A calculus of Scharlemann's manifold; sliding, isotopy, and upside-down

丹下 基生
(京大 数理研)

阪大トポロジーセミナー
2008 11/11

Contents of talk

§1 Introduction

'76 M. Scharlemann

Constructing strange manifolds with the dodecahedral space

(Duke math. J. 43 . 33–40)

1. Fake homotopy structure of $S^3 \times S^1 \# S^2 \times S^2$ (Scharlemann's mfd)

   $S^3 \times T^2$

2. Non-triangulable homotopy

   $\mathbb{C}P^2 \times S^1$

'99 S. Akbulut

Scharlemann's manifold is standard

(Annals of Math. 149 497–510)

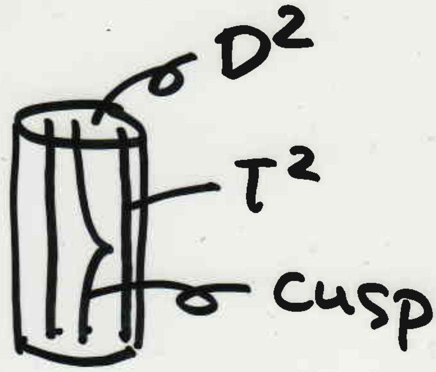
- Scharlemann's mfd is diffeo. to standard $S^3 \times S^1 \# S^2 \times S^2$.

Motivation, purpose, exposition, ... etc.

X : 4-dim smooth mfd.

$\cup$

C : cusp nbd. $=$



$\cup$

$\nu(T^2)$: nbd of regular fiber.

$K \subset S^3$: knot

$$\boxed{X_K := [X \setminus \nu(T^2)] \cup [\{S^3 \setminus \nu(K)\} \times S^1] \quad \text{knot surgery of } X}$$

Fact (Fintushel-Stern)

X : K3 surface (elliptic fibration)

$$X_K \simeq X \quad \text{(homeo)}$$

but

$$X_K \not\cong X \quad \text{(non-diffeo)}$$

(if $\Delta_K \neq 1$)

(Because $SW_{X_K} = \Delta_K SW_X$)

How about other examples?

$S^2 \times S^2 = C \cup (-C)$ $\qquad SW_{S^2\times S^2} = 0$

Question ('02 M. Akaho)

Is $C_K \cup (-C_K)$ standard?

Proposition (Akaho)

$$(X_K)_L = X_{K\#L}$$

Corollary

$$C_K \cup (-C_K) = C_{K\#\bar{K}} \cup (-C)$$

Question

Is $C_K \cup (-C)$ standard or non-standard?

Theorem (Akbulut)

$C_{\text{trefoil}} \cup (-C) \cong S^2 \times S^2$ (diffeo)

Sketch of Proof

K : trefoil

$$C_K(-C) = \text{surgery of } M$$

$$\underset{\text{Akbulut}}{=} \text{surgery of } S^3 \times S^1 \# S^2 \times S^2 \quad \text{(Scharlemann's mtd)}$$

$$= S^2 \times S^2$$

$\blacksquare$

Generalize the result of Akbulut for solving the Akaho's question!

We expose Akbulut's result.

Theorem (S. Akbulut)

$$M \cong S^3 \times S^1 \# S^2 \times S^2$$

§ 2 Scharlemann's manifold

Definition N : PL-mfd homotopy str on N

$\exists f : N' \to N$ (simple homotopy str)

Def $f_0 : N_0 \to N$

$f_1 : N_1 \to N$ $f_0 \sim f_1$

$\Leftrightarrow \exists F : \exists W \to N \times I$ (homotopy eq.)

W : s-cobordism of N_0 & N_1

$F|_{N_0} = f_0$, $F|_{N_1} = f_1$

Def $f : N_1 \to N$ fake homotopy str (homotopy str)

W : PL-cobordism of M_1 & M

$F : W \to M \times I$ (a map)

but

F : non-homotopy eq.

Σ : the Poincaré sphere

$\Sigma \times S^1 \supset \gamma$: a loop generating of $\pi_1(\Sigma)$

$\Sigma \times S^1 \setminus \nu(\gamma)$ removing

$[\Sigma \times S^1 \setminus \nu(\gamma)] \cup D^2 \times S^2$ attaching

Definition We define Scharlemann's manifold M to be the resulting mfd

Theorem (Scharlemann)

$\exists f : M \longrightarrow S^3 \times S^1 \# S^2 \times S^2$ (homotopy str.)

$f \not\simeq$ diffeo (homotopic)

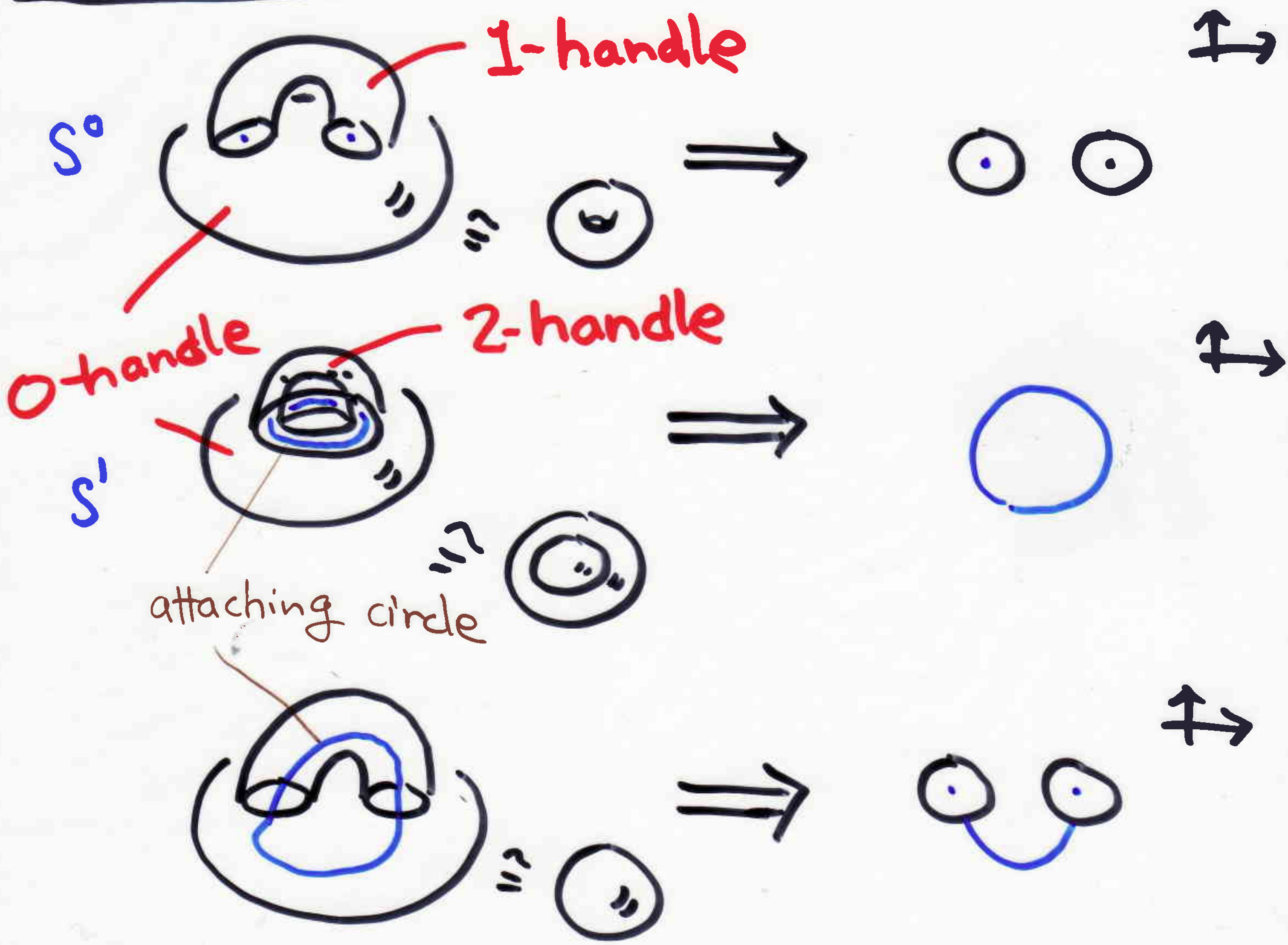
Q ('76–'99)

M is diffeo. to $S^3 \times S^1 \# S^2 \times S^2$ in the first place?

§ 3 Kirby calculus

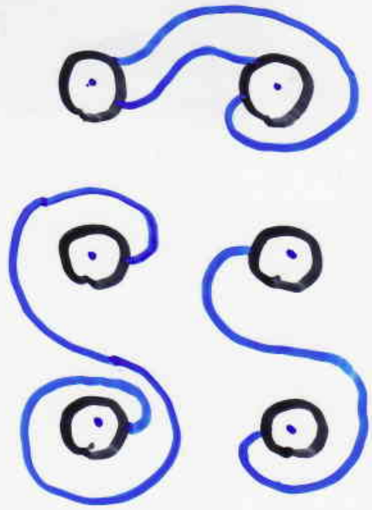
Design of 4-Manifold

3-Manifold



a 2-handle running through a 1-handle

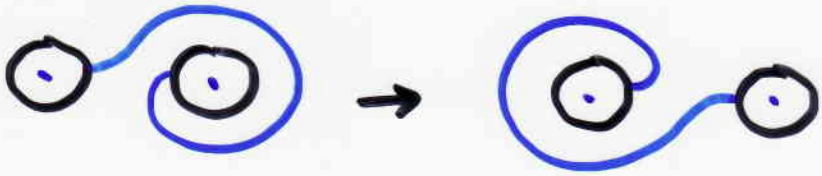
Example (Gompf-Stipsicz)



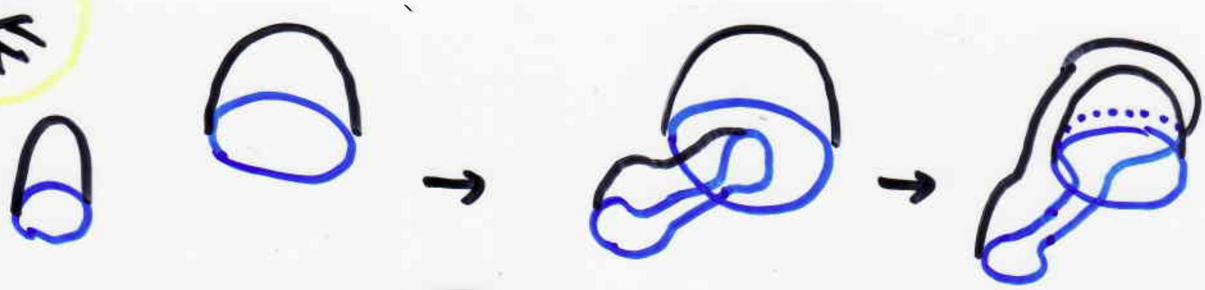
Kirby diagram

The fundamental move (Kirby move)

