レンズ空間手術を中心とした低次元幾何学について

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§1. Dehn surgery

Definition 1 (Dehn surgery)

Removing of the neighborhood of a knot K in M and gluing another solid torus $S^1 \times D^2$.

$$M' = [M - \mathit{nbd}(K)] \cup_\phi [S^1 \times D^2].$$

Remark 1

Gluing map ϕ is determined by the isotopy class only of the new meridian $\gamma := pt \times \partial D^2$.
γ is called slope of the Dehn surgery.

$$(M, K, \gamma) \to M_\gamma(K) \quad \text{(surgered manifold)}$$

Remark 2

When K is a null-homologous knot in M

$$\{\textit{slopes}\} \xleftrightarrow{1:1} \mathbb{Q} \cup \infty.$$

Remark 3

An iterated Dehn surgery along a link L is called Dehn surgery of the link L.

Theorem 1

Any 3-manifold is obtained from integral Dehn surgery of a link in S^3.

$$D_l : \{(S^3, \text{link})\} \times \{n\text{-tuple of slopes}\} \rightarrow \{3\text{-mfds}\}$$
$$(L, \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)) \mapsto M_{\boldsymbol{\gamma}}(L)$$

is surjective.
How about knots?

$$D_k : \{(S^3, \text{knot})\} \times \{\text{slope}\} \rightarrow \{3\text{-mfds}\}$$

Is this surjective?

The answer is NO.

Example 1

$L(17, 2)$ requires at least 2-component to construct by integral Dehn surgery.

Question1

Characterize manifolds which can be constructed by integral Dehn surgery of a knot.

What is image(D_k)?

Question2

Characterize knots which can construct a manifold by integral Dehn surgery of a knot.

What is $D_k^{-1}(M)$?

Nobody knows the solution completely.
What is image$(D_k)\cap\{$lens spaces$\}$?
What property does image$(D_k)\cap\{$lens spaces$\}$ have?
(lens surgery problem)

What is image$(D_k)\cap\{$Seifert manifolds$\}$?
What property does image$(D_k)\cap\{$Seifert manifolds$\}$ have?
(Seifert surgery problem)

Inverse image of D_k

$*D_k^{-1}(S^3) =$(unknot,± 1)

Theorem(Kronheimer-Mrowka-Ozsváth-Szabó)

If $S^3_r(K) = S^3$ then K is unknot.

$*D_k^{-1}(\Sigma(2,3,5)) =$(trefoil,$\pm 1$).

Theorem(Ghiggini)

If $S^3_r(K) = S^3_1$(right-handed trefoil)$= \Sigma(2,3,5)$ then K is trefoil.

If $S^3_r(K) = S^3_1$(figure-8)$= \Sigma(2,3,7)$ then K is figure-8.

The homology class of dual knot

We call $K^* \subset Y_\gamma(K)$ *dual knot*.
$[K^*] \in H_1(Y_\gamma(K))$ is invariant on Dehn surgery.

$$D_l : \{(S^3, \text{link})\} \times \{\text{slopes}\} \to \{(\text{3-mfds}, H_1^{\oplus n})\}$$
$$(L, \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)) \mapsto (M_{\boldsymbol{\gamma}}(L), h_1, \cdots, h_n)$$

$H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$
For $h \in H_1(L(p,q))$
$*D_k^{-1}(L(p,q),h) =$?

Conjecture(Berge)

