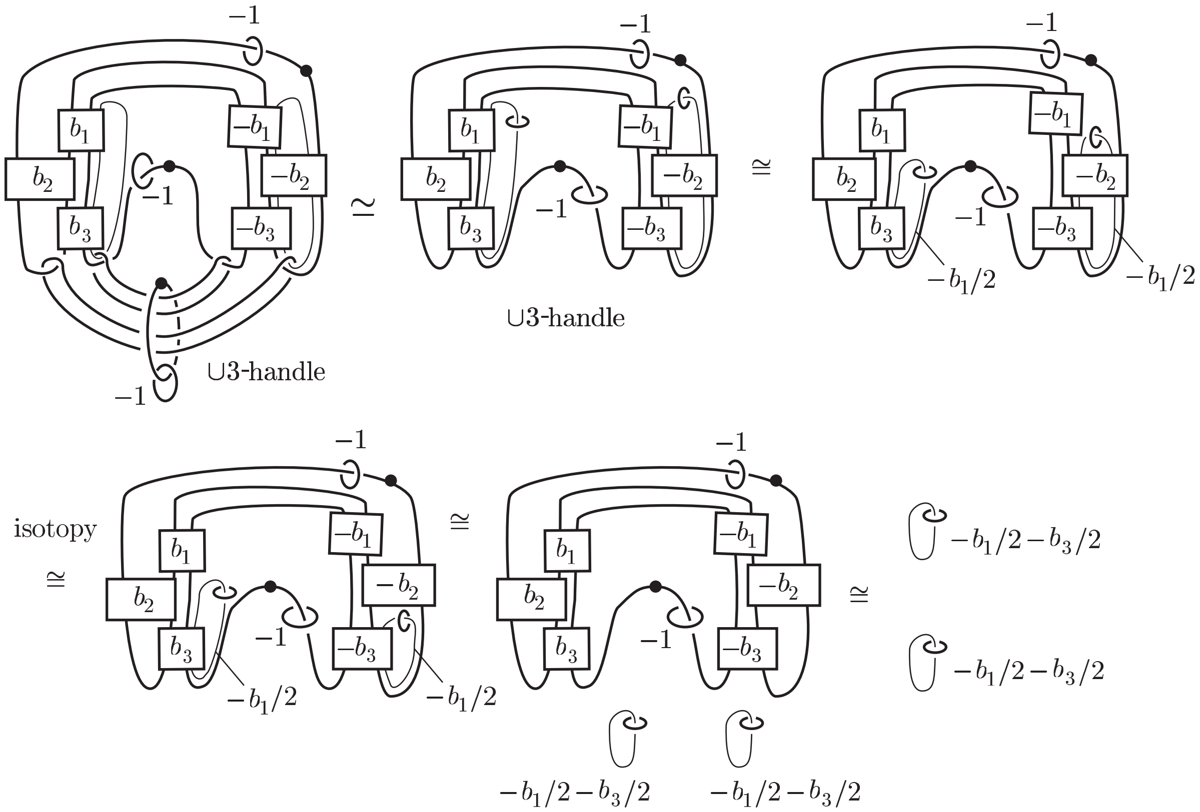

FIGURE 22. The homeomorphism type of $D_{\varphi_{p,q}}(P)$.

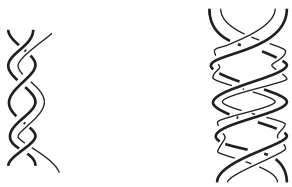

FIGURE 23. The local pictures of the fine curves in the box $\boxed{\pm b_{2k-1}}$ and $\boxed{\pm b_{2k}}$ in the first picture in FIGURE 22. (the cases of $b_{2k-1} = 4$ or $b_{2k} = 4$.

last picture in FIGURE 25. Here $-B_{p,q}$ is the mirror image of the braid $B_{p,q}$.