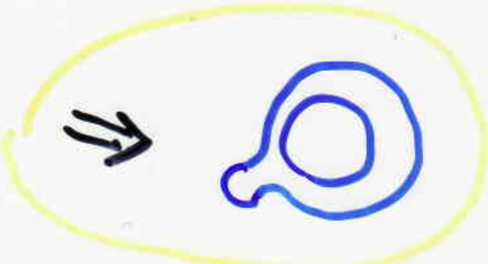
- isotopy of attaching sphere



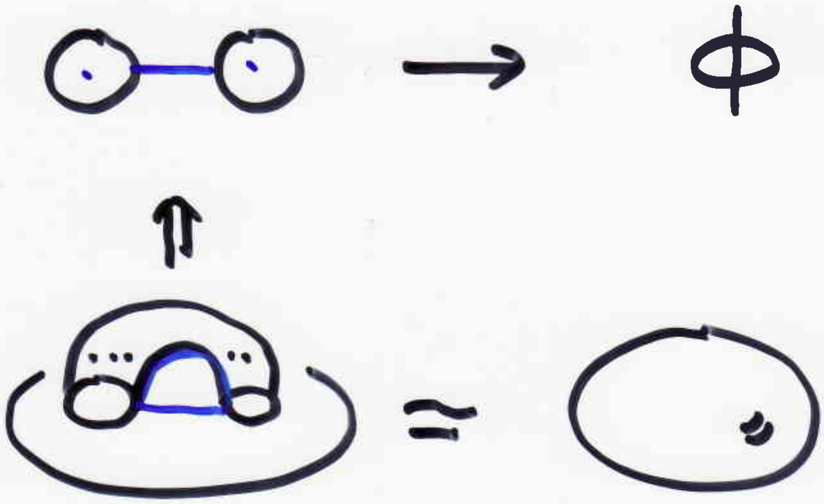
- sliding of attaching sphere



connected sum!



· cancelling of a pair of handles



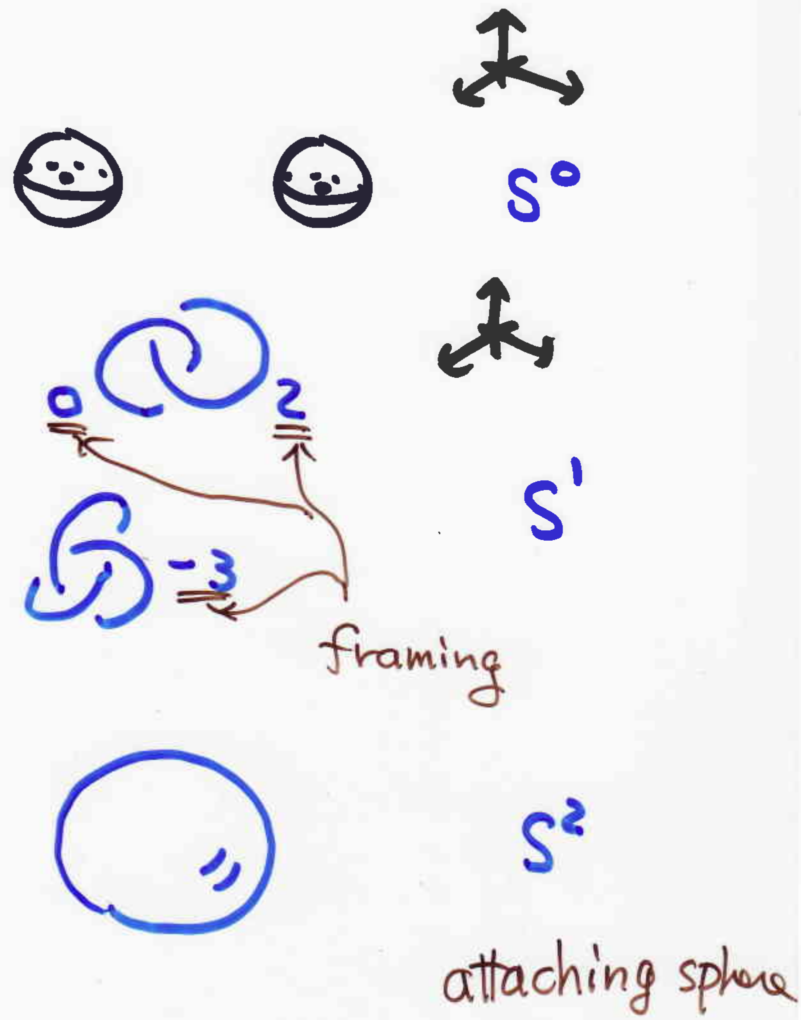
· creating

4-Manifold

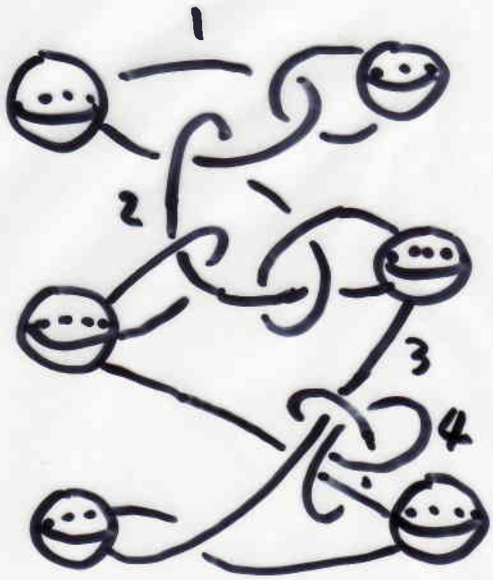
1-handle

2-handle

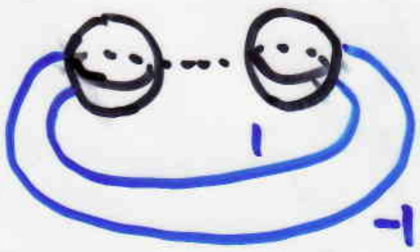
3-handle



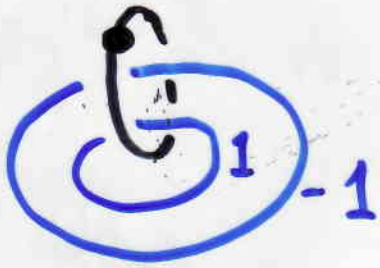
Examples (Gompf - Stipsicz)



1-handle description



$\approx$



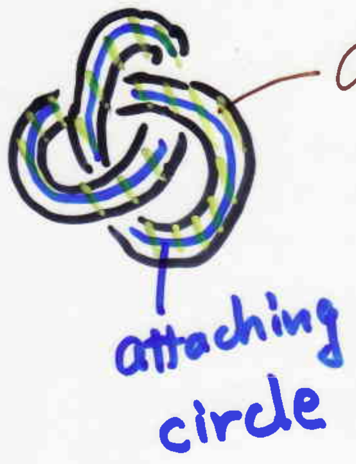
image

3-handle description

$\approx \quad \phi$

3-handle running through 2-handle

framing

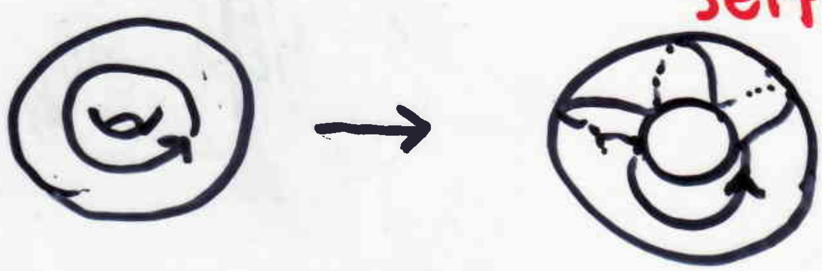


$\cong S^1 \times D^2$

Isotopy classes of diffeo. $S^1 \times D^2 \to S^1 \times D^2$

$\cong \mathbb{Z}$ (twisting #)

