Branched covering description of lens space surgery

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§1.1 **Lens Space Surgery on S^3**

J.O.Berge,

- *Some knots with surgeries giving lens spaces*

unpublished manuscript

He defined doubly primitive knot in this paper.
There exist many examples yielding lens space by Dehn surgery.
He listed many examples

Def.(Doubly primitive knot)
When a knot $K \subset S^3$ is the following condition, we call it *doubly primitive knot*(DP-knot).

1) K lies on genus 2 surface Σ which is trivially embedded in S^3.

$$S^3 - \Sigma = V_1 \amalg V_2$$
(two handlebodies)

2) K induces both primitive elements in $\pi_1(V_i) \cong F_2$.

Berge's Conjecture

Any knot yielding lens space by integral Dehn surgery is a DP-knot.

The case which lens space is $L(p, 1)$ is solved by KMOS.
The other cases are open.

Properties.

An integral Dehn surgery of any DP-knot gives rise to a lens space.

$$\text{genus 2 handleboldy} \stackrel{\text{surgery}}{\longrightarrow} \text{genus 1 handlebody}$$

Any lens space has a genus 1 Heegaard splitting.

Key Point We generalize the *phenomenon*.

$$S^3 \to \text{a 3-manifold with genus 2 Heegaard splitting}$$

Def.((generalized) Doubly primitive knot)
When a knot $K \subset Y$ is the following condition, we call it *(generalized) doubly primitive knot*(DP-knot).

1) K lies on genus 2 surface Σ which is a Heegaard surface of Y.

$$Y = V_1 \cup V_2$$
(Heegaard splitting)

2) K induces both primitive elements in $\pi_1(V_i) \cong F_2$.

generalized Berge's Conjecture

Let Y be Heegaard genus 2 knot.
Any knot in Y yielding a lens space by integral Dehn surgery is a (generalized) DP-knot.

Questions

What 3-manifolds are there for the possibility of Y?
Such 3-manifolds Y are restricted?
(What conditions are required for such 3-manifolds Y)
What are the knots K?
What is the condition which K satisfies?

§1.2 **Berge's list**

Berge listed lens spaces obtained by Dehn surgery of DP-knots in S^3.
Conj. This is the complete list of DP-knots.
$L(p,q)$
s.t.
$p = |Aa + Bb|$,$a^2 q = \pm b^{\pm 2}$ mod p
(1)$A = 1, a = \pm 1, \mathrm{gcd}(B, b) = 1, B \geq 2$
(2)$A = 1, a = \pm 1, \mathrm{gcd}(B, b) = 2, B \geq 4$
(3)$A > 1, a = \pm 1, \epsilon = \pm 1, \frac{B+\epsilon}{A}$ is odd, $b = -2\epsilon Aa$ mod B
(4)$A > 3, a = \pm 1, \epsilon = \pm 1, \frac{2B+\epsilon}{A}$ is integer, $b = -\epsilon Aa$ mod B
(5)$A > 1$ odd, $a = \pm 1, \epsilon = \pm 1, \frac{B-\epsilon}{A}$ is integer $b = -\epsilon Aa$ mod B
(6)$A > 2$ even, $a = \pm 1, B = 2A + 1, b = -a(A - 1)$ mod B
(7)$a = -(A + B), b = -B$

(8)$a = -(A+B), b = B$

(9)$(A, B, a, b) = (4J+1, 2J+1, 6J+1, -J)$ for some integer J

(10)$(A, B, a, b) = (6J+2, 2J+1, 4J+1, -J)$

for some integer J

In particular (9) and (10)

$p = 22J^2 + 9J + 1$

$p = 22J^2 + 13J + 21$

What is this exception?

§1.3 **The dual knots**

Let $K \subset Y$ be a knot.
$\tilde{K} \subset Y(K, p)$: the dual knot of K.

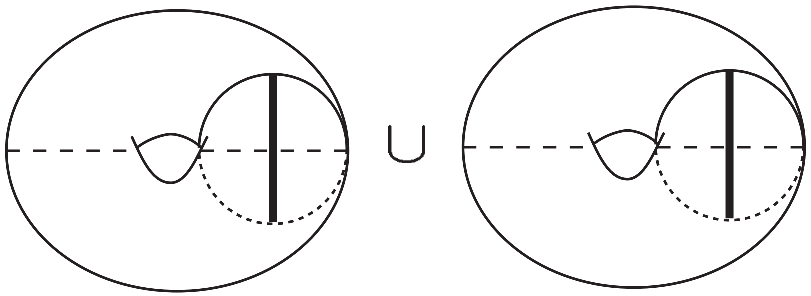
Suppose that $Y(K, p)$ is a lens space $L(p, q)$.
$[\tilde{K}] \in H_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$:a generator

$$h \in (\mathbb{Z}/p\mathbb{Z})^{\times}$$

The case of DP-knot

Fact

Let K be a DP-knot.
The dual knot $\tilde{K}$ of the lens space surgery is a 1-bridge knot in the lens space.



Remark

Let K_1 and K_2 be DP-knots in Y.
If $h_1 = h_2$ then $\tilde{K}_1$ and $\tilde{K}_2$ are isotopic.
Thus K_1 and K_2 are isotopic in Y.

$$\{\text{DP-knots}\} \leftrightarrow \cup_{p \in \mathbb{Z}_{>1}} (\mathbb{Z}/p\mathbb{Z})^{\times} / \sim$$

For any (p, h), what are Y and K?

§2.1 **Surgery on** $\Sigma(2,3,5)$ **and** $\Sigma(2,3,7)$

The speaker computed lens space surgery on $\Sigma(2,3,5)$ and $\Sigma(2,3,7)$ before.

Ozsvath-Szabo's formula for $Y(K,p) = L(p,q)$.
For any $i \in \mathbb{Z}/p\mathbb{Z}$

$$d(Y) - d(L(p,q), hi + c) + d(L(p,1), i) = l_i$$

$$\sum_{j \bmod p = i} l_j - \chi(HF_{\text{red}}(Y)) = t_i(K)$$

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m} (-1)^{m-j}(t^{n_j} + t_{-n_j})$$

The case of $Y = \Sigma(2,3,5)$

h:homology class

	p	h	$2g-p-1$	J
A_1	$14J^2+7J+1$	$7J+2$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
A_2	$20J^2+15J+3$	$5J+2$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
B	$30J^2+9J+1$	$6J+1$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
C_1	$42J^2+23J+3$	$7J+2$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
C_2	$42J^2+47J+13$	$7J+4$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
D_1	$52J^2+15J+1$	$13J+2$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
D_2	$52J^2+63J+19$	$13J+8$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
E_1	$54J^2+15J+1$	$27J+4$	$-\lvert J\rvert$	$\lvert J\rvert > 0$
E_2	$54J^2+39J+7$	$27J+10$	$-\lvert J\rvert$	$\lvert J\rvert > 0$

F_1	$69J^2 + 17J + 1$	$23J + 3$	$-2\|J\|$	$\|J\| > 0$
F_2	$69J^2 + 29J + 3$	$23J + 5$	$-2\|J\|$	$\|J\| > 0$
G_1	$85J^2 + 19J + 1$	$17J + 2$	$-2\|J\|$	$\|J\| > 0$
G_2	$85J^2 + 49J + 7$	$17J + 5$	$-2\|J\|$	$\|J\| > 0$
H_1	$99J^2 + 35J + 3$	$11J + 2$	$-2\|J\|$	$\|J\| > 0$
H_2	$99J^2 + 53J + 7$	$11J + 3$	$-2\|J\|$	$\|J\| > 0$
I_1	$120J^2 + 16J + 1$	$12J + 1$	$-2\|J\|$	$\|J\| > 0$
I_2	$120J^2 + 20J + 1$	$20J + 2$	$-2\|J\|$	$\|J\| > 0$
I_3	$120J^2 + 36J + 3$	$12J + 2$	$-2\|J\|$	$\|J\| > 0$
J	$120J^2 + 104J + 22$	$12J + 5$	$-\|2J + 1\|$	all integers
K	191	15	-2	