We compute the intersection form of the twisted double $D_{\varphi_{p,q}}(P)$. □

Remark 4. *In the similar way, we can also construct another embedding $i' : P \hookrightarrow V$ by changing the crossings in the broken circles in* FIGURE *24. This embedding is different from i, because the twist $(P, \varphi_{p-1,q})$ gives $V_{K_{p-2,q}}$. In general, the Alexander polynomials of $K_{p-2,q}$ and $K_{p,q}$ are different.*

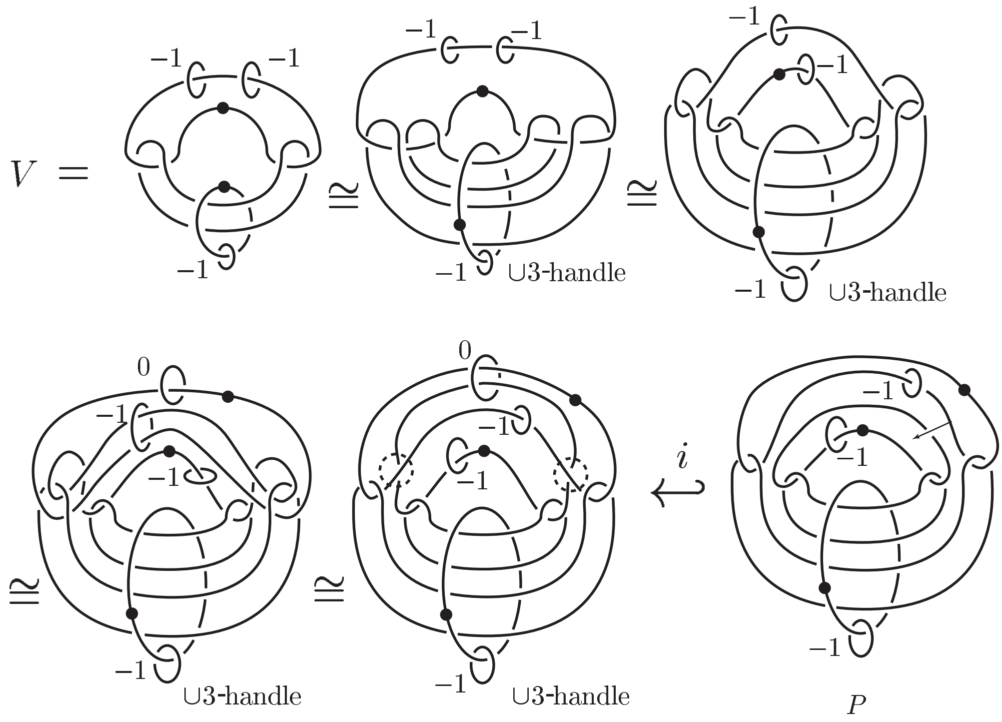

FIGURE 24. The embedding $i : P \hookrightarrow V$.

3.2. **2-bridge link-surgery.** We consider the case of link-surgery. Let C be a cusp neighborhood (i.e., Kodaira's singular fibration II). The handle decomposition of C is described in FIGURE 3. We denote $C\#C\#S^2 \times S^2$ by W.

Proposition 7. *Let X_i $(i = 1, 2)$ be two 4-manifolds containing C and let X be $X_1\#X_2\#S^2 \times S^2$. If $K_{p,q}$ is a 2-bridge link (i.e. p is an even number), then there exists an embedding $j : P \hookrightarrow W \subset X$ such that the twist $(P, \varphi_{p,q})$ gets*

$$X(P, \varphi_{p,q}) = (X_1, X_2)_{K_{p,q}},$$

where the embedding $j : P \hookrightarrow X := X_1\#X_2\#(S^2 \times S^2)$ is the one obtained by the same way as indicated in FIGURE *25.*

Proof. The application of $\varphi_{p,q}$ to FIGURE 26 in the same way as FIGURE 25 gives the twist $X(P, \varphi_{p,q}) = (X_1, X_2)_{K_{p,q}}$. □

Proof of Theorem 7. Let K be a 2-bridge knot or link. Then Proposition 6 and 7, it follows the required assertion. □

3.3. **A twist** (M, μ)**.** Let M be the manifold described in FIGURE 10. We factorize the knot mutation into the three processes as in FIGURE 27. According to this process, we define μ to be the map obtained by the process as described in FIGURE 28. Here $\tilde{\varphi}_1, \tilde{\varphi}_2 : \partial M \to \partial M$ are maps obtained by performing locally φ, φ^{-1} on ∂M.

