Branched covering description of lens space surgery
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Osaka City university
§1.1 Lens Space Surgery on $S^3$

J.O. Berge,

- Some knots with surgeries giving lens spaces
  unpublished manuscript

He defined doubly primitive knot in this paper.
There exist many examples yielding lens space by Dehn surgery.
He listed many examples
**Def.** (Doubly primitive knot)

When a knot $K \subset S^3$ is the following condition, we call it *doubly primitive knot* (DP-knot).

1) $K$ lies on genus 2 surface $\Sigma$ which is trivially embedded in $S^3$.

$$S^3 - \Sigma = V_1 \amalg V_2$$

(two handlebodies)

2) $K$ induces both primitive elements in $\pi_1(V_i) \cong F_2$. 
Berge’s Conjecture

Any knot yielding lens space by integral Dehn surgery is a DP-knot.

The case which lens space is $L(p, 1)$ is solved by KMOS. The other cases are open.
Properties.

An integral Dehn surgery of any DP-knot gives rise to a lens space.

\[ \text{genus 2 handlebody} \xrightarrow{\text{surgery}} \text{genus 1 handlebody} \]

Any lens space has a genus 1 Heegaard splitting.

**Key Point** We generalize the *phenomenon*.

\[ S^3 \rightarrow \text{a 3-manifold with genus 2 Heegaard splitting} \]
Def. ((generalized) Doubly primitive knot)
When a knot $K \subset Y$ is the following condition, we call it (generalized) doubly primitive knot (DP-knot).

1) $K$ lies on genus 2 surface $\Sigma$ which is a Heegaard surface of $Y$.

\[ Y = V_1 \cup V_2 \]
(Heegaard splitting)

2) $K$ induces both primitive elements in $\pi_1(V_i) \cong F_2$. 
generalized Berge’s Conjecture

Let $Y$ be Heegaard genus 2 knot. Any knot in $Y$ yielding a lens space by integral Dehn surgery is a (generalized) DP-knot.
Questions

What 3-manifolds are there for the possibility of $Y$?
Such 3-manifolds $Y$ are restricted?
(What conditions are required for such 3-manifolds $Y$)
What are the knots $K$?
What is the condition which $K$ satisfies?
§1.2 Berge’s list

Berge listed lens spaces obtained by Dehn surgery of DP-knots in $S^3$.
Conj. This is the complete list of DP-knots. $L(p, q)$
s.t.
$p = |Aa + Bb|, a^2q = \pm b^{\pm 2} \mod p$

(1) $A = 1, a = \pm 1, \gcd(B, b) = 1, B \geq 2$
(2) $A = 1, a = \pm 1, \gcd(B, b) = 2, B \geq 4$
(3) $A > 1, a = \pm 1, \epsilon = \pm 1, \frac{B + \epsilon}{A} \text{ is odd, } b = -2\epsilon Aa \mod B$
(4) $A > 3, a = \pm 1, \epsilon = \pm 1, \frac{2B + \epsilon}{A} \text{ is integer, } b = -\epsilon Aa \mod B$
(5) $A > 1 \text{ odd, } a = \pm 1, \epsilon = \pm 1, \frac{B - \epsilon}{A} \text{ is integer } b = -\epsilon Aa \mod B$
(6) $A > 2 \text{ even, } a = \pm 1, B = 2A + 1, b = -a(A - 1) \mod B$
(7) $a = -(A + B), b = -B$
\[(8) a = -(A + B), \quad b = B \]
\[(9) (A, B, a, b) = (4J + 1, 2J + 1, 6J + 1, -J) \text{ for some integer } J \]
\[(10) (A, B, a, b) = (6J + 2, 2J + 1, 4J + 1, -J) \text{ for some integer } J \]

In particular (9) and (10)

\[ p = 22J^2 + 9J + 1 \]
\[ p = 22J^2 + 13J + 21 \]

What is this exception?
§1.3 The dual knots

Let $K \subset Y$ be a knot.
$\tilde{K} \subset Y(K, p)$: the dual knot of $K$.