The case of $\Sigma(2,3,7)$

	p	h	$2g-p-1$	J
A_1	$18J^2+9J+1$	$9J+2$	$\lvert J\rvert$	$\lvert J\rvert>0$
A_2	$30J^2+25J+5$	$5J+2$	$\lvert J\rvert$	$\lvert J\rvert>0$
B	$42J^2+15J+1$	$6J+1$	$\lvert J\rvert$	$\lvert J\rvert>0$
C_1	$56J^2+33J+5$	$7J+2$	$\lvert J\rvert$	$\lvert J\rvert>0$
C_2	$56J^2+65J+19$	$7J+4$	$\lvert J\rvert$	$\lvert J\rvert>0$
D_1	$76J^2+17J+1$	$19J+2$	$\lvert J\rvert$	$\lvert J\rvert>0$
D_2	$76J^2+97J+31$	$19J+12$	$\lvert J\rvert$	$\lvert J\rvert>0$
E_1	$74J^2+17J+1$	$37J+4$	$\lvert J\rvert$	$\lvert J\rvert>0$
E_2	$74J^2+57J+11$	$37J+14$	$\lvert J\rvert$	$\lvert J\rvert>0$

F_1	$93J^2 + 19J + 1$	$31J + 3$	$\|2J\|$	$\|J\| > 0$
F_2	$93J^2 + 43J + 5$	$31J + 7$	$\|2J\|$	$\|J\| > 0$
G_1	$115J^2 + 21J + 1$	$23J + 2$	$\|2J\|$	$\|J\| > 0$
G_2	$115J^2 + 71J + 11$	$23J + 7$	$\|2J\|$	$\|J\| > 0$
H_1	$143J^2 + 53J + 5$	$11J + 2$	$\|2J\|$	$\|J\| > 0$
H_2	$143J^2 + 79J + 11$	$11J + 3$	$\|2J\|$	$\|J\| > 0$
I_1	$168J^2 + 32J + 1$	$12J + 1$	$\|2J\|$	$\|J\| > 0$
I_2	$168J^2 + 60J + 5$	$12J + 2$	$\|2J\|$	$\|J\| > 0$
I_3	$168J^2 + 28J + 1$	$28J + 2$	$\|2J\|$	$\|J\| > 0$
J	$168J^2 + 136J + 28$	$12J + 5$	$\|2J + 1\|$	$\|J\| > 0$
K	259	15	2	

Any lens space in this list can be realized as Dehn surgeries of DP-knots

Conj.1

Let Y be a homology sphere with $\lambda = -1$. If $Y(K, p)$ is lens space, then Y is $\Sigma(2,3,5)$ or $\Sigma(2,3,7)$ and this list is complete.

Conj.2

Let Y be a homology sphere with $\lambda = 0$. If $Y(K, p)$ is lens space, then Y is S^3 and Berge's list is complete.

Fact

Let Y be $\overline{\Sigma(2,3,5)}$.
Then *no* knot in Y gives rise to any lens space.

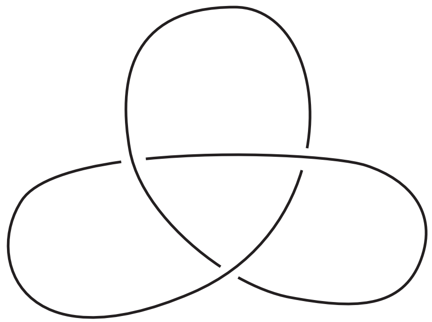
Conj.3

Let Y be a homology spehre with $\lambda > 0$.
Then *no* knot in Y gives rise to a lens space ($p > 0$).

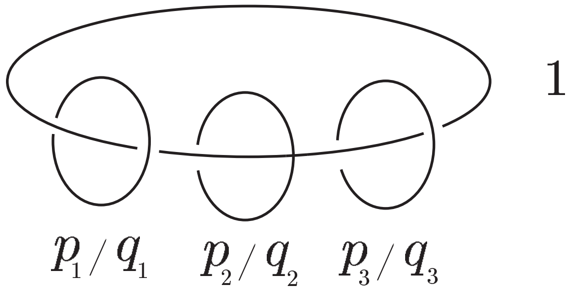
§2.2 **Knots in homology spheres**

Motivation

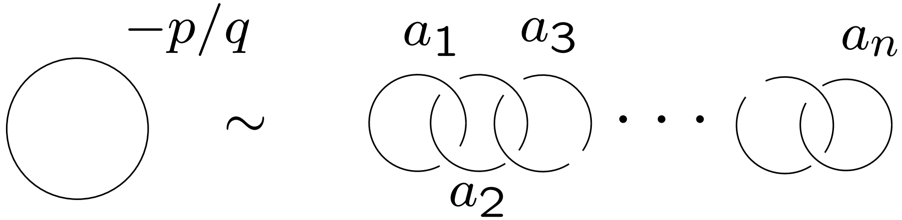
We want to see knots in a homology sphere like this.