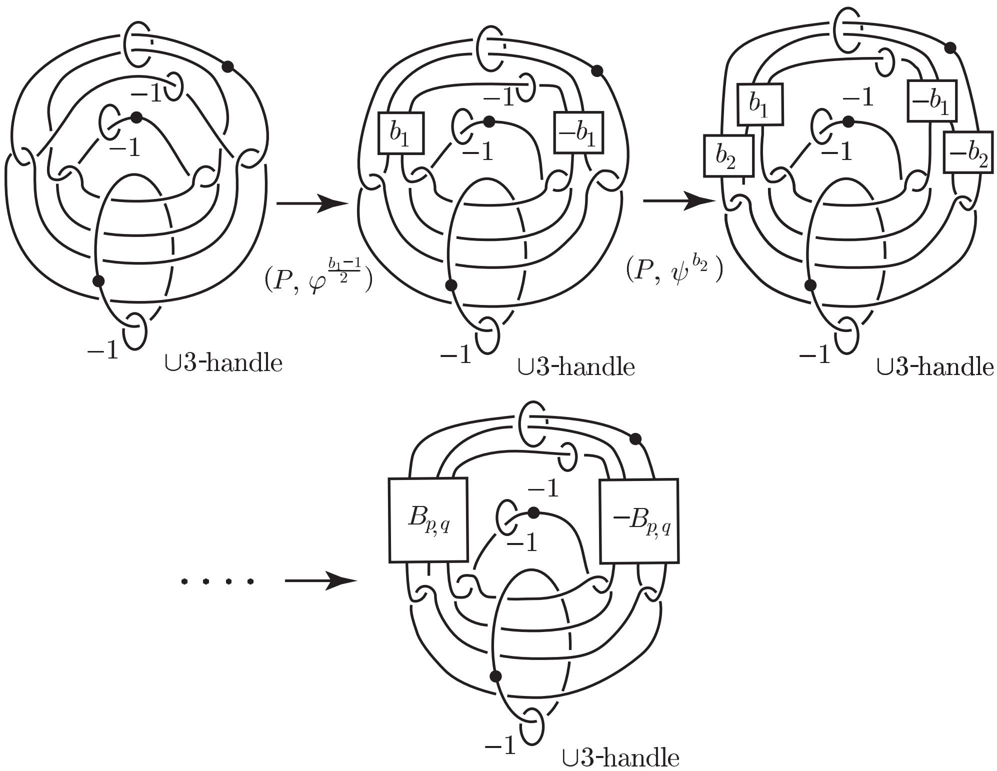

FIGURE 25. The construction of the twist by $(P, \varphi_{p-1,q})$ of V. The box $\boxed{n}$ stands for the n-half twist.

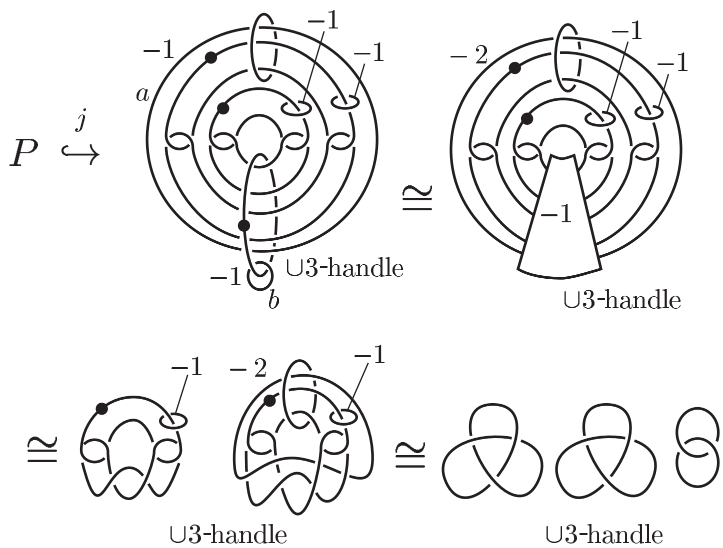

FIGURE 26. The embedding $j : P \hookrightarrow C\#C\#S^2 \times S^2 =: W$.