Suppose that $Y(K, p)$ is a lens space $L(p, q)$.
$[\tilde{K}] \in H_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$: a generator

$$h \in (\mathbb{Z}/p\mathbb{Z})^\times$$
The case of DP-knot

Fact

Let $K$ be a DP-knot. The dual knot $\tilde{K}$ of the lens space surgery is a 1-bridge knot in the lens space.
Remark

Let $K_1$ and $K_2$ be DP-knots in $Y$. If $h_1 = h_2$ then $\tilde{K}_1$ and $\tilde{K}_2$ are isotopic. Thus $K_1$ and $K_2$ are isotopic in $Y$.

\[
\{\text{DP-knots}\} \leftrightarrow \bigcup_{p \in \mathbb{Z}_{>1}} (\mathbb{Z}/p\mathbb{Z})^\times / \sim
\]

For any $(p, h)$, what are $Y$ and $K$?
§2.1 Surgery on \( \Sigma(2, 3, 5) \) and \( \Sigma(2, 3, 7) \)

The speaker computed lens space surgery on \( \Sigma(2, 3, 5) \) and \( \Sigma(2, 3, 7) \) before.

Ozsvath-Szabo’s formula for \( Y(K, p) = L(p, q) \).
For any \( i \in \mathbb{Z}/p\mathbb{Z} \)

\[
d(Y) - d(L(p, q), hi + c) + d(L(p, 1), i) = l_i
\]

\[
\sum_{j \mod p = i} l_j - \chi(HF_{\text{red}}(Y)) = t_i(K)
\]

\[
\Delta_K(t) = (-1)^m + \sum_{j=1}^{m} (-1)^{m-j}(t^{n_j} + t^{-n_j})
\]
The case of $Y = \Sigma(2, 3, 5)$

$h$: homology class

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$h$</th>
<th>$2g - p - 1$</th>
<th>$J$</th>
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<tbody>
<tr>
<td>A1</td>
<td>$14J^2 + 7J + 1$</td>
<td>$7J + 2$</td>
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</tr>
<tr>
<td>A2</td>
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<tr>
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<td>$6J + 1$</td>
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</tr>
<tr>
<td>C1</td>
<td>$42J^2 + 23J + 3$</td>
<td>$7J + 2$</td>
<td>$-</td>
<td>J</td>
</tr>
<tr>
<td>C2</td>
<td>$42J^2 + 47J + 13$</td>
<td>$7J + 4$</td>
<td>$-</td>
<td>J</td>
</tr>
<tr>
<td>D1</td>
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<tr>
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<tr>
<td>E2</td>
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<td>$27J + 10$</td>
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<td></td>
<td>Formula</td>
<td>Coefficient</td>
<td>Constraint</td>
<td>Notes</td>
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</tr>
<tr>
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<tr>
<td>F₂</td>
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<td>$23J + 5$</td>
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<tr>
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<tr>
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<td>K</td>
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<td>15</td>
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### The case of $\Sigma(2,3,7)$

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<th>$h$</th>
<th>$2g - p - 1$</th>
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<td>$B$</td>
<td>$42J^2 + 15J + 1$</td>
<td>$6J + 1$</td>
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<tr>
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<tr>
<td>$D_2$</td>
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<td>(J)</td>
<td>(168J^2 + 136J + 28)</td>
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<tr>
<td>(K)</td>
<td>(259)</td>
<td>(15)</td>
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</table>
Any lens space in this list can be realized as Dehn surgeries of DP-knots

--- Conjecture 1 ---
Let $Y$ be a homology sphere with $\lambda = -1$. If $Y(K,p)$ is lens space, then $Y$ is $\Sigma(2,3,5)$ or $\Sigma(2,3,7)$ and this list is complete.

--- Conjecture 2 ---
Let $Y$ be a homology sphere with $\lambda = 0$. If $Y(K,p)$ is lens space, then $Y$ is $S^3$ and Berge’s list is complete.
Fact

Let $Y$ be $\Sigma(2,3,5)$. Then no knot in $Y$ gives rise to any lens space.

Conj.3

Let $Y$ be a homology sphere with $\lambda > 0$. Then no knot in $Y$ gives rise to a lens space ($p > 0$).
§2.2 Knots in homology spheres

Motivation
We want to see knots in a homology sphere like this.
Seifert manifolds

Lens space $L(p, q)$

\[-\frac{p}{q} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\ldots - \frac{1}{a_n}}}}\]
The case of $A$

\[-\ell \quad 7 \quad \ell+1 \quad 2\]

\[-\ell \quad 5 \quad \ell+1 \quad 4\]
\[
((a, b) = (3, 5), (5, 3))
\]
The case of $I_1, I_2, I_3$
Is this knot DP-knot?
§2.3 Branched covering