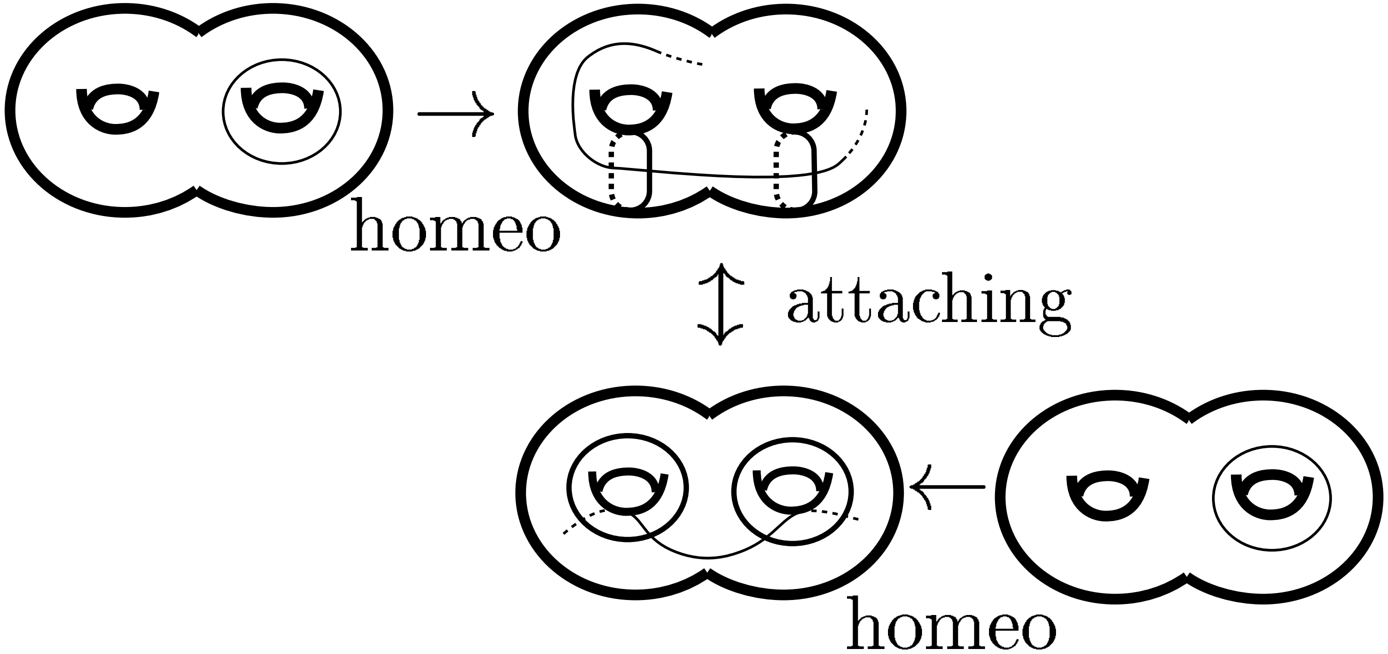
If $S_p^3(K) = L(p,q)$ then K is doubly primitive knot.
$*D_k^{-1}(L(p,q),h) =$ a doubly primitive knot or $\emptyset$. (uniqueness)

Definition 2 (doubly primitive knot)

$K \subset Y$ *is doubly primitive knot*
$\overset{def}{\Leftrightarrow} Y = V \cup_{\Sigma_2} U$ *(genus 2 Heegaard decomposition)*
$K \subset \Sigma_2$, *and*
$[K] \in \pi_1(V) \cong F_2$ *and* $[K] \in \pi_1(U) \cong F_2$ *are both primitive.*

Remark 4

Any doubly primitive knot obtains a lens space by an integral Dehn surgery.



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$*|D_k^{-1}(^{\exists}\text{ hyperbolic mfd})| = \infty$

Theorem(Teragaito)

$^{\exists}$Hyperbolic manifolds M' such that $M' = S^3_4(K_n)$ for any natural number n

$*|D_k^{-1}(^{\forall}\text{ elliptic mfd})| = 1$ or $< \infty$?

Perspective

If any elliptic manifold is obtained by a Dehn surgery, then such a knot is unique.

ex) elliptic case

$*D_k^{-1}(\Sigma(2,3,4), h) =$(trefoil,$\pm$4).(Ghiggini)

$*D_k^{-1}(\Sigma(2,3,3), h) =$(trefoil,$\pm$3).(Ghiggini)

Genus bounds

Conjecture(Goda-Teragaito)

Let $L(p,q) = S^3_p(K)$ be a hyperbolic lens surgery.
Then $2g(K) + 8 \leq p \leq 4g(K) - 1$

Theorem(Goda-Teragaito)

Let $L(p,q) = S^3_p(K)$ be a hyperbolic lens sugery
Then $|p| \leq 12g(K) - 7$

The proof is by *Gordon-Luecke method*.

- Consider *the graph* which is intersection between two surfaces.
- Investigate the behaivior of Scharlemann cycle or S-cycle.

Strategy 2

Find a Ozsváth Szabó's counterpart of Gordon-Luecke method?

- Can we generalize Heegaard Floer homology to graph theory on Riemann surface?
- Are there relasionship between Hempel's curve distance and Alexander grading of differential of Heegaard Floer homology?