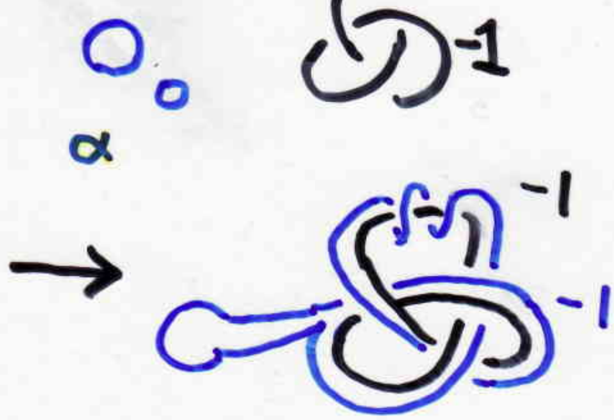
self-linking #



twisting # $= 3$

Kirby move

- isotopy
- sliding



$(\alpha+\beta)^2$

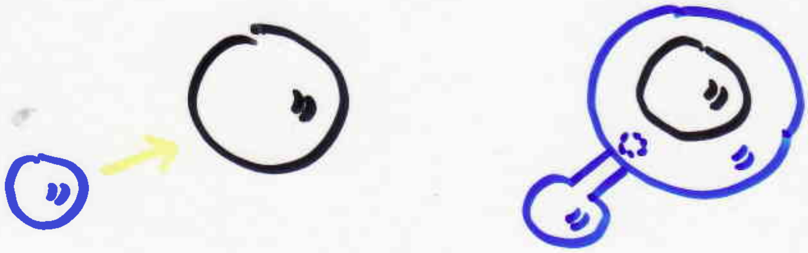
$= \alpha^2 + \beta^2$

$+ 2lk(\alpha.\beta)$

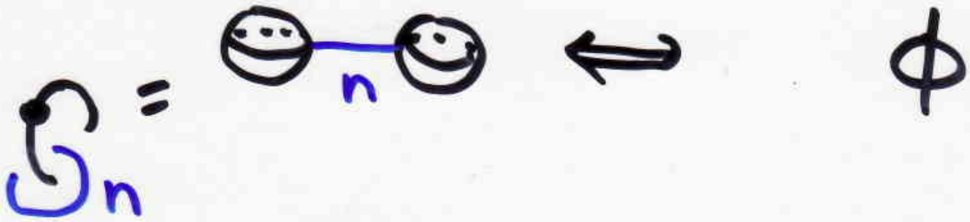
$= 0 + (-1)$

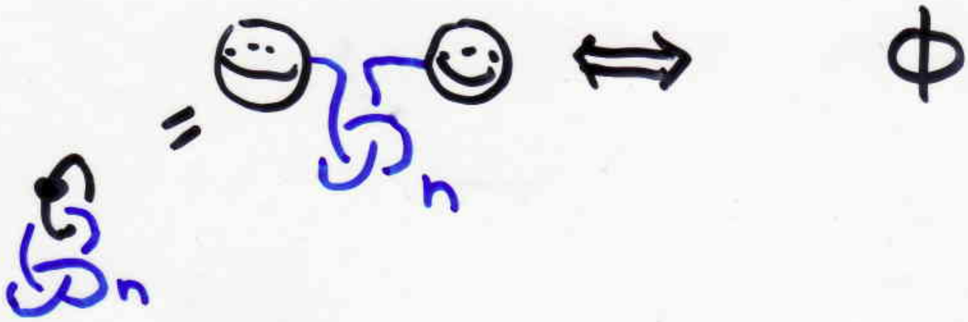
$+ 2\cdot 0$

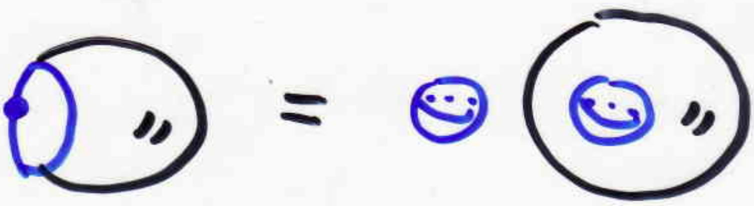
$= -1$



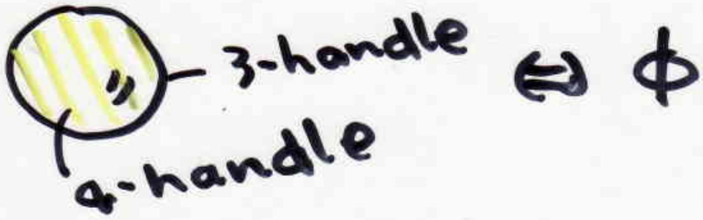
· Cancelling and creating







·



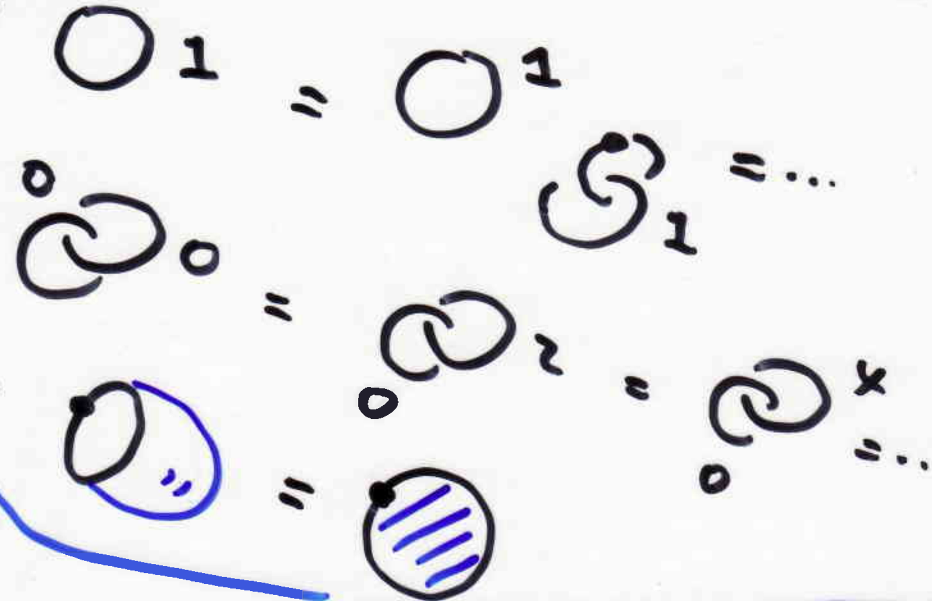
Theorem

$$\left\{\begin{matrix}\text{cpt. ori-ble}\\ \text{4-Manifolds}\end{matrix}\right\} \longleftrightarrow \{\text{framed links}\} \Big/ \begin{matrix}\text{isotopy.}\\ \text{sliding.}\\ \text{creating \&}\\ \text{cancelling}\end{matrix}$$

$\mathbb{CP}^2 =$

$S^2 \times S^2 =$

$S^3 \times S^1 =$

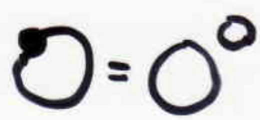


Theorem

$$\left\{\begin{matrix}\text{cpt. ori-ble}\\ \text{4-Manifolds}\end{matrix}\right\} \longleftrightarrow \{\text{framed links}\} \Big/ \sim$$

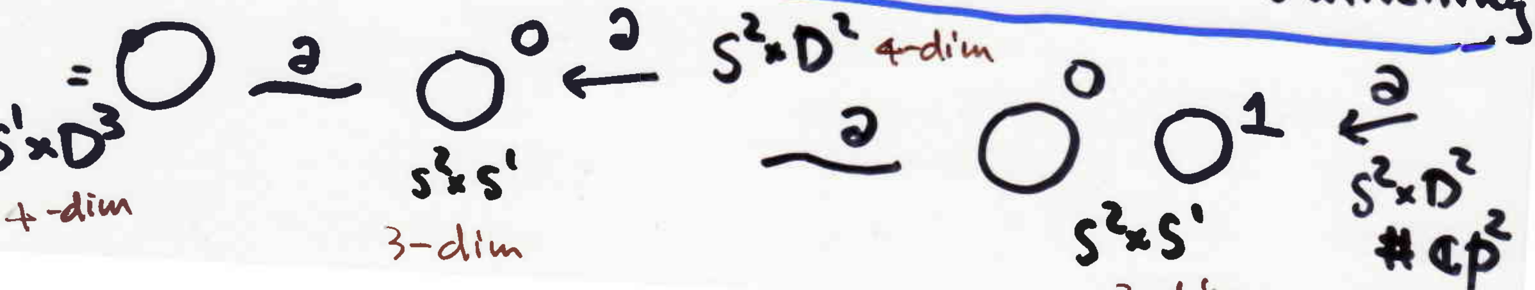
$$\downarrow \partial : \text{boundary} \qquad\qquad \downarrow$$

$$\{\text{3-Manifolds}\} \longleftrightarrow \{\text{framed links}\} \Big/ \sim$$

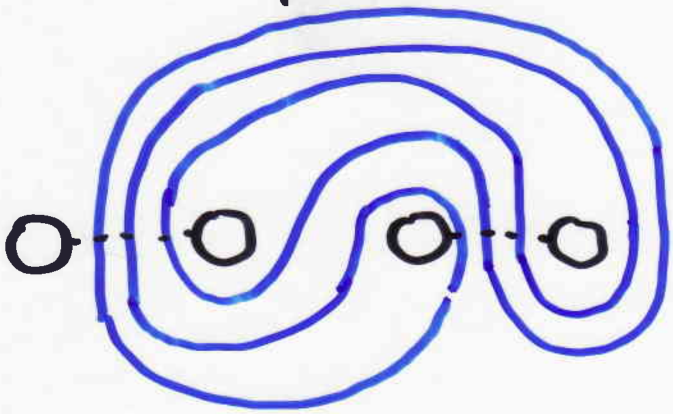


& $O^{\pm 1}$ - creating

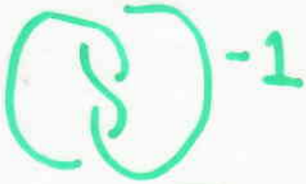
" - vanishing



§4 Description of Scharlemann's mfd



$S^3 \setminus \nu(\ )$

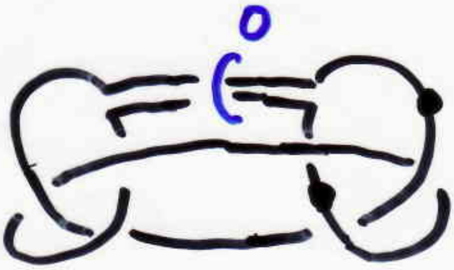


$\Sigma(2,3,5)$

$\downarrow \times I$

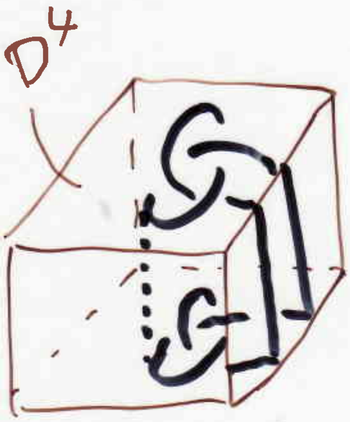
$\{S^3 \setminus \nu(\ )\} \times I$

$\sim$



$\longrightarrow$

slice 1-handle



$\sim$

the surgery of [ι ×]