Seifert manifolds

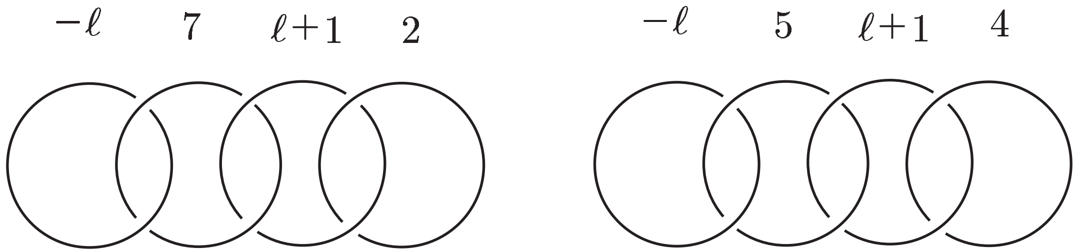


Lens space $L(p,q)$

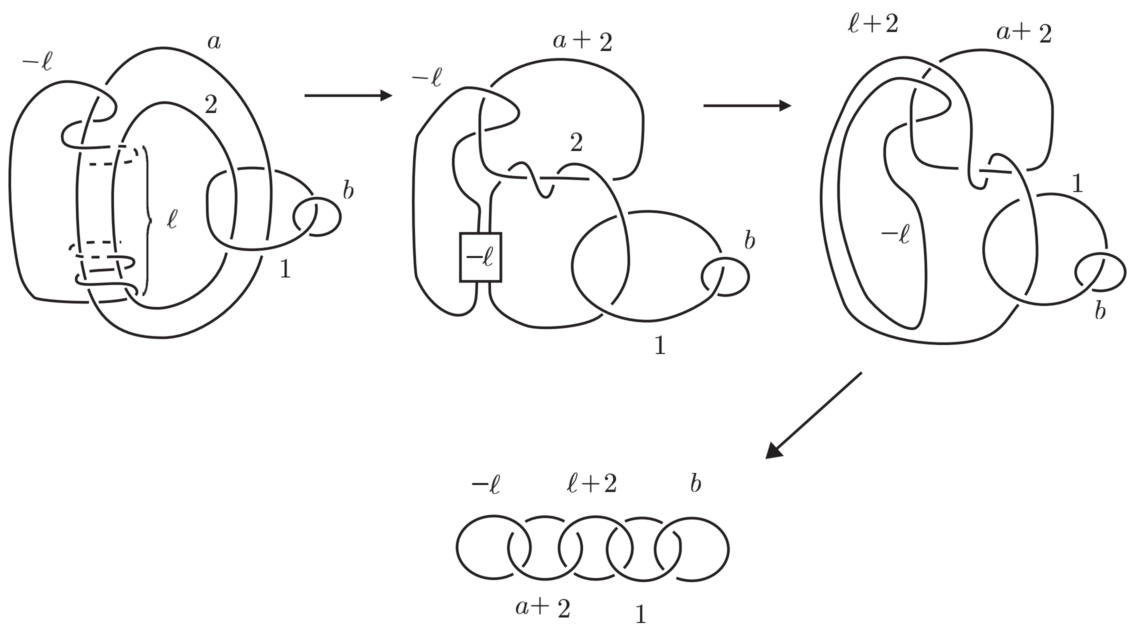


$$-\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}}}$$

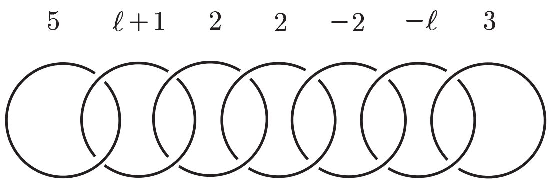
The case of A

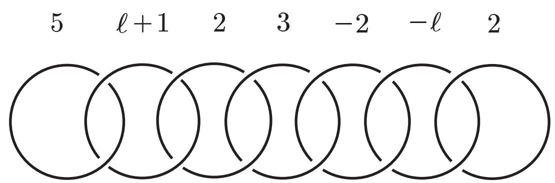


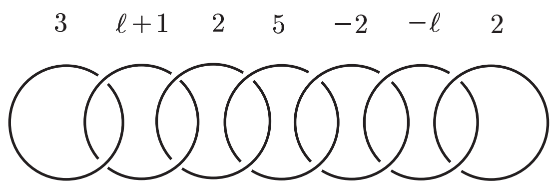
$((a, b) = (3, 5), (5, 3))$

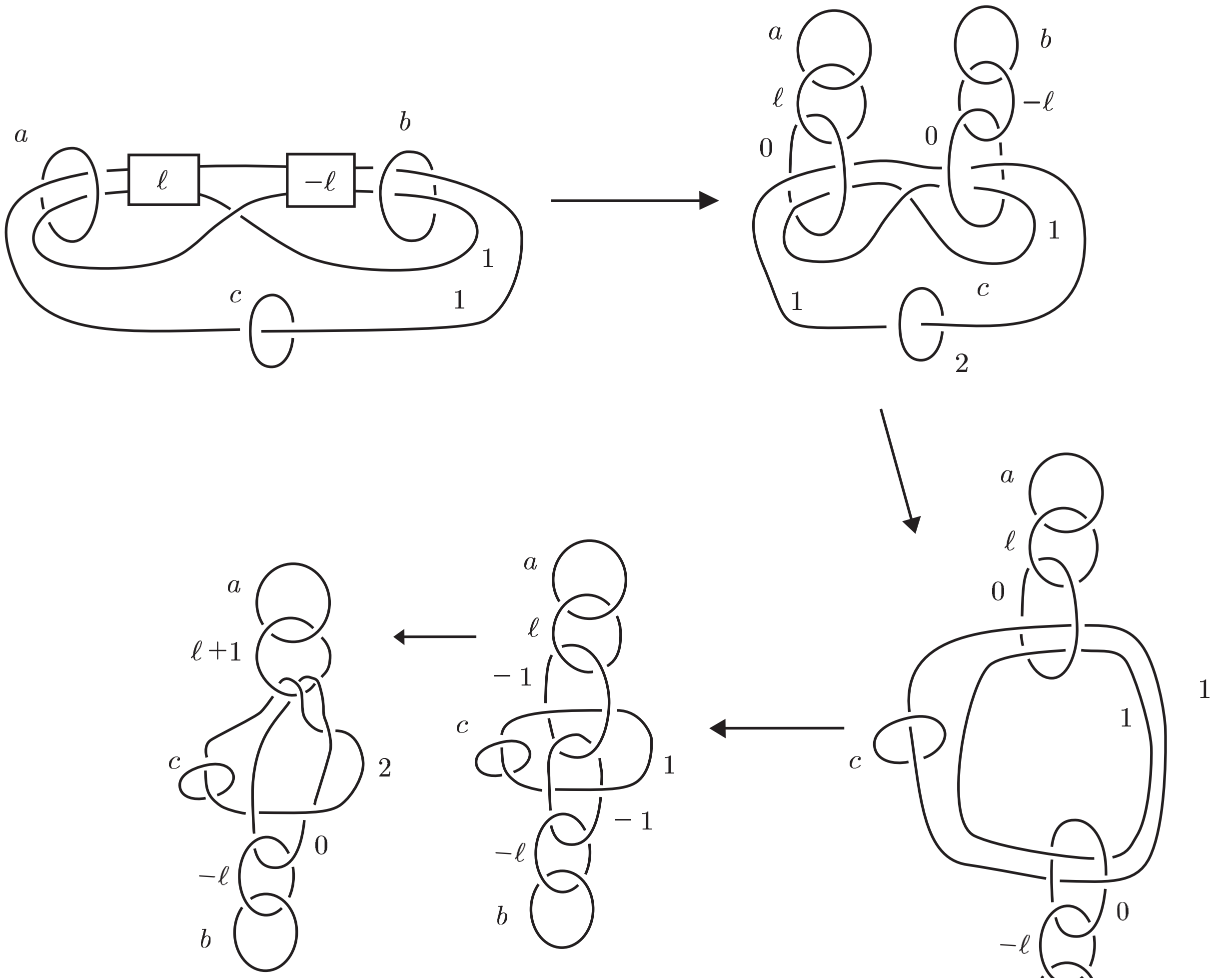


The case of I_1, I_2, I_3








a
ℓ
−ℓ
b
1
c
1
a
b
ℓ
−ℓ
0
0
1
1
c
2
a
ℓ
0
1
1
c
0
−ℓ
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1
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b
a
ℓ +1
c
2
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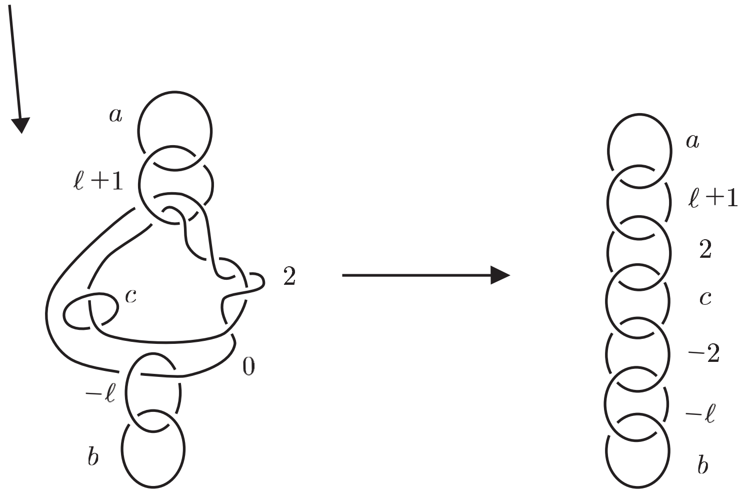
b



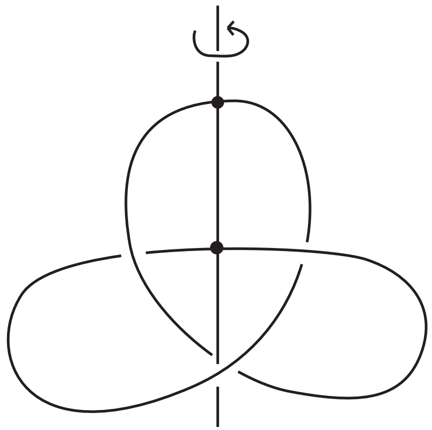
Is this knot DP-knot?