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Dehn surgery of Poincaré homology sphere

$$D_k^{P_\pm} : (\pm\Sigma(2,3,5), \{\text{knots}\}) \times \{\text{slopes}\} \to \{\text{3-mfds}\}$$

Main theorem

Let Y be $-\Sigma(2,3,5)$.
Then there *never exists* any K in Y s.t. $Y_p(K) = L(p,q)$ for some positive integer p.

	positive slope	negative slope
$\Sigma(2,3,5)$	$\lvert\{K \mid D_k^{P_+} = \text{lens space}\}\rvert = \infty$	$\lvert\{K \mid D_k^{P_+} = \text{lens space}\}\rvert = 0$
$-\Sigma(2,3,5)$	$\lvert\{K \mid D_k^{P_-} = \text{lens space}\}\rvert = 0$	$\lvert\{K \mid D_k^{P_-} = \text{lens space}\}\rvert = \infty$

Example 2

$\Sigma(2,3,5)_p(K) = L(8,1), L(22,3), L(38,7), L(40,9), L(43,15), L(53,11)\cdots$

For any $\ell \in \mathbb{Z} - \{0\}$

$p = 14\ell^2 + 7\ell + 1, h = 7\ell + 2, g = \frac{p+1-|\ell|}{2}$

$p = 20\ell^2 + 15\ell + 3, h = 5\ell + 2, g = \frac{p+1-|\ell|}{2}$

$\cdots$

The quadratic families are more than or equal to 20 kinds.

Corollary

$\Sigma(2,3,5)\# - \Sigma(2,3,5)$ never includes any knot yielding lens surgery with positive or negative slopes.

$|\{K | D^{P_+ \# P_-}{}_k = \text{lens spce}\}| = 0$

Main Lemma

Y: L-space homology sphere

K: a knot in Y.

If $Y_p(K)$ is an L-space for $p \in \mathbb{Z}_{>0}$

$\Rightarrow Y$ admits a tight contact structure

Theorem(Fintushel-Stern)[constraint 1]

There exists a homology sphere and a knot K in Y satisfying $Y_p(K) = L(p,q)$, if and only if $q = x^2 \bmod p$ for some integer x.

$x \in \mathbb{Z}/p\mathbb{Z}$: a homology class $[K^*] \in H_1(L(p,q))$.

$(p,q,x) \to L(p,q) = {}^{\exists}Y_p({}^{\exists}K)$

$(p,1,1) \to L(p,1) = S^3_p$(unknot): unique

$(5,4,2) \to L(5,4) = S^3_5$(trefoil): unique

$(7,2,3) \to L(7,2) = S^3_7$(trefoil): unique

$(8,1,3) \to L(8,1) = \Sigma(2,3,5)_8(K)$

$(9,4,2) \to L(9,4) = S^3_9$((2,5)-torus knot)

$(10,9,3) \to L(10,9) = \Sigma(2,3,7)_{10}(K)$

$(11,3,5) \to L(11,3) = S^3_{11}$((2,5)-torus knot)
$(11,5,4) \to L(11,5) = S^3_{11}$((3,4)-torus knot)
$(12,1,5) \to L(12,1) = \Sigma(3,5,7)_{12}(K)$
$(13,3,4) \to L(13,3) = S^3_{13}$((3,4)-torus knot)
$(13,4,2) \to L(13,4) = S^3_{14}$((2,7)-torus knot)
$(13,12,5) \to L(13,12) = \Sigma(3,5,8)_{13}(K)$
$(14,9,3) \to L(14,9) = S^3_{14}$((3,5)-torus knot)
$(15,1,4) \to L(15,1) = \Sigma(3,4,11)_{15}(K)$
$(15,4,2) \to L(15,4) = S^3_{15}$((2,7)-torus knot)
$(16,1,7) \to L(16,1) = \Sigma(4,7,9)_{16}(K)$
$(16,9,3) \to L(16,9) = S^3_{16}$((3,5)-torus knot)
$(n^2-1,1,n) \to L(n^2-1,1) = \Sigma(n-1,n,n^2-n-1)_{n^2+1}(K)$
$(n^2+1,n^2,n) \to L(n^2+1,n^2) = \Sigma(n-1,n,n^2-n+1)_{n^2+1}(K)$
$(2n^2+2n,1,2n+1) \to L(2n^2+2n,1) = \Sigma(2n-1,2n+1,2n^2-1)_{2n^2+2n}(K)$

A strategy at lens surgery

Research the mechanism of the degeneracy in homology sphere surgery!