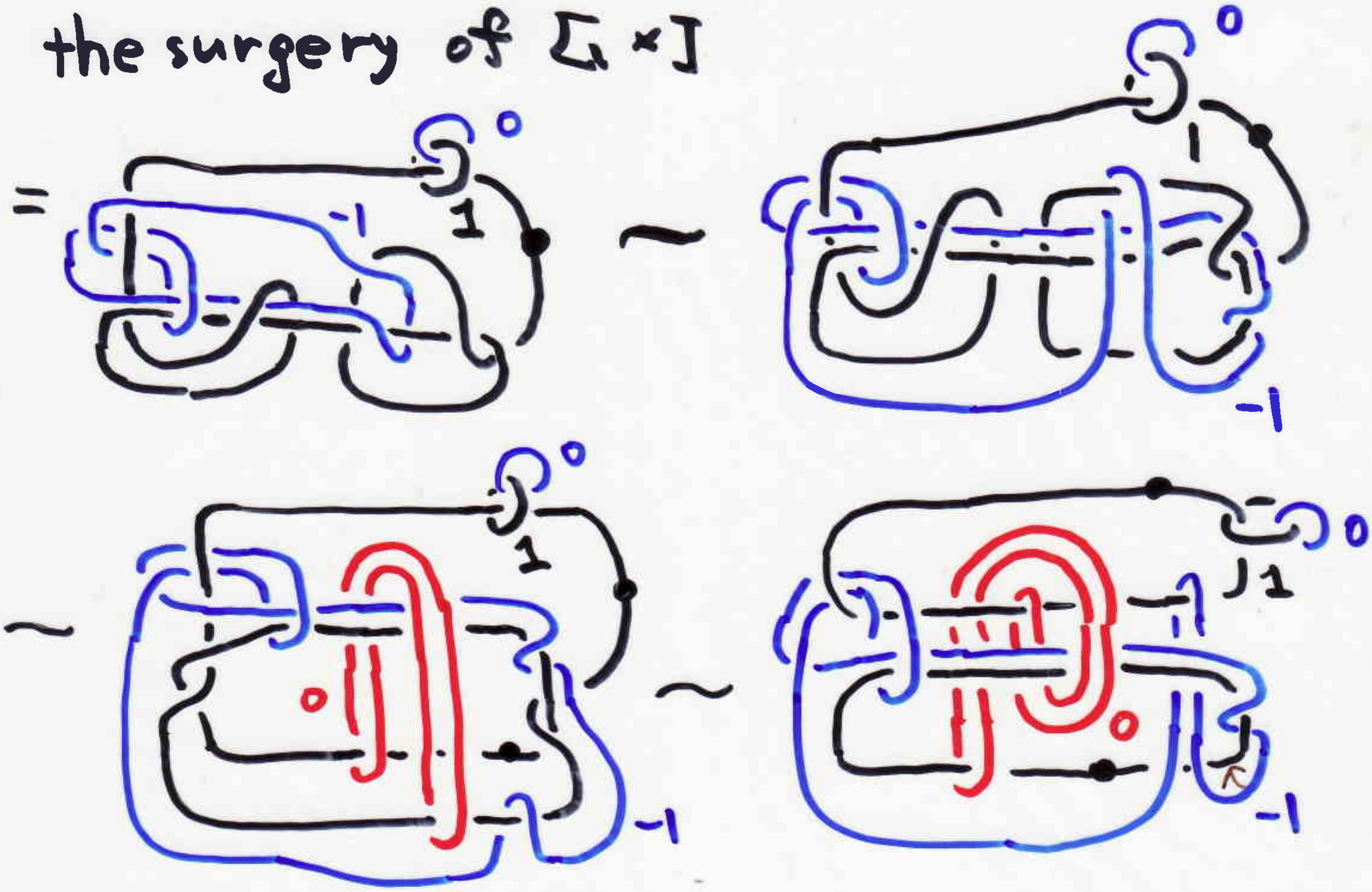
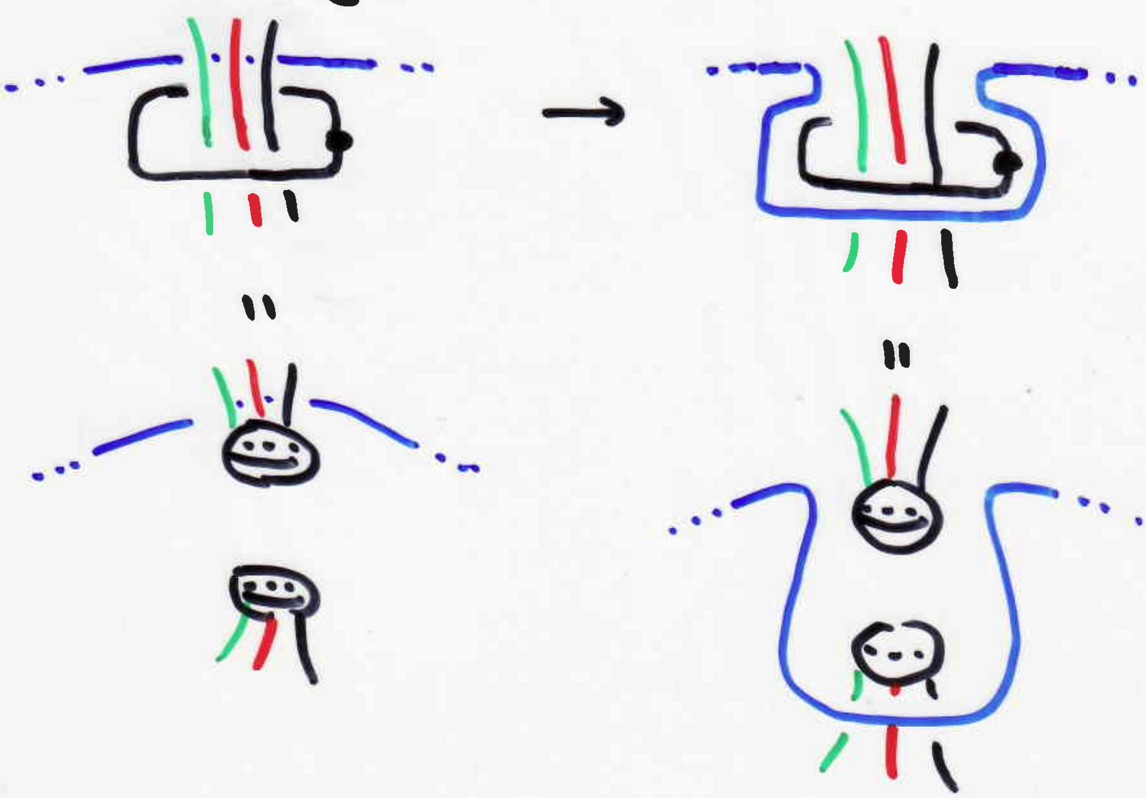
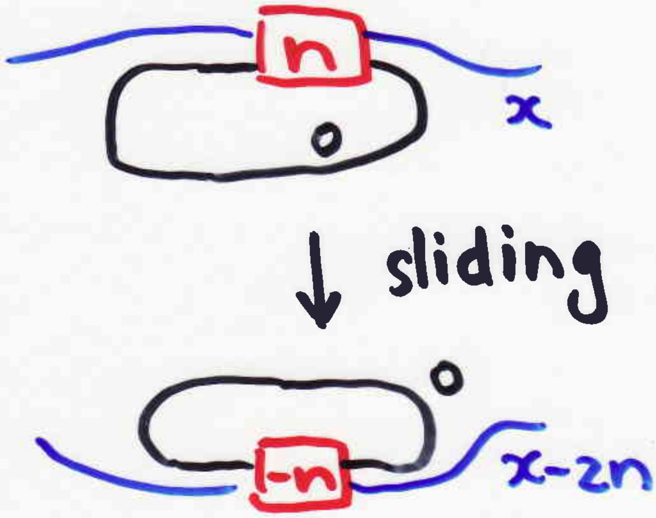
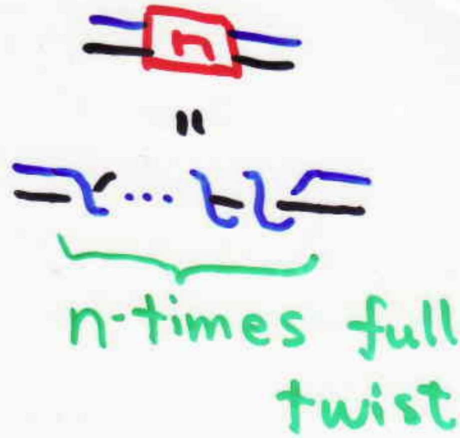


Fig. 5

sliding formulae







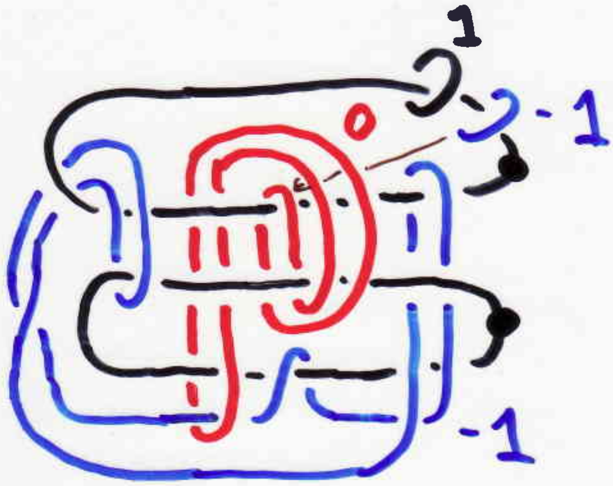


Fig. 6

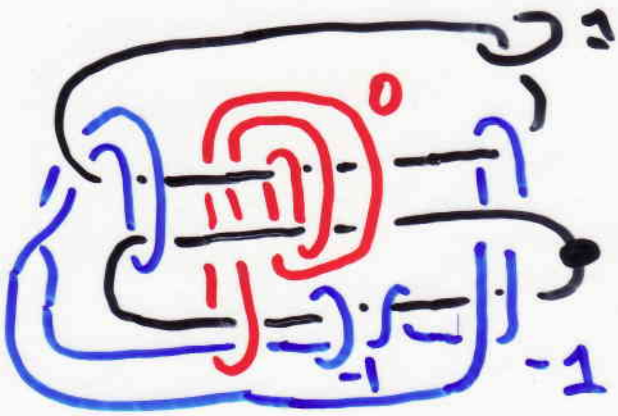
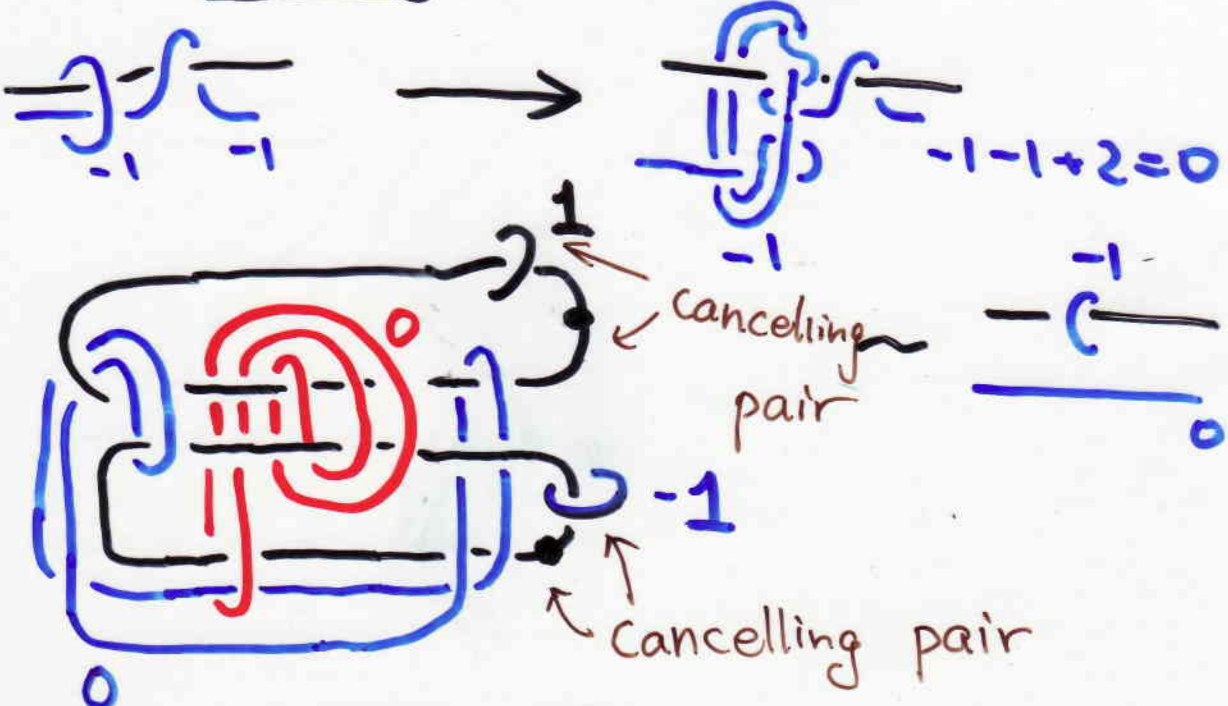