§2.3 **Branched covering**

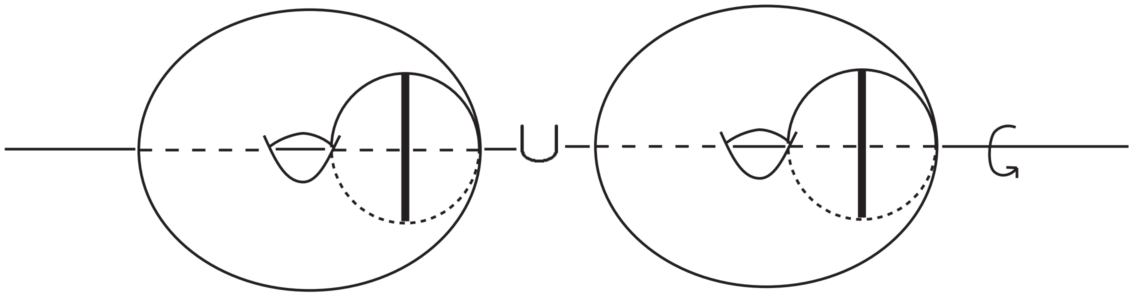
$L(p,q) \to S^3$: $^\exists$double branched covering
$K_{p,q}$:the branched locus(the 2-bridge link)

Let K be a DP-knot
Y: Heegaard genus two
$Y \to S^3$: $^\exists$double branched covering
K: *strongly invertible* by this involution

trefoil



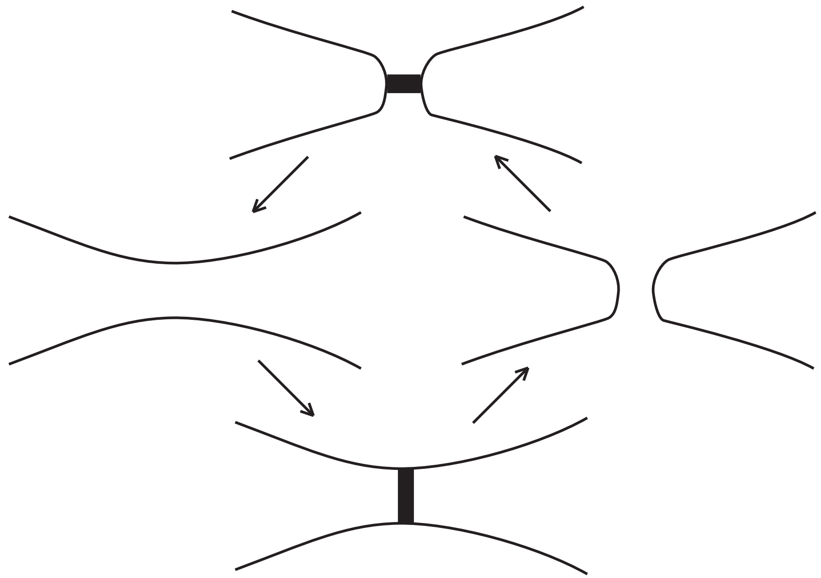
DP-knots



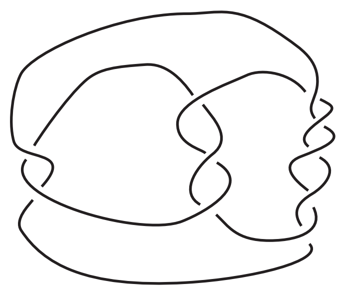
$Y(K, p) = L(p, q)$

$$
\begin{array}{ccc}
Y & \xrightarrow{\text{Dehn sergery}} & L(p,q) \\
\Big\downarrow \text{branched covering} & & \Big\downarrow \text{branced covering} \\
(S^3, B) & \longrightarrow & (S^3, K_{p,q})
\end{array}
$$

the lower arrow is *the band sum*

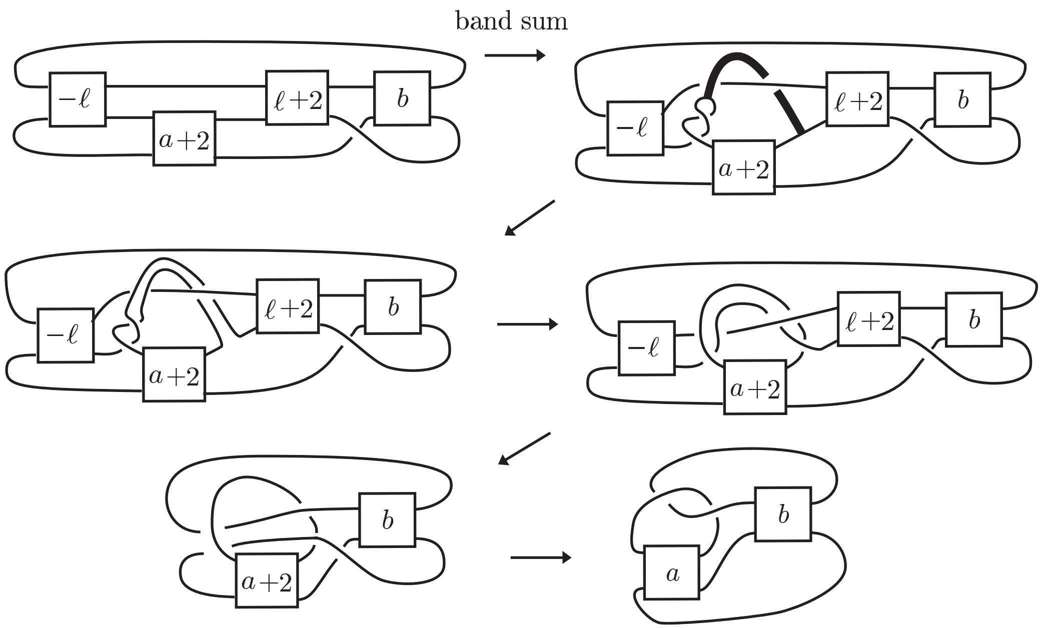


For example we see the surgeries lens space surgery on $\Sigma(2,3,5)$.