Theorem(Kadokami-Yamada)

If $S^3_p(K) = L(p,q)$ then

$$\Delta_K(t) = \frac{(t^{hg}-1)(t-1)}{(t^h-1)(t^g-1)} \mod t^p - 1$$

Use the surgery formula of Reidemeister torsion.

Theorem(Ozsváth-Szabó)[constraint 2]

If $S^3_p(K) = L(p,q)$ then

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{n} (-1)^{m-j}(t^{n_j} + t^{-n_j})$$

$(0 < n_1 < n_2 < \cdots < n_m = d)$

Remark 5

The two conditions only are Not sufficient for lens surgery.
For example $L(22,3), h = 5$ holds two constraints, but is never realized by Dehn surgery of a knot in S^3.

$t_i(K) := \tau_i(Y_0(K))$ i-th Turaev torsion

Theorem(Ozsváth-Szabó)[constraint 3]

If $S^3_p(K) = L(p,q)$ then, $t_i(K) \geq 0$.

Due to Heegaard Floer homology argument.

Conjecture(Ozsváth-Szabó)

These three conditions completely cover lens spaces coming from a Dehn surgery.

Main motivation

Does Heegaard Floer homology know all lens surgery information?

Genus bounds Let Y be L-space $\mathbb{Z}HS^3$

Theorem(T.)

If $Y_p(K) = L(p,q)$, then $g > 4\lambda(Y)$ and

$$2g-1 \leq p < \frac{4g(g+1)}{g-4\lambda(Y)}$$

. $\lambda(Y)$: Casson invariant.

$2g-1 = TB(K) := \max\{tb(L) | K \sim L\}$ (in this case).

Expectation

$$Y_{TB(K)}(K) \neq L(p,q).$$

i.e. $2g \leq p$.

§2. **Heegaard Floer homology and contact invariant**

$\mathfrak{t} \in \mathrm{Spin}^c(Y)$ $K \subset Y$ null-homologous knot.

$$\begin{cases} HF^+(Y,\mathfrak{t}), & HF^-(Y,\mathfrak{t}), \quad HF^\infty(Y,\mathfrak{t}), \\ \widehat{HF}(Y,\mathfrak{t}), & HF_{\mathrm{red}}(Y,\mathfrak{t}), \qquad \text{(topological invariants)} \\ \widehat{HFK}(Y,K,i) & \end{cases}$$

Definition 3

$Y = U_0 \cup_\Sigma U_1$ *(Heegaard decomposition)*

$\rightsquigarrow (\Sigma, \alpha, \beta, z)$ *(pointed Heegaard diagram)*

$\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_g\}$, $\beta = \{\beta_1, \beta_2, \cdots, \beta_g\}$ *circles compressible in* U_0 *and* U_1 *respectively*

$z \in \Sigma - \alpha_1 - \cdots - \beta_g$

$$((\mathrm{Sym}^g(\Sigma_g), \omega^g), \mathbb{T}_\alpha, \mathbb{T}_\beta) \quad \text{(Lagrangian intersection)}$$

In general

$$[(\mathcal{S},\omega),\mathcal{L}_1,\mathcal{L}_2] \overset{\text{symplectic package}}{\Longrightarrow} HF((Y,\omega),\mathcal{L}_1,\mathcal{L}_2)$$

$$[(\mathrm{Sym}^g(\Sigma_g),\omega^g),\mathbb{T}_\alpha,\mathbb{T}_\beta] \Rightarrow \widehat{HF}(\Sigma,\alpha,\beta,z)$$

Definition(Ozsváth-Szabó)

$\widehat{HF}(\Sigma,\alpha,\beta,z) := \widehat{HF}(Y)$ (topological invariant)

Definition(Ozsváth-Szabó)

$HF^\infty(Y)$, $HF^+(Y)$, $HF^-(Y)$ are equivariant versoins of $\widehat{HF}(Y)$.

Definition(Ozsváth-Szabó)

$HF_{\text{red}}(Y) := HF^{+}(Y)/HF^{\infty}$

Definition($\mathbb{Q}$-grading)

$Y : \mathbb{Q}HS^3$,

HF^* $(* = \hat{}\ , \pm, \infty, \text{red})$ admits $\mathbb{Q}$-grading.

$d(Y, \mathfrak{t}) = \frac{c_1^2(\mathfrak{s}) - 2\chi(X) - 3\sigma(X)}{4}$

$K \subset Y$ null-homologous knot

$\rightsquigarrow$ we can find a natural filtration on $\widehat{CF}(Y)$.