Fig. 7

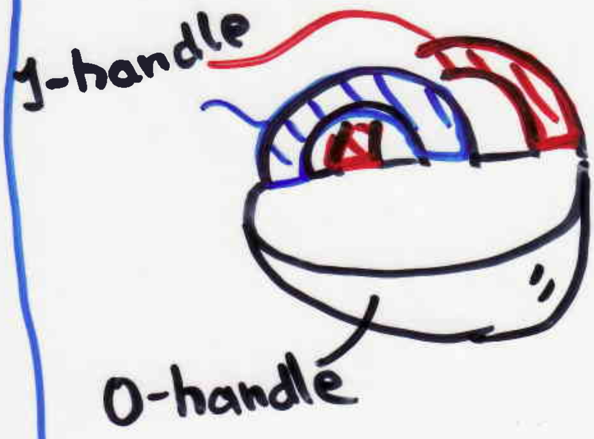
sliding formula II

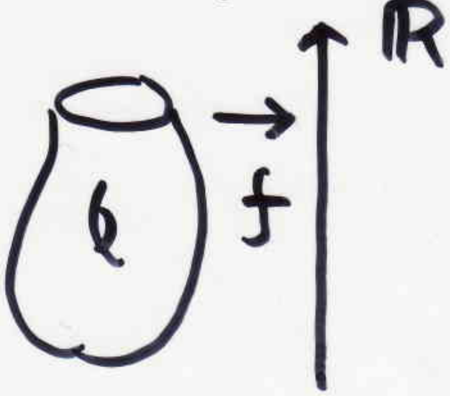


∪ 3-handle

upside-down

Kirby diagram $\longleftrightarrow$ Morse function

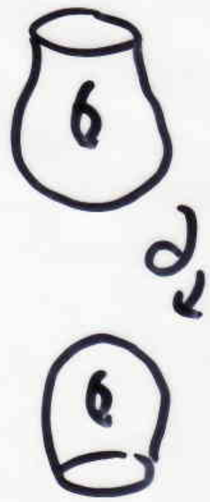


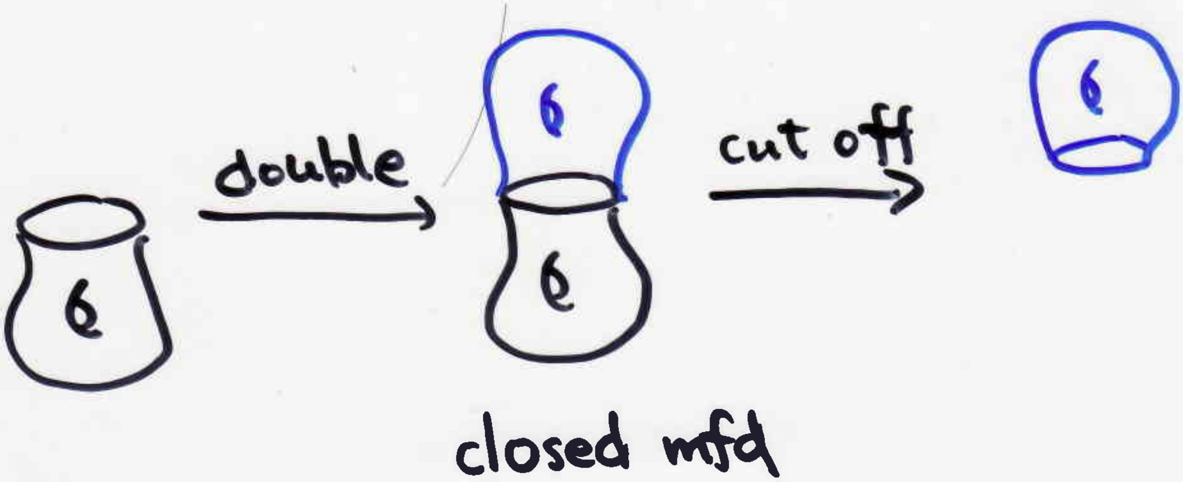


a Kirby diagram $\longrightarrow$ f

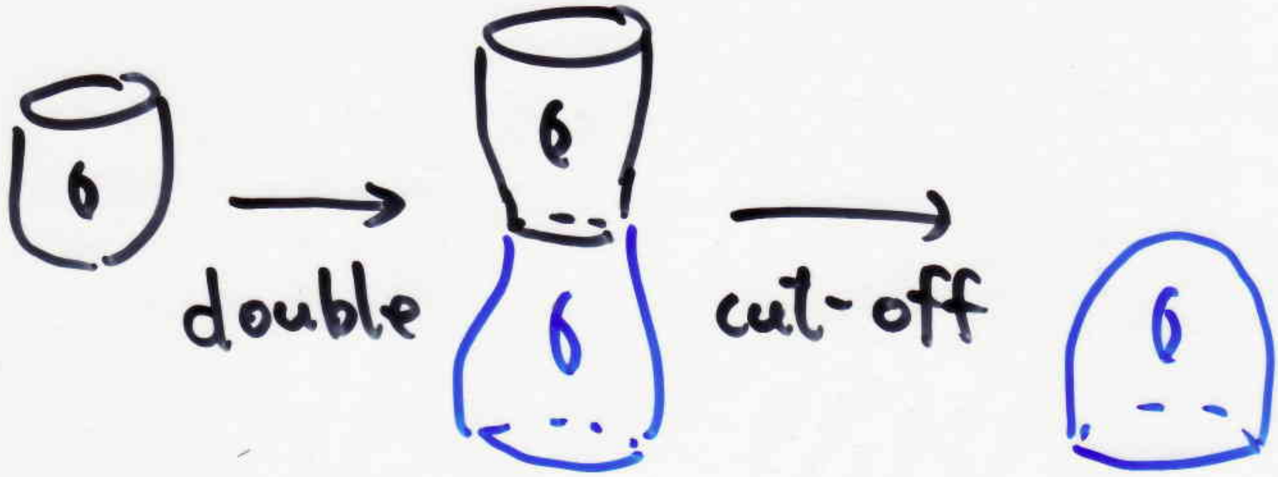
$\downarrow \partial$

the upside-down diagram of the Kirby diagram. $\longleftarrow$ $-f$



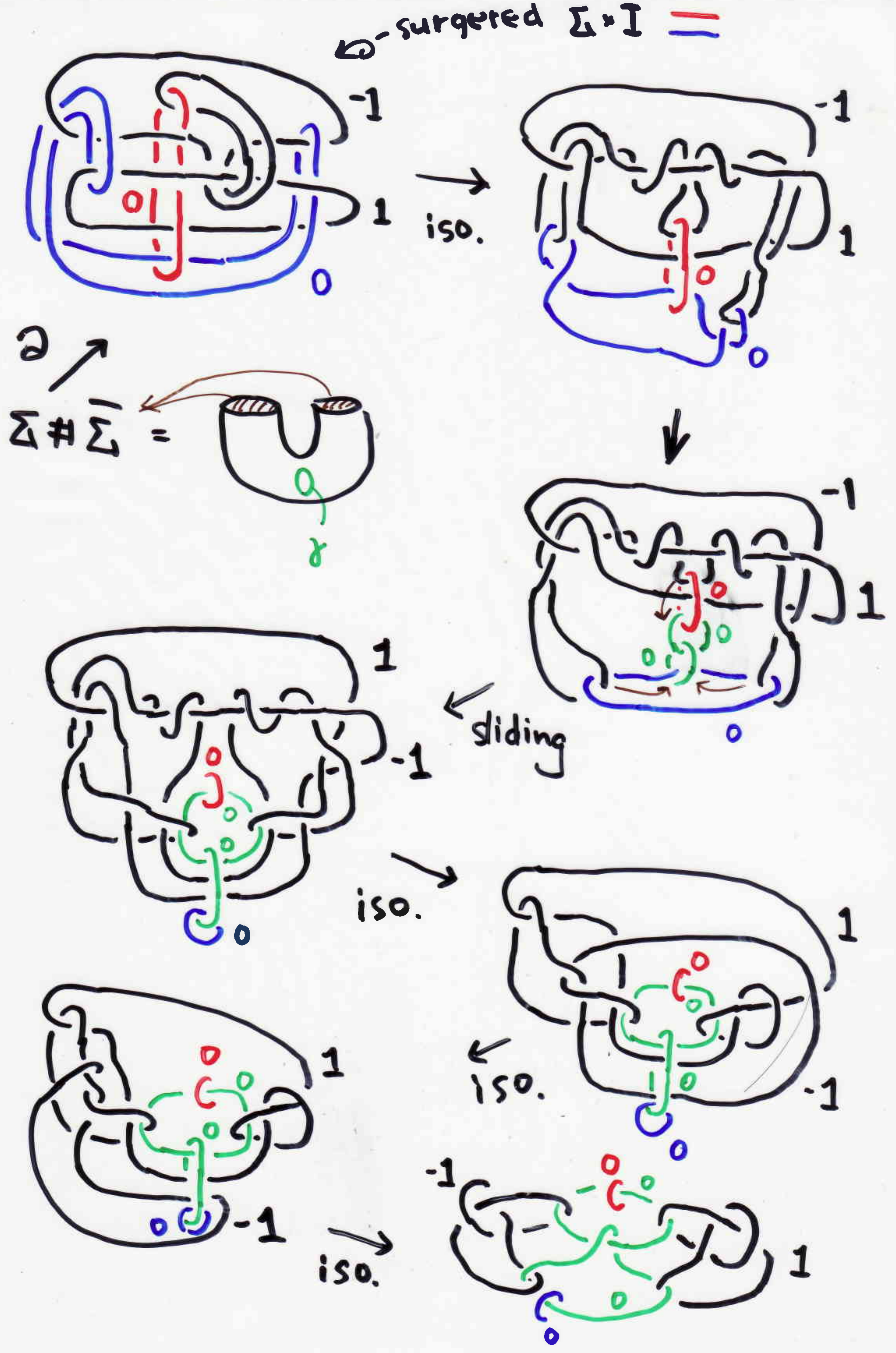


Actually this upside-down is the following pattern:



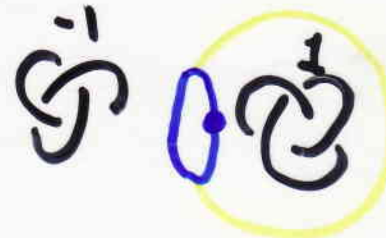
A later upside-down in this talk is the former.