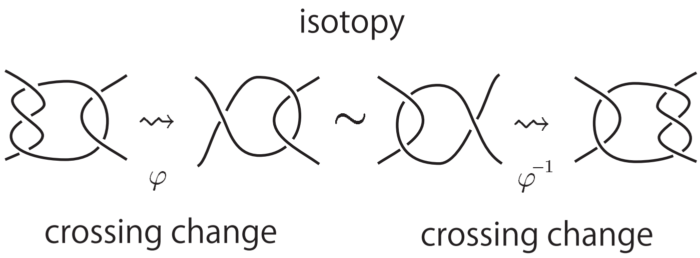

FIGURE 27. The factorization of the knot mutation.

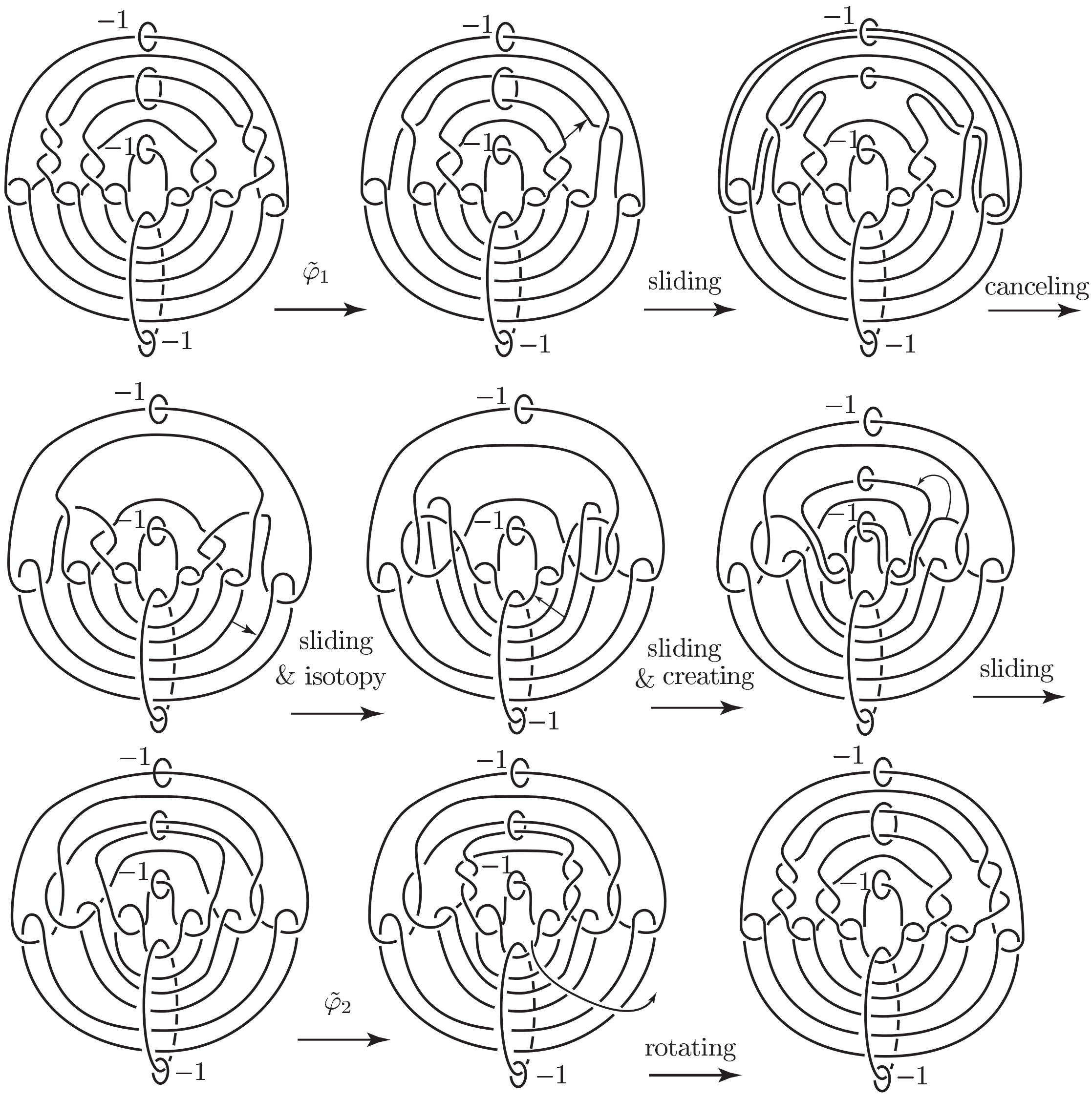

FIGURE 28. The definition of μ.