$\widehat{CFK}(Y, K, i)$: i-th filtered chain complex(Alexander grading).

Proposition(Ozsváth-Szabó)

$\widehat{HFK}(Y, K, i)$ is knot invariant of K.

$\chi(\widehat{HFK}(Y, K, i)) = a_i(K)$

Topological invariance means independence of how to take Heegaard decomposition.
But

<u>Want to know information of</u>
<u>Heegaard decomposition from Heegaard Floer homology!</u>

Theorem(O-S)

- $HF^+(Y)$ detects Thurston norm.
- $\widehat{HFK}(S^3, K)$ detects knot genus

Here Thurson norm is
$||\alpha|| := \min\{2g(F) - 2 \geq 0 | F = \alpha \in H_2(Y, \mathbb{Z})\}$.

Theorem(Y.Ni)

$\widehat{HFK}(Y, K)$ detects fiberness of K.

compuation of other invariant

Definition[correction term](O-S)

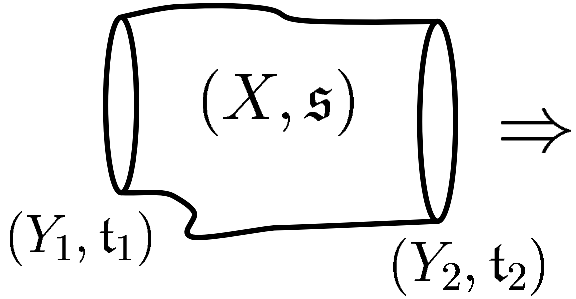
Let Y be $\mathbb{Q}HS^3$ and $(Y,\mathfrak{t}) = \partial(X,\mathfrak{s})$.

$$d(Y,\mathfrak{t}) = \frac{c_1^2(\mathfrak{s}) - 2\chi(X) - 3\sigma(X)}{4} \in \mathbb{Q}$$

(rationally spin^c-h-cobordism invariant)

$\chi(\widehat{HF}(Y)) = |H_1(Y)|$ (Y is $\mathbb{Q}HS^3$)
$\chi(HF_{\mathrm{red}}(Y)) - \frac{1}{2}d(Y) = \lambda(Y)$ (Casson invariant on $\mathbb{Z}HS^3$)
$\sum_i \chi(\widehat{HFK}(Y,K,i)t^i = \Delta_K(t)$
$\chi(HF^+(Y_0,\mathfrak{s})) = -\tau_i(Y_0)$ (rank$(H_1(Y_0)) = 1$)

Heegaard Floer homologies are a functor between 3-dimensional cobordism category and $\mathbb{F}_2[U]$-module.



$$\Rightarrow \quad HF^+(Y_1, \mathfrak{t}_1) \overset{F_{W,\mathfrak{s}}}{\to} HF^+(Y_2, \mathfrak{t}_2)$$

<u>Dehn surgery formula</u>

Let $K \subset Y$ be null-homologous knot

$$\begin{array}{ccc} HF^+(Y) & \xrightarrow{F_1} & \oplus_{\mathfrak{t} \in Q^{-1}(\mathfrak{s})} HF^+(Y_0(K), \mathfrak{t}) \\ {\scriptstyle F_3} \nwarrow & & \swarrow {\scriptstyle F_{2,\mathfrak{s}}} \\ & HF^+(Y_p(K), \mathfrak{s}) & \end{array}$$

(exact)

This exact triangle is a refinement of surgery formula of Casson invariant.
By summing up through spin^c-structures

$$\lambda_{CW}(Y_p(K)) = \lambda(Y) + \frac{1}{2p}\Delta''_K(1) - s(1,p)$$

$$\lambda_{CW}(L(p,q)) = -s(q,p)$$

$$s(q,p) := \sum_{k=1}^{p} \left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{kq}{p}\right)\right)$$

$$((\alpha)) = \begin{cases} 0 & \alpha \in \mathbb{Z} \\ \alpha - [\alpha] - \frac{1}{2} & \text{otherwise} \end{cases}$$

§3.**Invariants of lens space**

$\mathbb{Z}/p\mathbb{Z} \curvearrowright S^3 =$ unit sphere in $\mathbb{C}^2$ $\quad (z_1, z_2) \mapsto (\zeta z_1, \zeta^q z_1)$ $(\zeta^p = 1)$
$L(p,q) :=$ the quotient of the action

$L(p,q) \cong_{\text{homeo}} L(p', q') \Leftrightarrow p = p'$ and $(q = q' \bmod p$ or $qq' = 1 \bmod p)$