$L(p, q) \to S^3$: \exists\text{double branched covering}
$K_{p,q}$ :the branched locus(the 2-bridge link)

Let $K$ be a DP-knot
$Y$: Heegaard genus two
$Y \to S^3$: \exists\text{double branched covering}
$K$: \textit{strongly invertible} by this involution
trefoil

DP-knots
\[ Y(K, p) = L(p, q) \]

\[
\begin{array}{c}
Y \xrightarrow{\text{Dehn surgery}} L(p, q) \\
\text{branched covering} \quad \text{branched covering} \\
(S^3, B) \longrightarrow (S^3, K_{p,q})
\end{array}
\]

the lower arrow is the band sum
For example we see the surgeries lens space surgery on $\Sigma(2, 3, 5)$. 
$Pr(-2, 3, 5) = (3, 5)$-torus knot
The case of A
The case of $B$
The case of C and D  The case of E
The case of F, G, and H
The case of I
The case of $J$
In fact these are the bund sum of DP-knot

A criterion of DP-knot

Let $K_{p,q}$ be a 2-bridge knot

$S^3 = B_1 \cup B_2$: genus 0 Heeegaard decomposition

$B_i \cap K_{p,q}$ are two arcs.

Let $b$: a band of $K_{p,q}$ $B_i \cap b$ is single arc like the following:
Corollary

We put $a, b \in \mathbb{Q}$, $c \in \mathbb{Z}$.
Let $Y(a, b, c)$ be a Brieskorn homology spheres as in the figure below.

Then $Y$ contains a DP-knot.

Proof
In the case of $I$ put $a, b \in \mathbb{Q}, c \in \mathbb{Z}$. 
Conj.4

$Y(a, b, c)$: a Brieskorn homology sphere
If $Y(a, b, c)$ contains a DP-knot, then one of $a, b$ and $c$ must be an integer.

Lens space surgery
but not lens space surgery
If a homology sphere $Y$ contains DP-knot, then is $Y$ a Briskorn homology sphere?
Answer

There exist non-Brieskorn homology spheres which contain DP-knots.

Examples (graph manifolds)

\[
\begin{array}{c}
\begin{array}{c}
\lambda = -2 \\
\lambda = -5 \\
\end{array}
\end{array}
\]
The case of IV₂

\[
\begin{array}{|c|c|c|}
\hline
& p & h \\
\hline
I & 30\ell^2 + 5\ell & 10\ell + 1 \\
\hline
II_1 & 42\ell^2 + 19\ell + 2 & 21\ell + 5 \\
\hline
II_2 & 42\ell^2 + 5\ell & 14\ell + 1 \\
\hline
III_1 & 69\ell^2 + 55\ell + 11 & 23\ell + 9 \\
\hline
III_2 & 69\ell^2 + 37\ell + 5 & 23\ell + 6 \\
\hline
IV_1 & 85\ell^2 + 61\ell + 11 & 17\ell + 6 \\
\hline
IV_2 & 85\ell^2 + 41\ell + 5 & 17\ell + 4 \\
\hline
\end{array}
\]

The case of IV₂

\[
\begin{array}{c}
-2 \\
\hline
2 \\
\hline
0 \\
\hline
2 \\
\hline
-2 \\
\hline
3 \\
\hline
-3 \\
\hline
\lambda = -2 \\
\hline
\end{array}
\]

\[\rightarrow [3, 2, -\ell, -4, -2, -3, \ell]\]
\[ [a, b, -\ell, c - 1, -2, d - 1, \ell] \]

\[ a = 3, b = 2, c = -3, d = -2 \]
Corollary

Let $Y$ be a graph homology sphere with the plumbing graph as below:

Here $a \in \mathbb{Q}$, $b, c, d \in \mathbb{Z}$.

Then $Y$ contains a DP-knot.

Such manifold is the splicing of two Brieskorn homology spheres.
Conj.5

Let $Y$ be a homology sphere with plumbing graph as below:

\[
\begin{array}{c}
\text{d} \\
\text{f} \\
\text{e} \\
\text{b} \\
\text{c} \\
\text{a}
\end{array}
\]

where $a, c \in \mathbb{Q}$, $b, d, e, f \in \mathbb{Z}$. Then $Y$ contains a DP-knot.
Splicing(Σ(p, q, r), Σ(p′, q′, r′), r, r′)

pq′q′ = rr′ ± 1