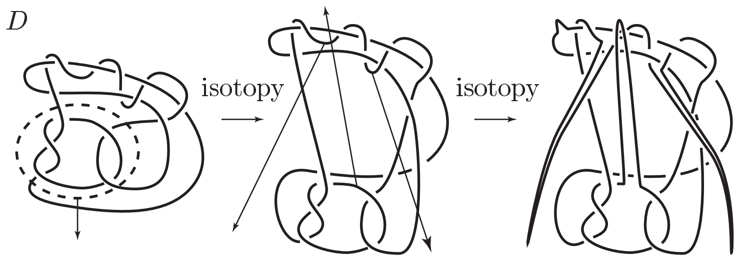

FIGURE 29. Moving the local tangle with respect to the mutant move.

Proof of Theorem 8. Let K, K' be a mutant pair. We find an embedding $M \hookrightarrow V_K$. Let D be a knot diagram of the knot K containing the local tangle of the right in FIGURE 9. For example, the first picture in FIGURE 29 is such a diagram. We move the local tangle surrounded by the broken line to a bottom position by some isotopy (the second picture). The resulting diagram gives a plate presentation with keeping the local picture in the bottom (the third picture).

We prove that (M, μ) is a twist between knot-surgeries for mutant pair K and K' by illustrating the case of $V_{KT} \rightsquigarrow V_C$ in FIGURE 30, where KT is the Kinoshita-Terasaka knot and C is the Conway knot.

By keeping track of the processes in FIGURE 28, the square μ^2 is the two times of the last move in FIGURE 28. This means a $360°$ rotation of ∂M along the torus. This is homotopic to the identity. □

Proof of Proposition 3. The first picture in FIGURE 31 presents the untwisted double $D(M) := M \cup_{\mathrm{id}} (-M)$. We can easily check the diffeomorphism $D(M) \cong \#^3 S^2 \times S^2$ by handle calculus. Removing M in $D(M)$, regluing by μ, we get the next picture in FIGURE 31. The intersection form of the twisted double $D_\mu(M)$ is isomorphic to $\oplus^3 H$. Thus, by using Boyer's result in [5], μ can extend to a self-homeomorphism $M \to M$. □

Here we define M_0 to be M with a -1-framed 2-handle deleted (the left of FIGURE 32). The boundary map $\mu_0 : \partial M_0 \to \partial M_0$ is naturally induced from the map μ, because the -1-framed 2-handle in M is fixed via the map μ. The diffeomorphism $D_{\mathrm{id}}(M_0) \cong \#^2 S^2 \times S^2 \# S^3 \times S^1$ and the homeomorphism $D_{\mu_0}(M_0) \simeq \#^2 S^2 \times S^2$ hold due to easy calculation.

Proof of Proposition 4. The outmost (Hopf-linked) pair of -1-framed 2-handle and 0-framed 2-handle in FIGURE 31 can be moved to the parallel position of the other Hopf-linked pair by several handle slides. Such handle slides are indicated in FIGURE 33. Hence, the pair can be removed as one Hopf link component with both framings 0. See the bottom row in FIGURE 33. The same deformation is seen in Fig.15 in [15]. The remaining part is $D_{\mu_0}(M_0)$. □