<u>homotopy, homology</u>
$\pi_1(L(p,q)) \cong H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$
<u>Reidemeister torsion</u>
For ${}^{\exists}t \in H := H_1(L(p,q), \mathbb{Z})$: generator

$$\mathcal{T}_{L(p,q)} \doteq \frac{1}{(t^{-1}-1)(t^{-q}-1)}$$

$\underline{\eta\text{-invariant}}$ $\mathfrak{t} \in \mathsf{Spin}^c(L(p,q))$ g: standard metric

$$\begin{aligned}
\eta^{\mathsf{sign}}(L(p,q)) &= \delta^{\mathsf{sign}}(L(p,q)) \quad \text{(signature defect)} \\
&:= \sigma(Y) - \frac{1}{3}\int_Y p_1(Y) \\
&= -\frac{1}{p}\sum_{i=1}^{p-1} \cot\frac{\pi k}{p}\cot\frac{\pi kq}{p}
\end{aligned}$$

$$\begin{aligned}
\eta^{\mathsf{Dirac},\mathfrak{t}_j}(L(p,q)) &= 2\delta^{\mathsf{Dirac}}(L(p,q),\mathfrak{t}_j) \quad \text{(eta of Dirac operator)} \\
&:= 2\cdot \mathrm{ind}(\partial\!\!\!/)_Y + \frac{1}{12}\int_Y p_1(Y) \\
&= -\frac{1}{4p}\sum_{k=1}^{p-1} \cos\frac{2\pi ki}{p}\csc\frac{\pi k}{p}\csc\frac{\pi qk}{p}
\end{aligned}$$

$$
\begin{aligned}
w(L(p,q), cone, c) &:= -8 \cdot \mathsf{ind}(\not{\partial})_{X \cup Y} + \mathsf{Sign}(Y) \quad \text{(Fukumoto-Furuta)} \\
&= -8\delta^{\mathsf{Dirac}}(L(p,q), \mathfrak{t}) - \delta^{\mathsf{sign}}(L(p,q)) \\
&= \frac{1}{p}\sum_{k=1}^{p-1}\left(\cot\frac{\pi k}{p}\cot\frac{\pi q k}{p} + 2\epsilon^k \csc\frac{\pi k}{p}\csc\frac{\pi q k}{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
d(L(p,q), \mathfrak{s}_j) &:= \frac{c_1^2(\mathfrak{s}) - 3\sigma(Y) - 2\chi(Y)}{4} \\
&\quad \text{(Ozsváth Szabó's correction term)} \\
&= -\frac{1}{2p} - \frac{j}{p} + \frac{1}{2} - 3s(q,p) - 2\sum_{k \le j}\left(\left(\frac{q'k}{p}\right)\right)
\end{aligned}
$$

In fact

$$
w(L(p,q), cone, c) = -4d(L(p,q), c) \quad \text{(Ue)}
$$

For any spherical manifolds M

$$w(M, cone, c) = -4d(M, c) \quad \text{(Ue)}$$

$$d(L(p,q), \mathfrak{t}_j) = -4w(L(p,q), \mathfrak{t}_j) \quad \text{(Ue, T)}$$

<u>Ozsváth Szabó's reciprocity formula</u>

$$d(L(p,q), \mathfrak{t}_i) = \frac{pq - (2i + 1 - p - q)^2}{4pq} - d(L(q,r), \mathfrak{s}_j)$$

where $0 \leq i \leq p + q$, $i = j \pmod q$ and $p = r \pmod q$.
Due to counting holomorphic disks and calculating $c_1(\mathfrak{s})$.

<u>Casson Walker invariant</u>

$$\lambda(L(p,q)) = -s(p,q) := -(\sum_{k=1}^{p-1} \left(\left(\frac{k}{p}\right)\right) \left(\left(\frac{qk}{p}\right)\right)) \quad \text{(Dedekind sum)}$$

§4.**Reidemeister torsion**

<u>Torsion of chain complex</u>

chain complex over a field $\mathbb{F}$

$C : \cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to \cdots$

$c_i = c_i^1 \wedge c_i^2 \wedge \cdots \wedge c_i^{n_i}$: exterior product of basis of C_i

$\mathbf{c} = \wedge_i c_i$: a basis of C

$\{b_i\}$ is a set of basis in C_i s.t. ∂ is one to one.