(This sheet is interpolated after the seminar.)



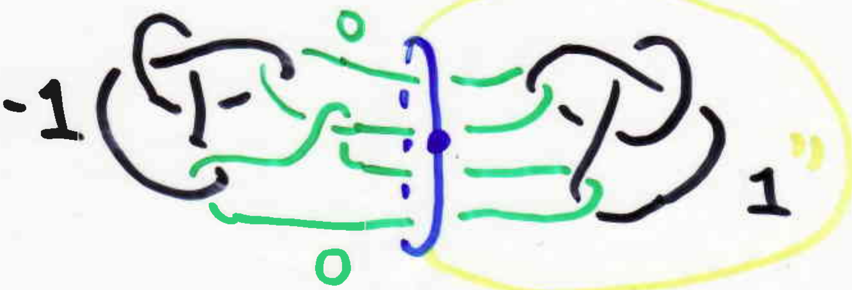
∞-surgered Σ×I
-1
1
0
iso.
∂
Σ # Σ̄ =
γ
sliding

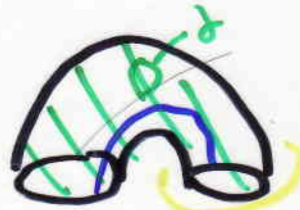
$\Sigma \amalg \bar{\Sigma} \cong_\partial$ 

attach $\longrightarrow$ 

$\Sigma \# \bar{\Sigma} \cong_\partial$

Remove $\left\{ \begin{array}{l} \text{red} \\ \text{blue} \end{array} \text{ \& curve} \right\}$!

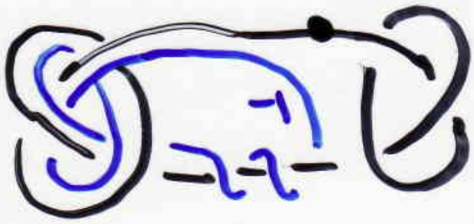


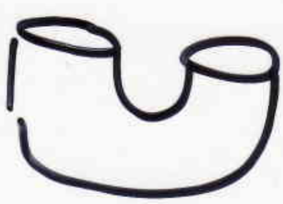
$=$ 

4-handle

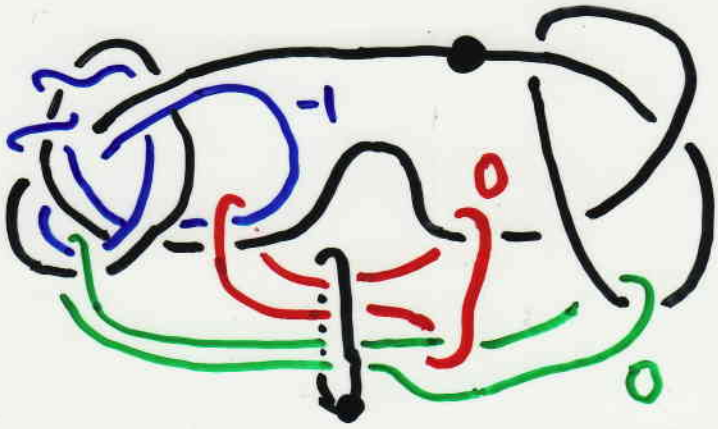
the upside-down of the surgered $\Sigma \times I$

Recall the diagram of $\Sigma \times I$!



$=$ 

$\cup$ 3-handle



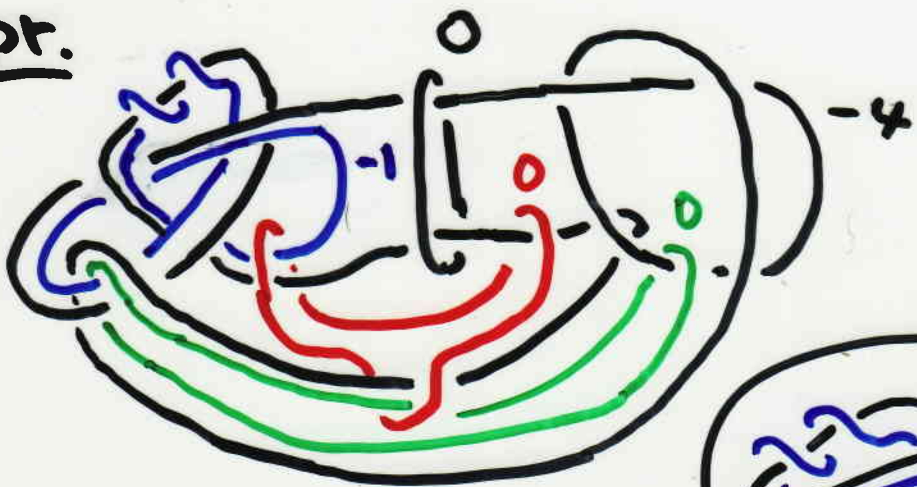
$\cup$ 3-handle

$= M$ (Scharlemann's mfd)

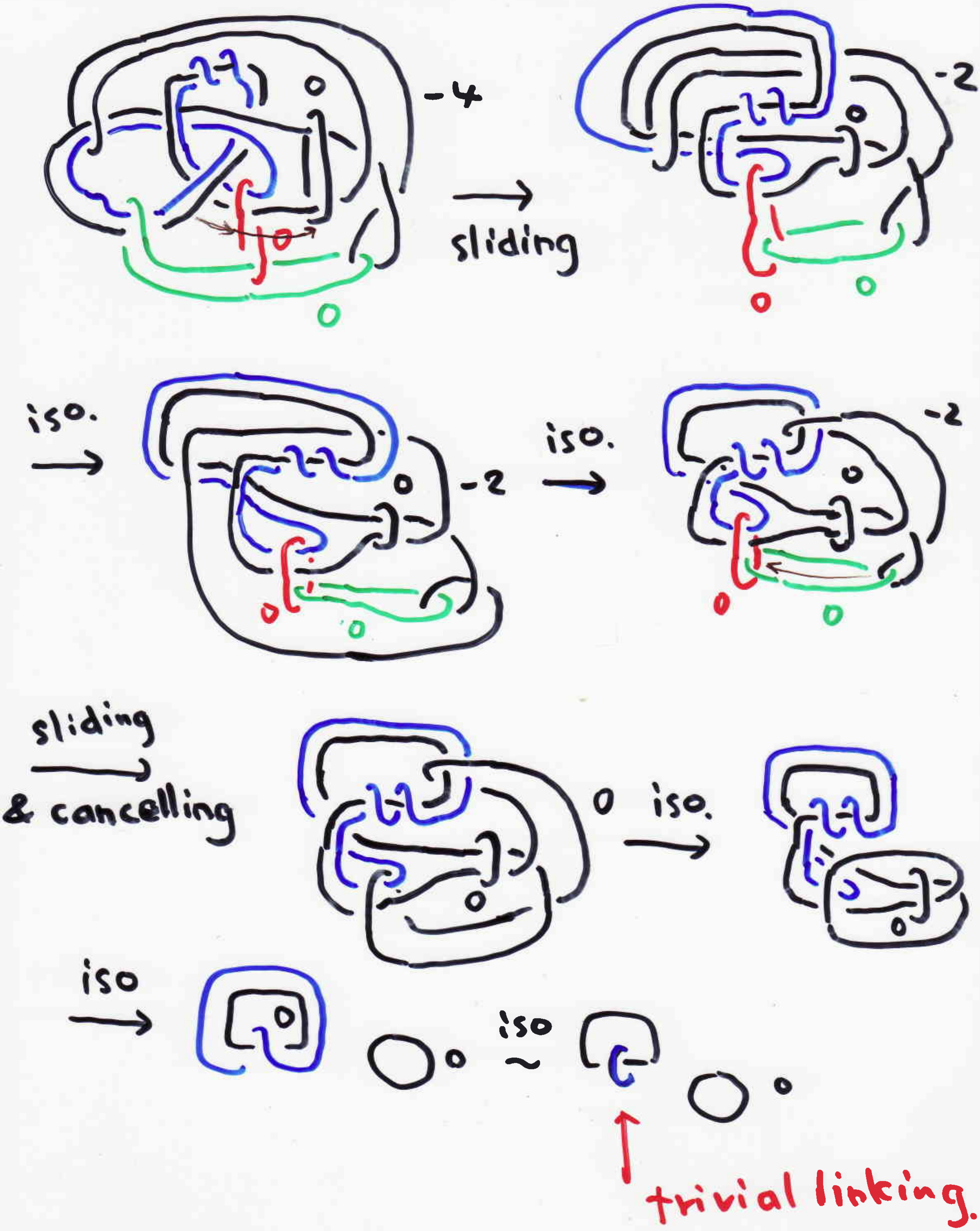
§ 5 Proofof Akbulut's Theorem.

<u>Lemma</u> The blue cirle is isotopic to trivial linking circle of the slice 1-handle.