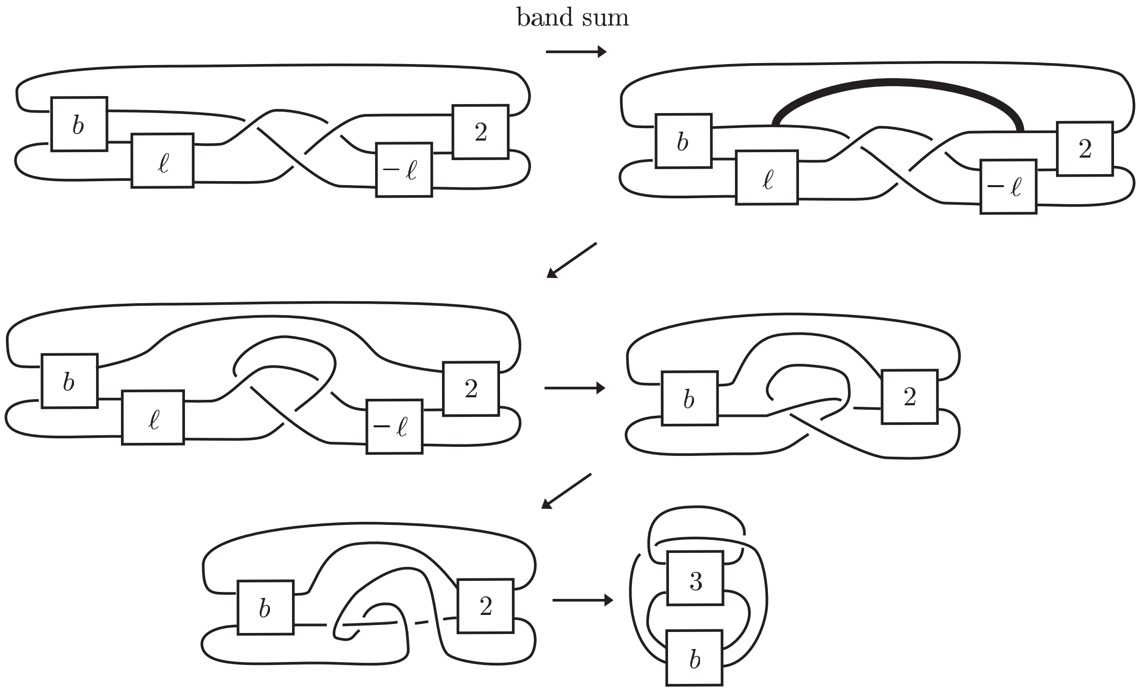


$Pr(-2, 3, 5) =$(3,5)-torus knot

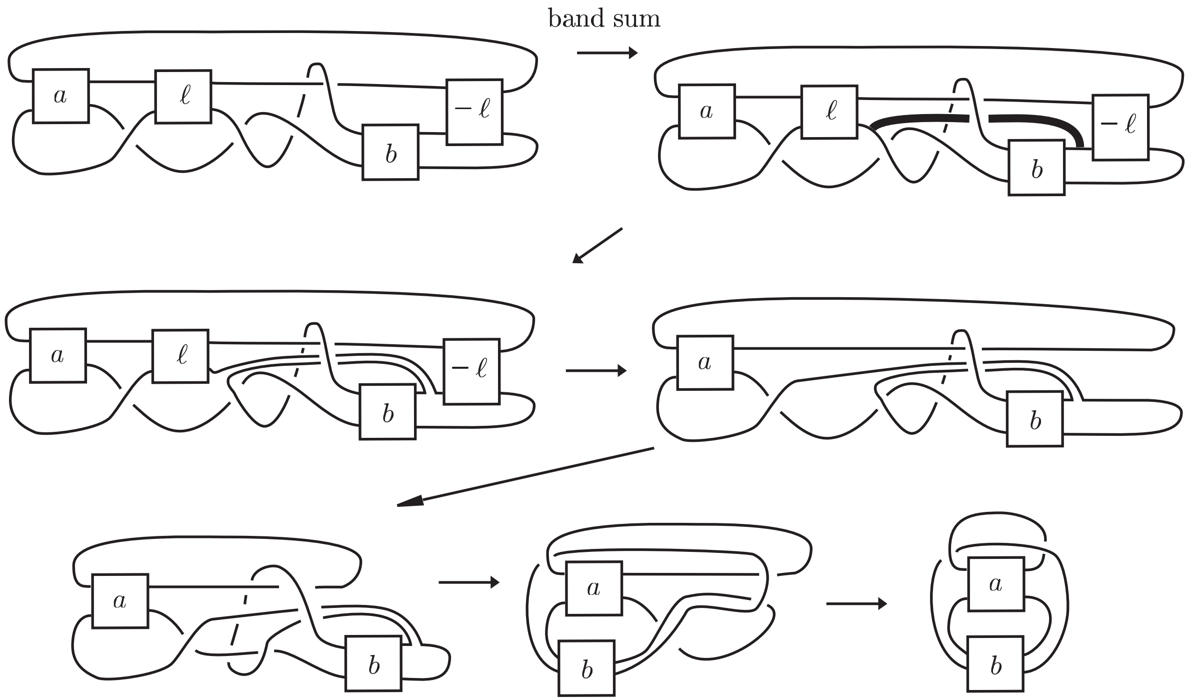
The case of A

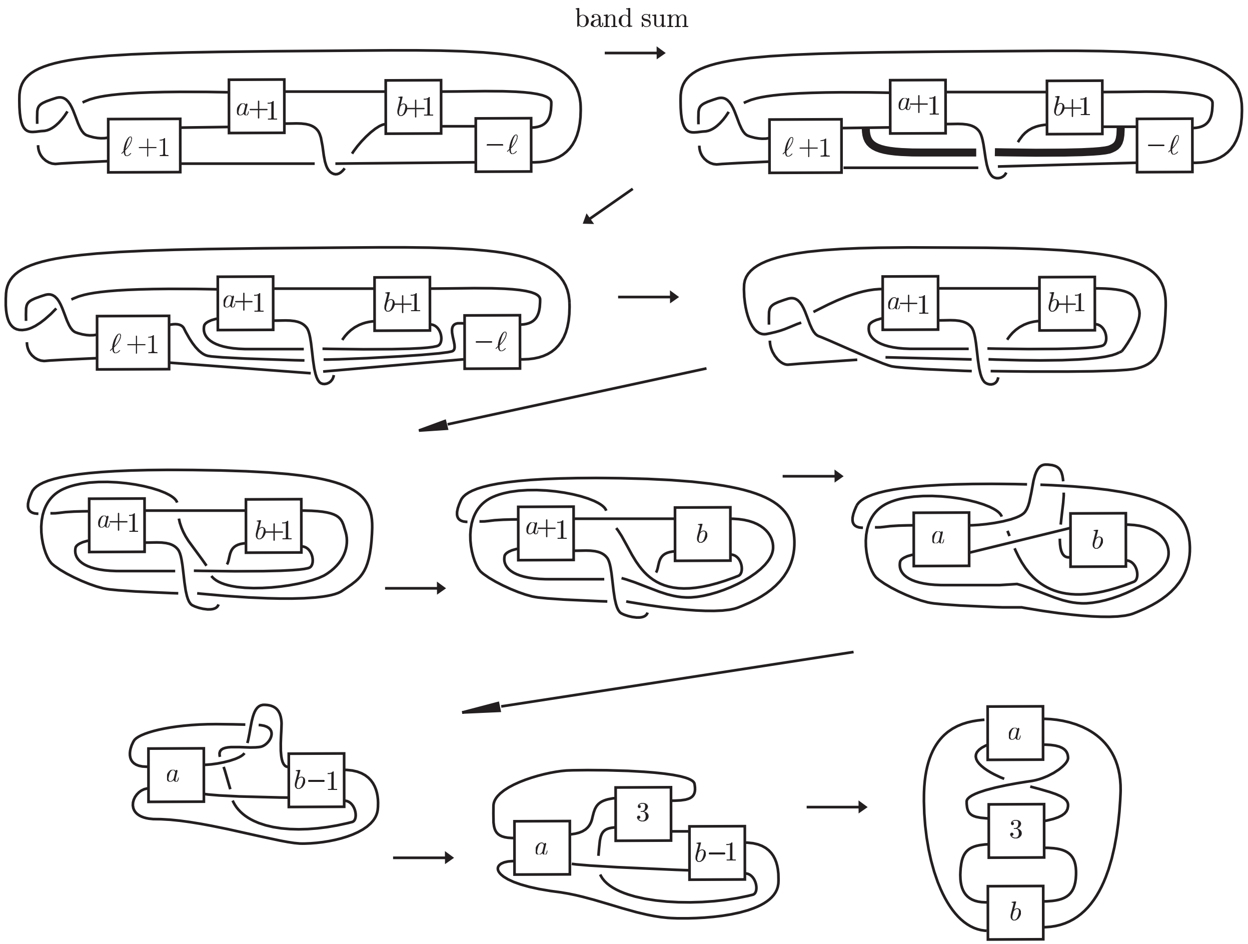


The case of B

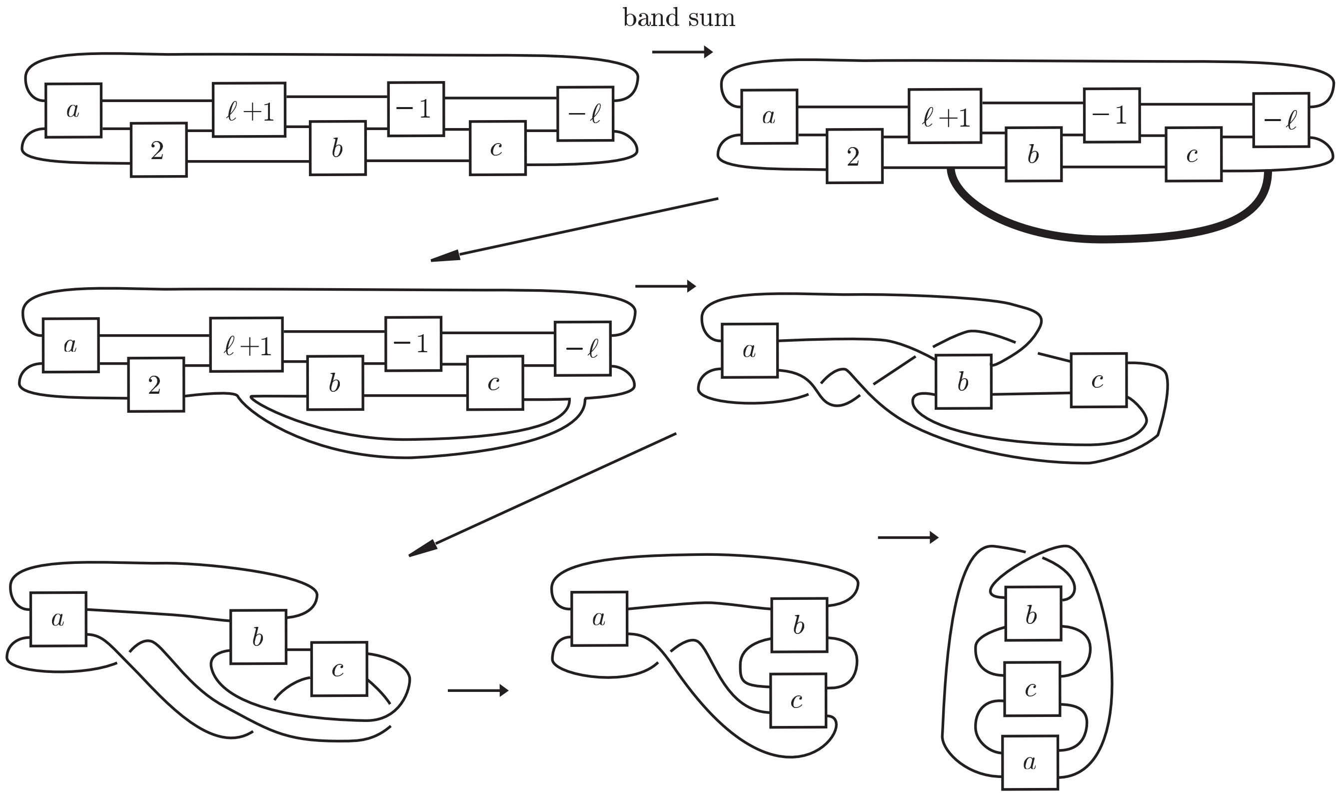


The case of C and D The case of E

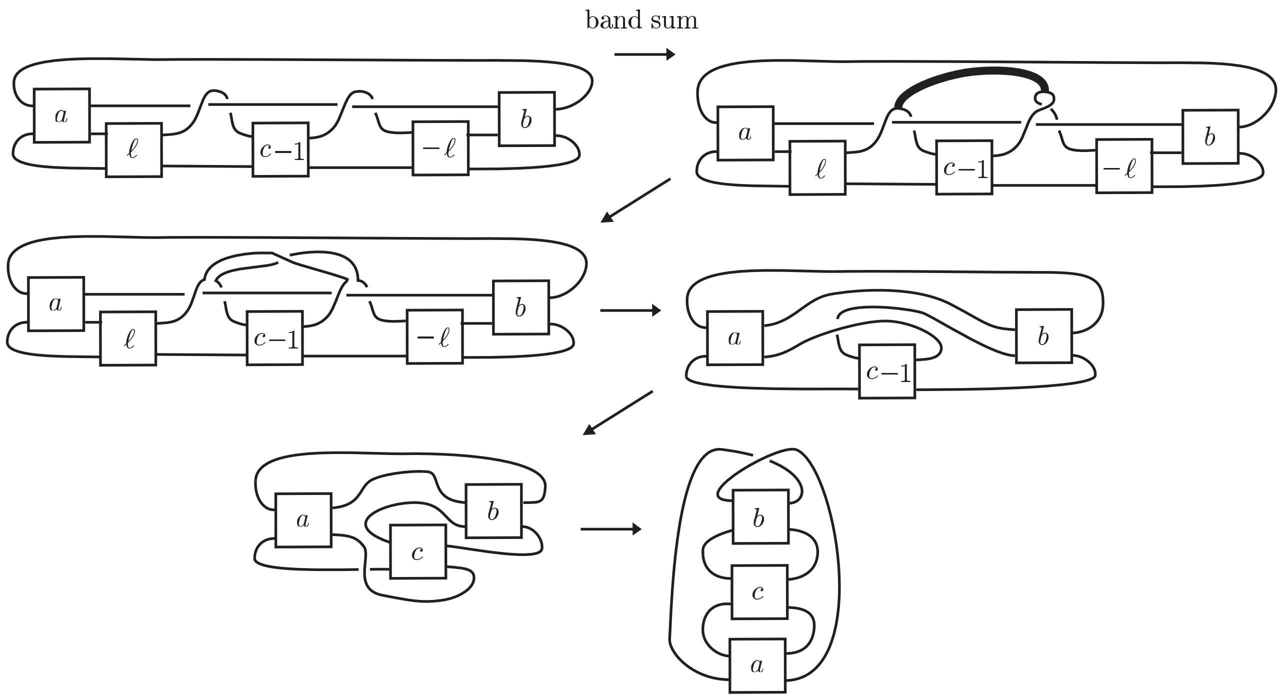



band sum
a+1
b+1
ℓ +1
−ℓ
a
b
b−1
3

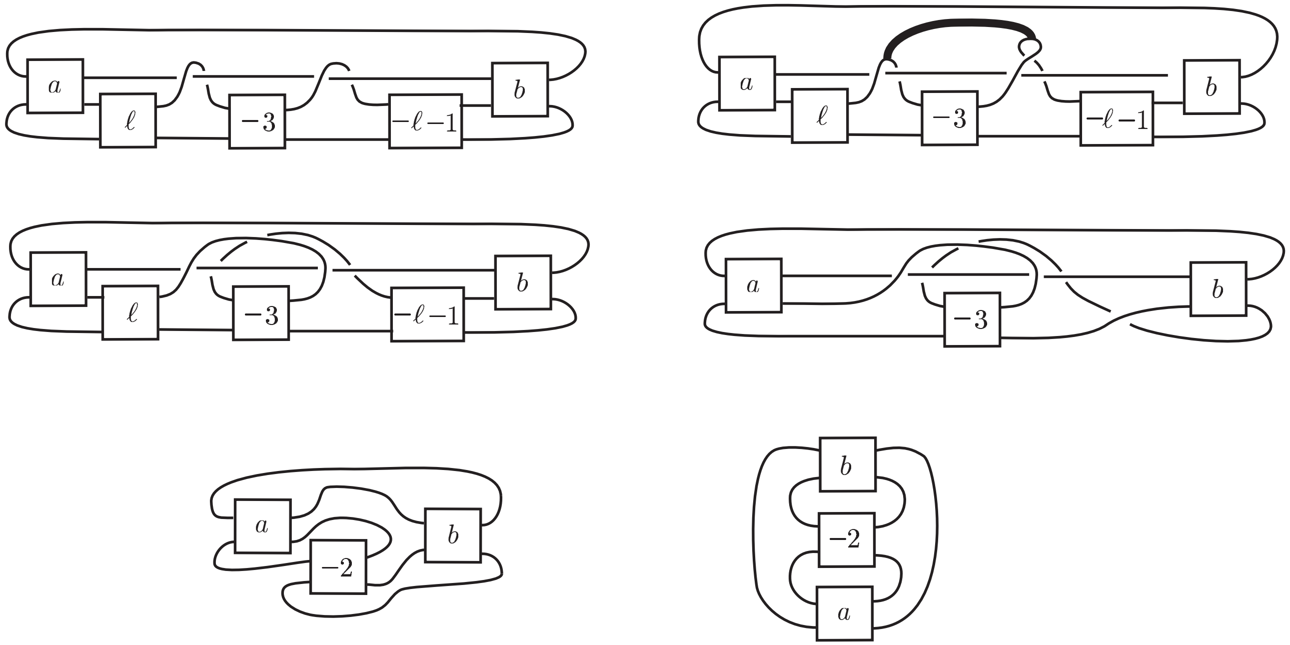
The case of F, G, and H