$\mathcal{T}(C, \mathbf{c}) := \prod_{i=0}^{n} [(\partial b_{i+1} b_i)/c_i]^{(-1)^{i+1}}$ when C acyclic

$\mathcal{T}(C, \mathbf{c}) := 0$ when C is non-acyclic

<u>Reidemeister torsion of CW-coplex</u>

X: topological space

$S(X)$: CW decomposition

$H := H_1(X)$: homology group of CW complex

$\det \mathfrak{A} = \pm H$ (as multiplicative group)

$\mathbb{Q}(H) := Q(\mathbb{Z}[H])$ $\mathbb{K}$: the field of fraction

$\hat{X} \to X \cong \hat{X}/H$ (maximal abelian cover)
Let H be finite
$\chi \in \mathrm{Hom}(H, U(1)) =: \hat{H}$: a character
image(χ)=R_χ
$C_*(X, R_\chi) := C_*(\hat{X}) \otimes R_\chi$: chain complex with local coefficient χ
$\hat{\mathcal{T}}_t(\chi) := \mathcal{T}(C(X, R_\chi), t \cdot \mathfrak{s})$

Fourier transform

$$(\mathbb{C}[G], +, *) \xrightarrow{\mathcal{F}} (C(\widehat{G}, \mathbb{C}), +, \cdot)$$

$$\mathcal{F}(f)(\chi) = \int_G f(g)\overline{\chi}(g) d\mu_G(g) = \widehat{f}(\chi)$$

$$\mathcal{F}^{-1}(\widehat{f})(g) = \frac{1}{\widehat{\mu}(G)} \int_{\widehat{G}} \widehat{f}(\chi)\chi(g) d\widehat{\mu}_G(\chi) = f(g)$$

$\mathfrak{s} \in \mathrm{Spin}^c(Y)$,$t \in H$

$$\widehat{\mathcal{T}}_t(\chi) := \mathcal{T}(C(X, R_\chi), t \cdot \mathfrak{s}) = \frac{1}{(\overline{\chi}(t) - 1)(\overline{\chi}(t)^q - 1)}$$

Fourier transform of Dedekind symbol

$$\Delta_p = \sum_{j=0}^{p} \left(\left(\frac{k}{p}\right)\right) t^j$$

$$\widehat{\Delta}_p(\zeta) = \begin{cases} \frac{1}{2} - \frac{\zeta}{\zeta - 1} & \chi \neq 1 \\ 0 & \chi = 1 \end{cases}$$

(exercise!)

$$\widehat{V}(\zeta) = \begin{cases} 0 & \zeta = 1 \\ \frac{\zeta}{\zeta - 1} & \zeta \neq 1 \end{cases}$$

$$\mathcal{F}^{-1}(\widehat{V}) = \frac{1}{2}(1 - \frac{1}{p}\sum_{j=1}^{p} t^j) - \sum_{j=1}^{p}\left(\left(\frac{k}{p}\right)\right) t^j$$

$$\begin{aligned} \mathcal{T}_{L(p,q)} &= \left\{\frac{1}{2}(1 - \frac{1}{p}\sum_{j=1}^{p} t^j) - \sum_{j=1}^{p}\left(\left(\frac{j}{p}\right)\right) t^j\right\}\left\{\frac{1}{2}(1 - \frac{1}{p}\sum_{j=1}^{p} t^j) - \sum_{j=1}^{p}\left(\left(\frac{j}{p}\right)\right) t^{qj}\right\} \\ &= \sum_{j=1}^{p}\left(\frac{1}{4p} - \frac{j}{4p} + \frac{1}{4} - s(q,p) - \sum_{k \le j}\left(\left(\frac{q'k}{p}\right)\right)\right) t^j \quad \text{(exercise!!)} \end{aligned}$$

where $qq' = 1 \bmod p$

$$\frac{1}{2}d(L(p,q), h \cdot \mathfrak{s}) - \frac{1}{2}\lambda(L(p,q)) = \mathcal{T}(h) \quad \text{(Nemethi)}$$

Contact Geometry

ξ is contact structure
$\overset{def}{\Leftrightarrow}$ the 2-plane field is nowhere integrable.
$\{$contact structures$\} \ni \xi \to \mathfrak{s}_\xi \in$ Spin^c
$^\exists c(\xi) \in HF^+(-Y, \mathfrak{s}_\xi)$ (contact invariant)
$K \subset Y$: fibered knot $\overset{def}{\Leftrightarrow} Y - K \overset{\pi}{\to} S^1$
$\mathcal{D} = (Y, K, \pi)$: open book decomposition