<u>pr.</u>

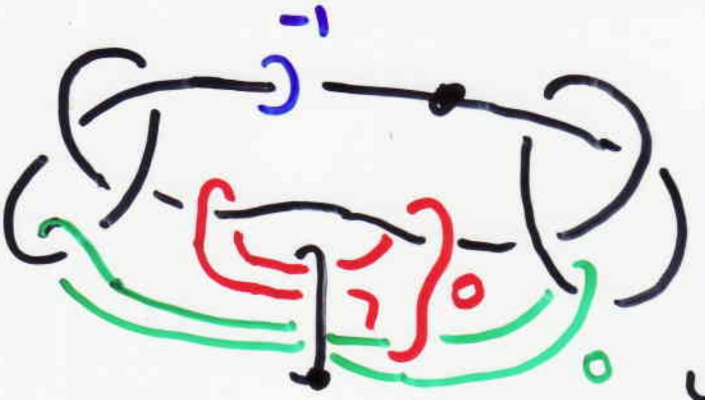


$\sim$



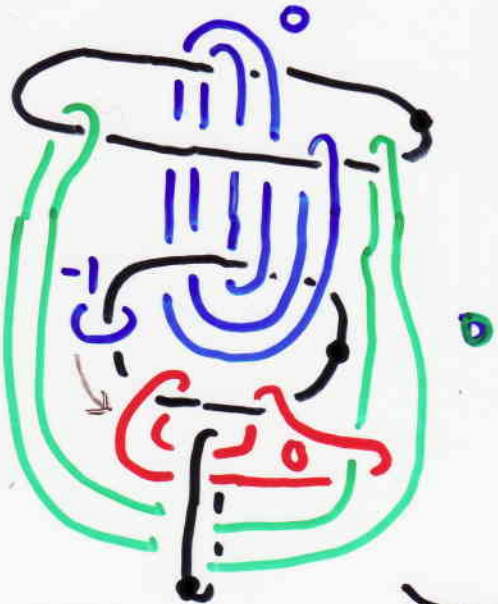
−4
0
0
sliding
−2
0
0
0
iso.
0
−2
0
0
iso.
−2
0
0
0
sliding
& cancelling
0
0
iso.
0
iso
0
0
iso
~
0
trivial linking.

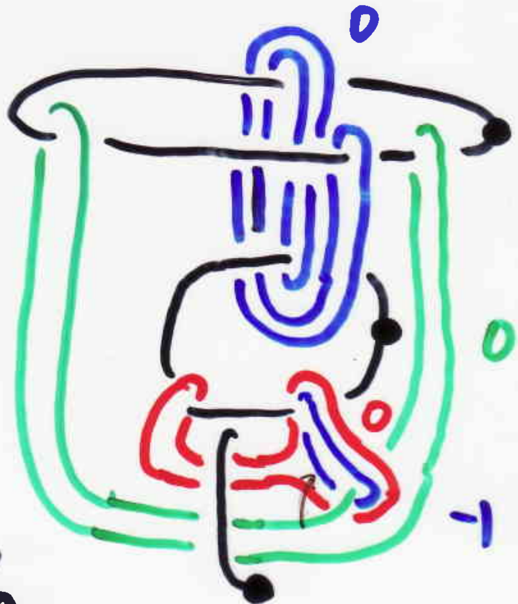
$\therefore$ **M** =



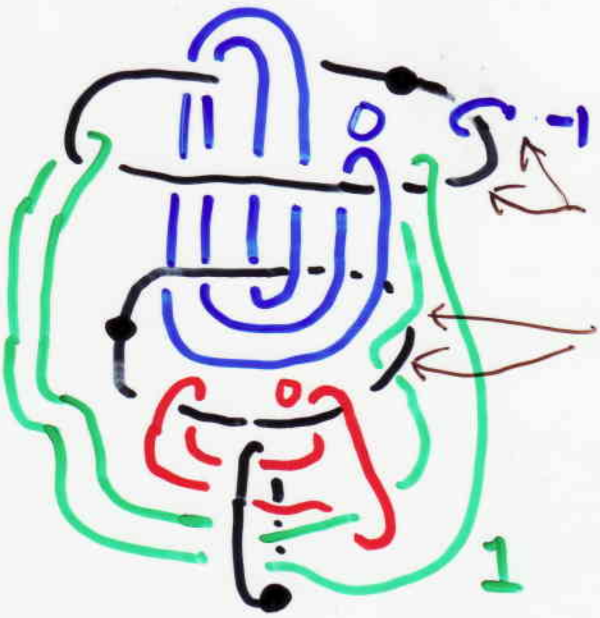
∪ 3-handle

ordinary $\longrightarrow$ 1-handle





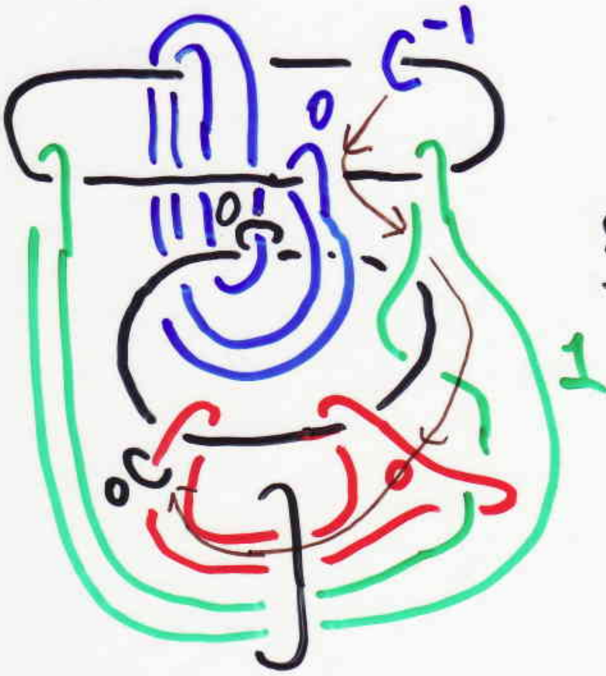
$\longrightarrow$ sliding



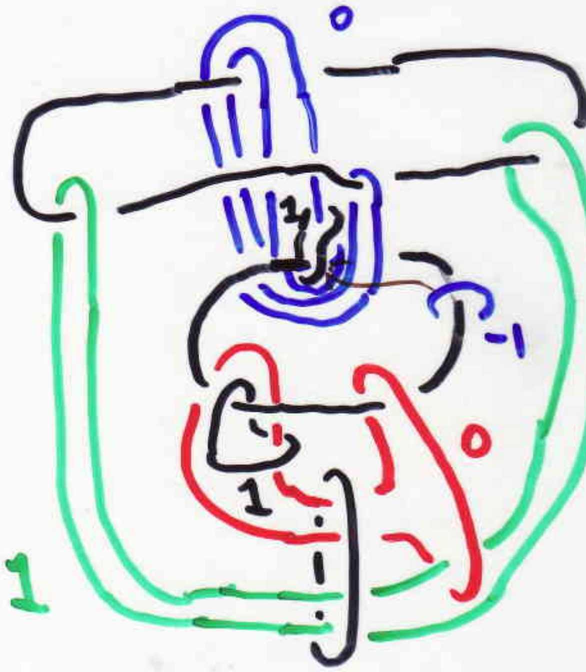
cancelling pair

cancelling pair

upside-down



sliding →



sliding →

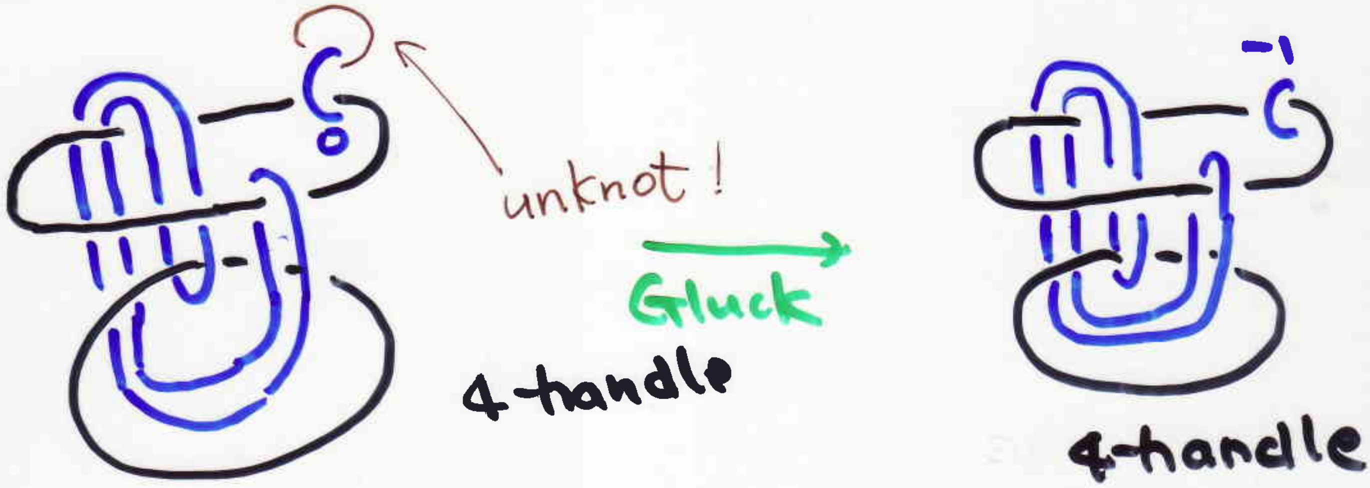
3-handle

4-handle.