The case of I



The case of J

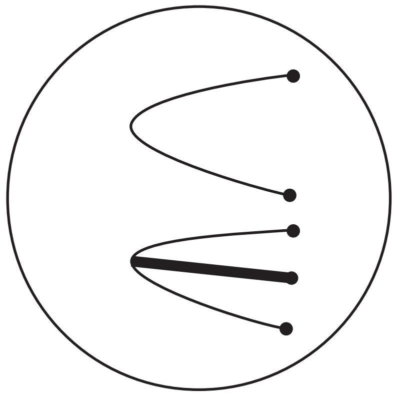


In fact these are the bund sum of DP-knot

A criterion of DP-knot

Let $K_{p,q}$ be a 2-bridge knot
$S^3 = B_1 \cup B_2$: genus 0 Heeegaard decomposition
$B_i \cap K_{p,q}$ are two arcs.

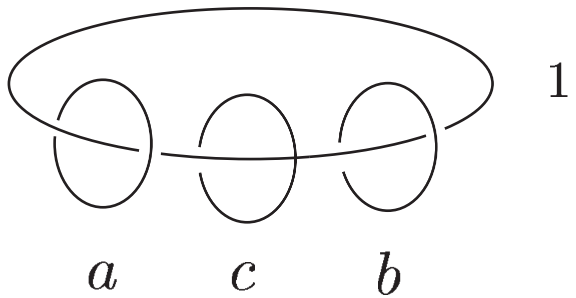
Let b: a band of $K_{p,q}$ $B_i \cap b$ is single arc like the following:



Corollary