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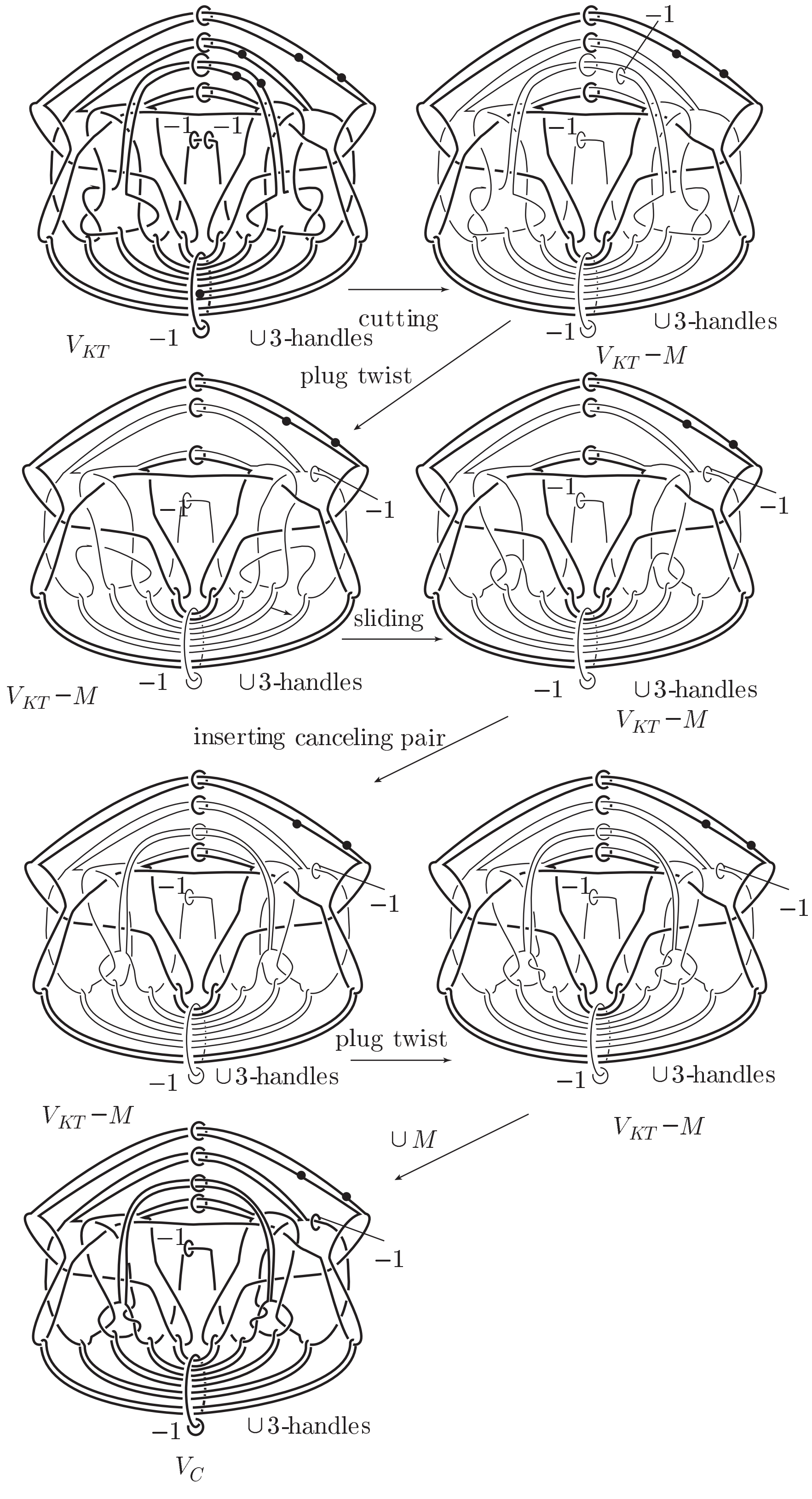

FIGURE 30. The performance $V_{KT} \rightsquigarrow V_{KT} - M \rightsquigarrow (V_{KT} - M) \cup_\mu M = V_C$. The fine curve presents the removed handles for $V_{KT} - M$.

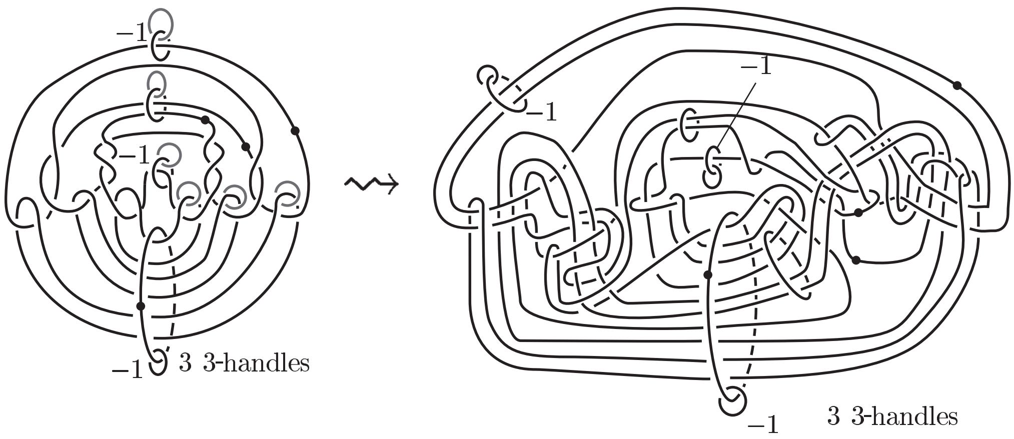

FIGURE 31. $D(M) \rightsquigarrow D_\mu(M)$ (via the local move (M, μ)).