Theorem(Thurston-Winkelnkemper)

$\mathcal{D}$: open book decomposition
$\Rightarrow \xi_{\mathcal{D}} = \ker \alpha$: $^\exists$ a contact structure
s.t. $d\alpha$ is a volume form on π^{-1}(pt) and $\alpha(K^{\#}) > 0$

Theorem(Giroux)

$$\left\{\begin{array}{c}\text{open book}\\ \text{decompositions}\end{array}\right\} \Big/ \left\{\begin{array}{c}\text{positive}\\ \text{stabilization}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{contact}\\ \text{structures}\end{array}\right\} \Big/ \text{isotopy}$$

$$[\mathcal{D}] \leftrightarrow \xi_{\mathcal{D}}$$

(Y, ξ): a contact 3-mfd

$\mathcal{D} = (Y, K, \pi)$: an open book decomposition associating ξ.

Attaching a 2-handle, $-Y_0(K) \to -Y$ (surgery cobordism).

$\Rightarrow \widehat{HF}(-Y_0(K), \mathfrak{s}_0^{can}) \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \xrightarrow{F} \widehat{HF}(-Y, \mathfrak{s}^{can})$

Here $HF^+(-Y_0(K), \mathfrak{s}_0^{can}) \twoheadrightarrow c_0(\xi) \in \mathbb{F}_2 \subset \widehat{HF}(-Y_0(K), \mathfrak{s}_0^{can})$

Definition(Ozsváth-Szabó)

$c(\xi) := F(c_0(\xi))$ (contact invariant)

Theorem(Ozsváth-Szabó)

Let (Y, ξ) be an overtwisted contact manifold

Then $c(\xi) = 0$.

Definition 4

Y is an L-space 3-mfd. $\overset{def}{\Leftrightarrow} HF_{red}(Y) \cong 0$,

Example 3 (L-space)

S^3, $L(p,q)$, elliptic manifolds, and some hyperbolic manifolds and so on.

Proof of Main lemma

Y: an L-space homology sphere

Suppose that $Y_p(K) = L(p,q)$ (or L-space)

Then $\widehat{HFK}(Y,K,g) \cong \mathbb{F}_2$. (Ozsváth-Szabó)

Theorem(Y.Ni)

Y: 3-manifold; K: a knot in Y.

Y, $Y - K$ is irreducible and $\widehat{HFK}(Y,K,g) \cong \mathbb{F}_2$

$\Rightarrow K$ is fibered.

In particular K yields lens surgery $\Rightarrow K$ is fibered.

$\widehat{HFK}(Y,K,g) = \widehat{HFS}(M,\gamma)$ (sutured Floer homology)

When $\widehat{HFK}(Y,K,g) \cong \mathbb{F}_2$,
what of monodromy map $\phi : F \to F$ does
$\widehat{HFK}(Y,K,i) \quad (i = 0, 1, \cdots, g-1)$ reflect?
In general how can we utilize $\widehat{HFK}(Y,K)$ or Juhász's $\widehat{SFH}(M,\gamma)$
for foliation research?

$$HF^+(-Y) = \mathcal{T}^+_{-d} \xleftarrow{\;F_1\;} F_2[U]/U^{t^i} = HF(-Y_0, Q^{-1}(i))$$

$$\mathcal{T}^+_{-d} \xrightarrow{\;F_3\;} \mathcal{T}^+_{d(-L(p,q),i)} \xrightarrow{\;F_{2,\mathfrak{s}_i}\;} F_2[U]/U^{t^i}$$

(exact)

where $\mathcal{T}_d^+ \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \cdot U^{-1} \oplus \mathbb{F}_2 \cdot U^{-2} \oplus \cdots$.

(degree) $\quad d \qquad d+2 \qquad d+4 \qquad \cdots$

F_1 is injective.

Hence $c(\xi) \neq 0$. Therefore ξ is tight contact structure. $\square$

Proof of Main theorem $-\Sigma(2,3,5)$ is L-space and never exists any tight contact structure by Etnyre and K. Honda. $\square$

How about $Y = \Sigma(2,3,5) \# \Sigma(2,3,5)$ or $\#3\Sigma(2,3,5)$?

Conjecture

$\{D_k^{\# n P_+} = \text{lens space}\} = \emptyset$

Due to Maple's evidence in the region $|n| < 40$ and $p < 1000$.