We put $a, b \in \mathbb{Q}$, $c \in \mathbb{Z}$.

Let $Y(a,b,c)$ be a Brieskorn homology spheres as in the figure below.



Then Y contains a DP-knot.

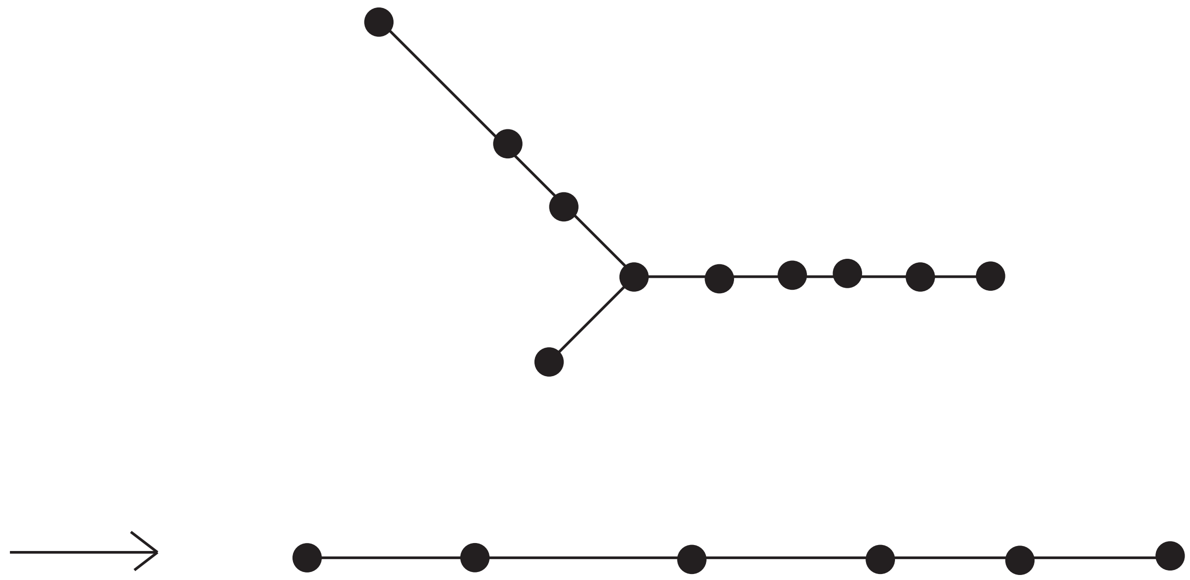
Proof

In the case of I put $a, b \in \mathbb{Q}$,$c \in \mathbb{Z}$.

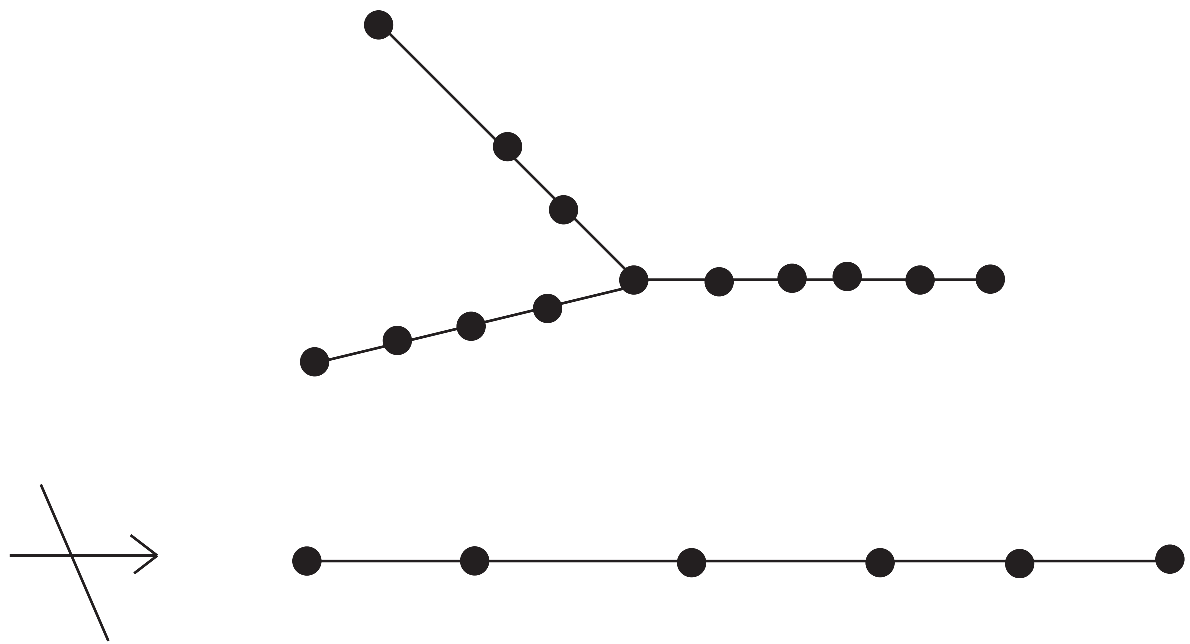
Conj.4

$Y(a,b,c)$: a Brieskorn homology sphere
If $Y(a,b,c)$ contains a DP-knot, then
one of a, b and c must be an integer.