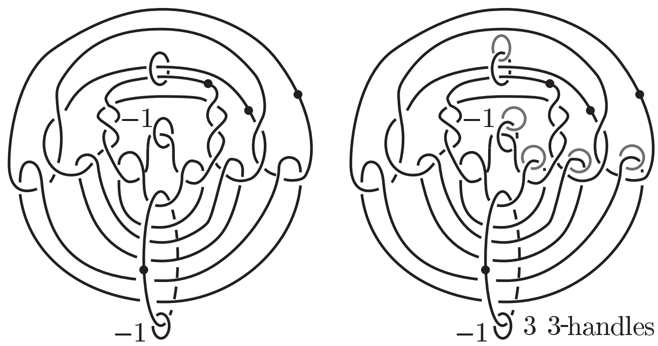

FIGURE 32. M_0 and $D(M_0) = \#^3 S^2 \times S^2 \# S^3 \times S^1$.

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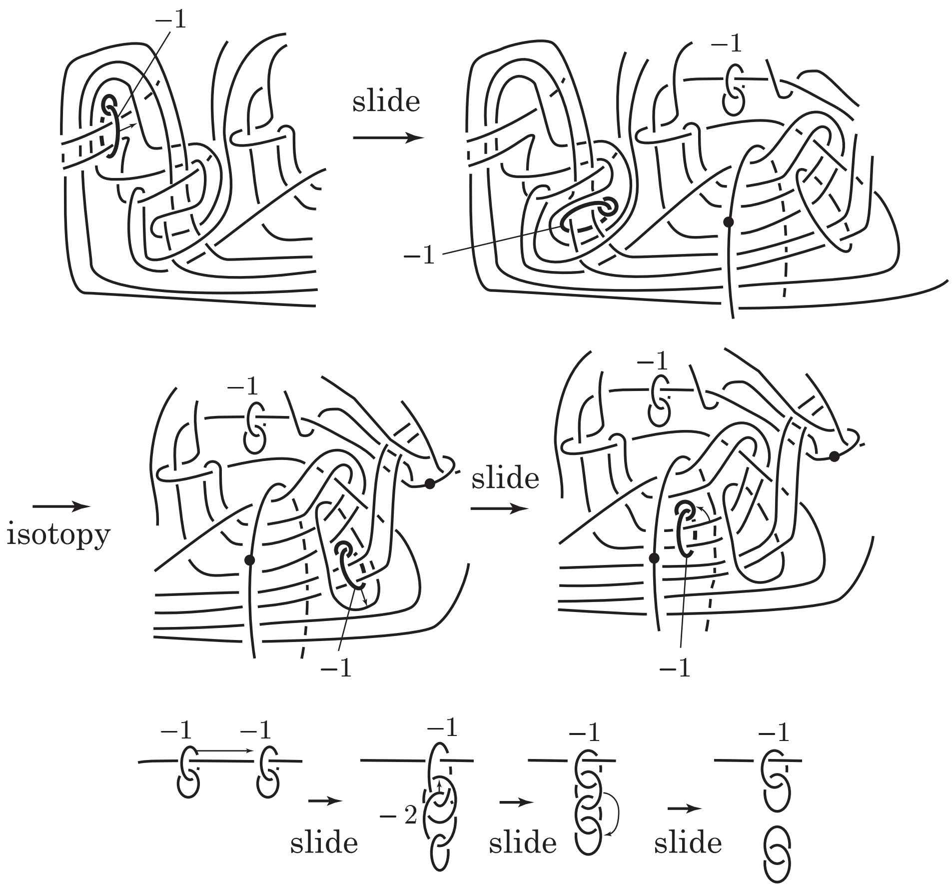

FIGURE 33. To move a pair of 2-handles to the position of the other pair.

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