Lens space surgery



but not lens space surgery



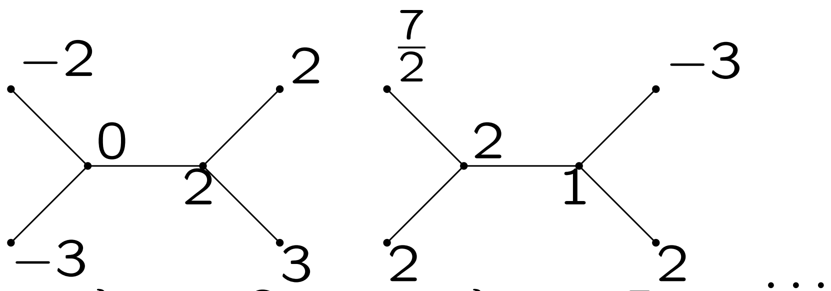
Question

If a homology sphere Y contains DP-knot, then is Y a Briskorn homology sphere?

Answer

There exist non-Brieskorn homology spheres which contain DP-knots.

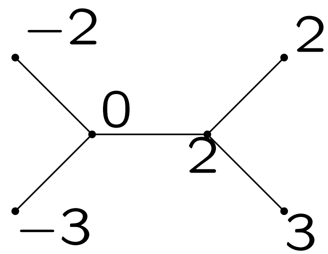
Examples(graph manifolds)



$\lambda = -2$ $\lambda = -5$ $\cdots$

	p	h
I	$30\ell^2 + 5\ell$	$10\ell + 1$
II_1	$42\ell^2 + 19\ell + 2$	$21\ell + 5$
II_2	$42\ell^2 + 5\ell$	$14\ell + 1$
III_1	$69\ell^2 + 55\ell + 11$	$23\ell + 9$
III_2	$69\ell^2 + 37\ell + 5$	$23\ell + 6$
IV_1	$85\ell^2 + 61\ell + 11$	$17\ell + 6$
IV_2	$85\ell^2 + 41\ell + 5$	$17\ell + 4$

<u>The case of IV_2</u>

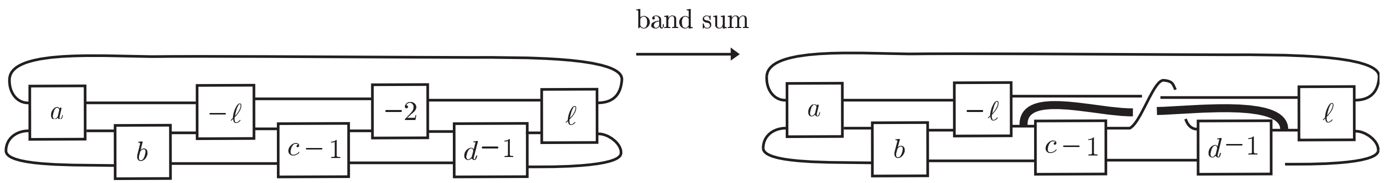


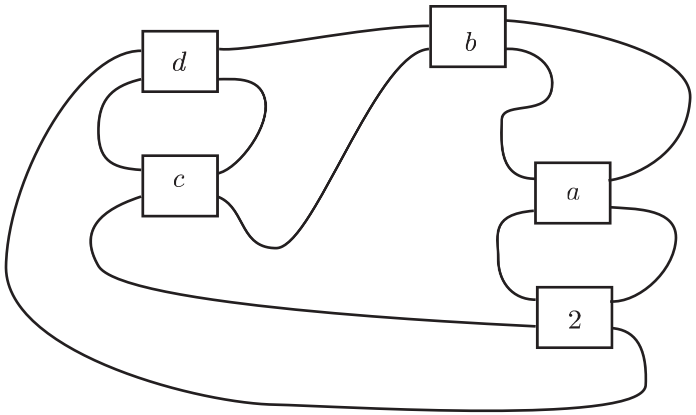
$$\lambda = -2$$

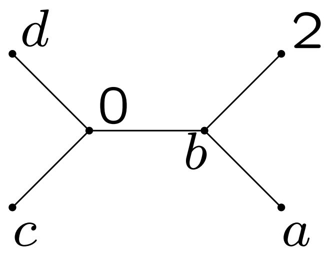
$$\rightarrow [3, 2, -\ell, -4, -2, -3, \ell]$$

$$[a, b, -\ell, c-1, -2, d-1, \ell]$$

$$a = 3, b = 2, c = -3, d = -2$$

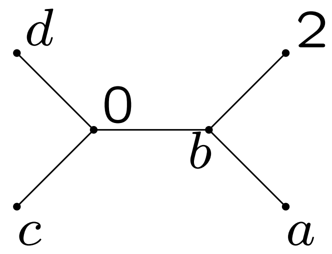






Corollary

Let Y be a graph homology sphere with the plumbing graph as below:

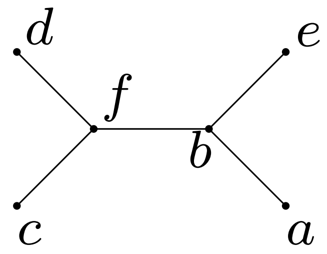


Here $a \in \mathbb{Q}$ $b, c, d \in \mathbb{Z}$.
Then Y contains a DP-knot.

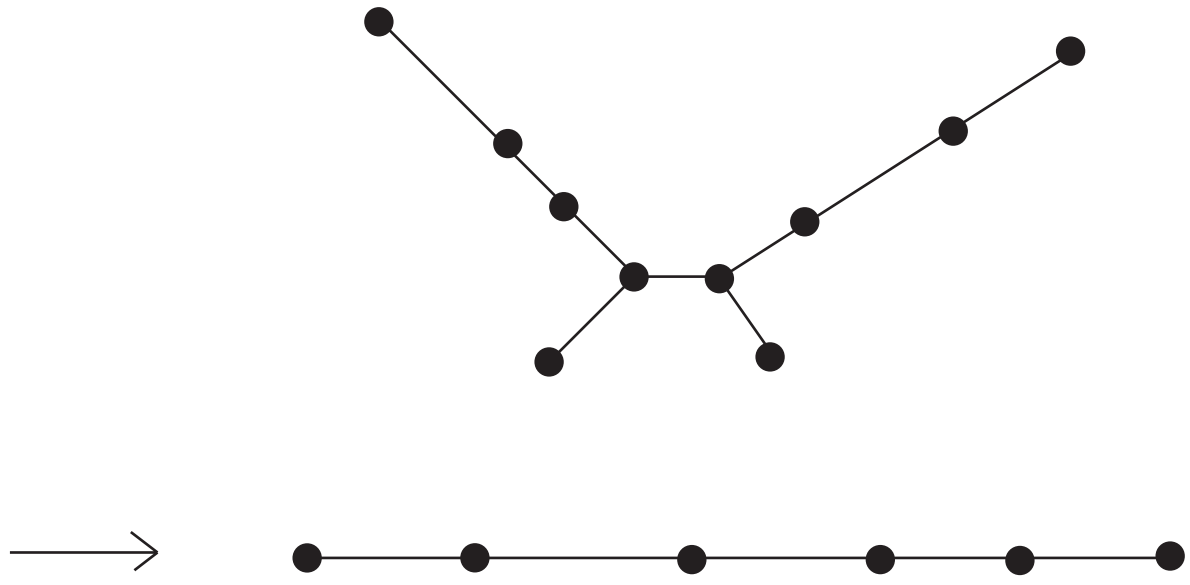
Such manifold is the splicing of two Brieskorn homology spheres

Conj.5

Let Y be a homology sphere with plumbing graph as below:



where $a, c \in \mathbb{Q}$ $b, d, e, f \in \mathbb{Z}$. Then Y contains a DP-knot.



$\text{Splicing}(\Sigma(p,q,r),\Sigma(p',q',r'),r,r')$

$$pqq'q' = rr' \pm 1$$