Gluck surgery

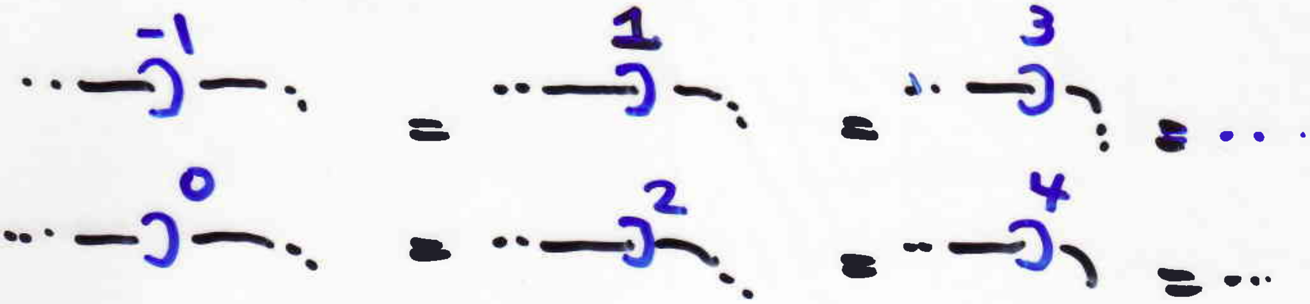
$$\left[X \setminus \nu(S^2)\right] \cup D^2 \times S^2$$

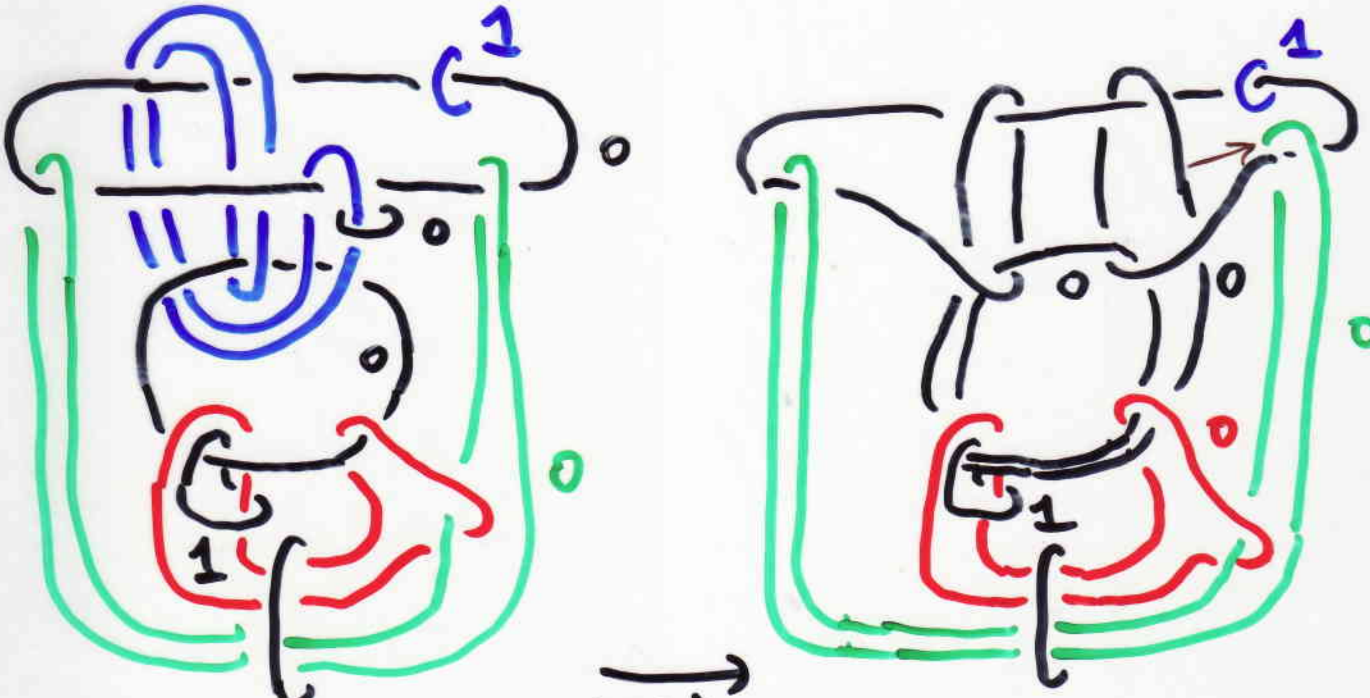
Gluck surgery



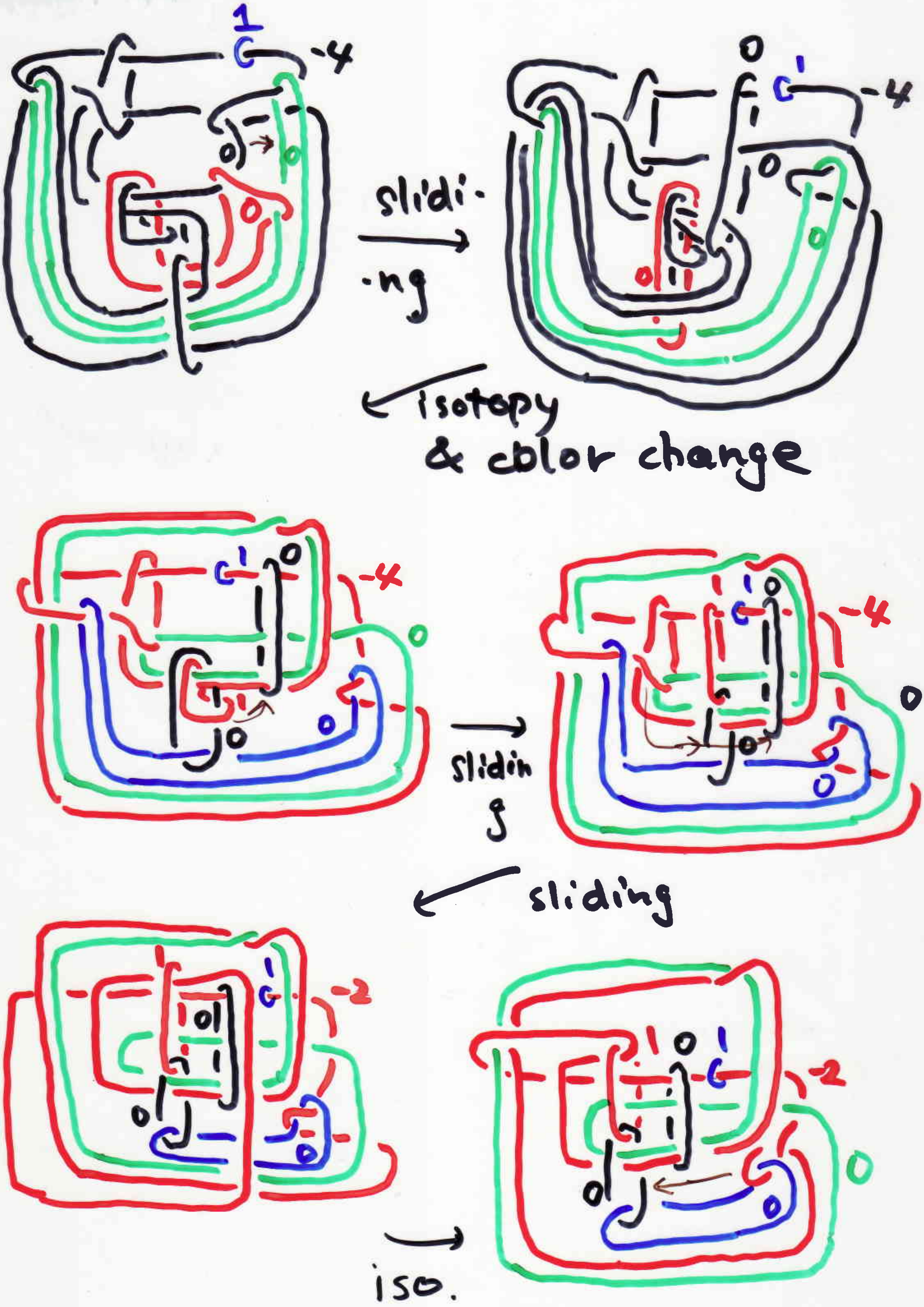
S²- complement   Gluck twist

The homotopy class$_{es}$ of $S^2 \times S^1$ ↻ are $\mathbb{Z}/2\mathbb{Z}$.

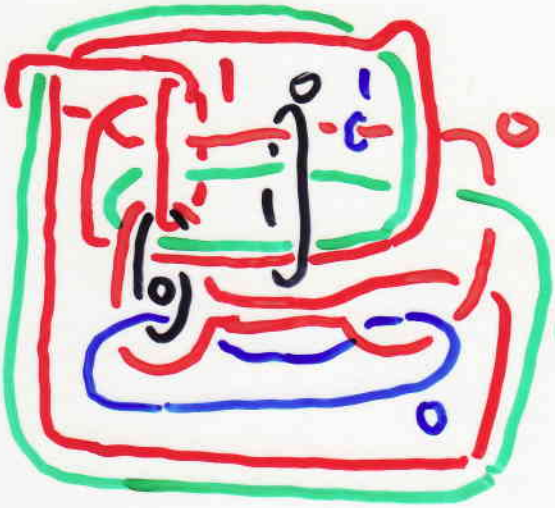




sliding & cancelling



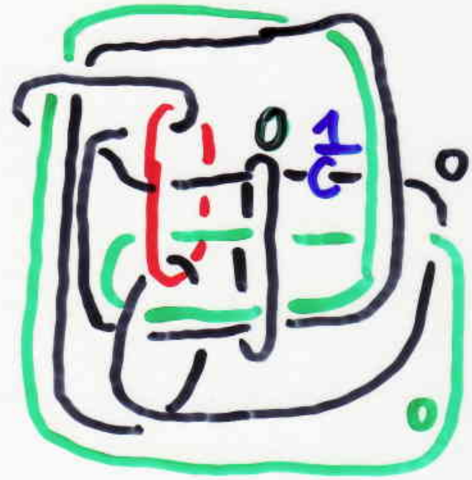
sliding
isotopy & color change
sliding
sliding
iso.



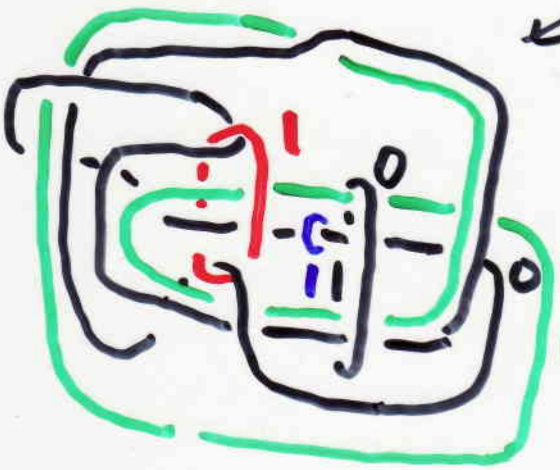
cancelling

&

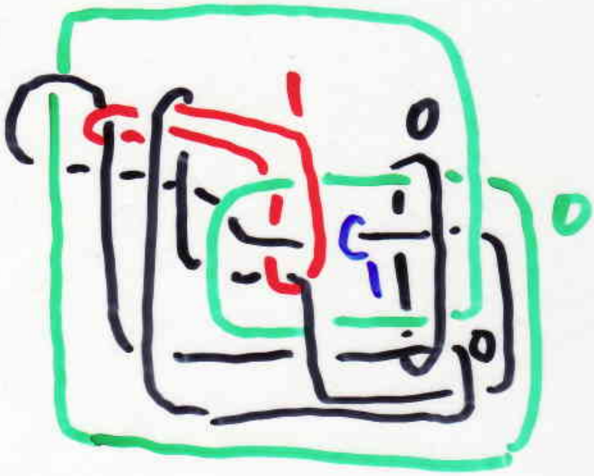
color change



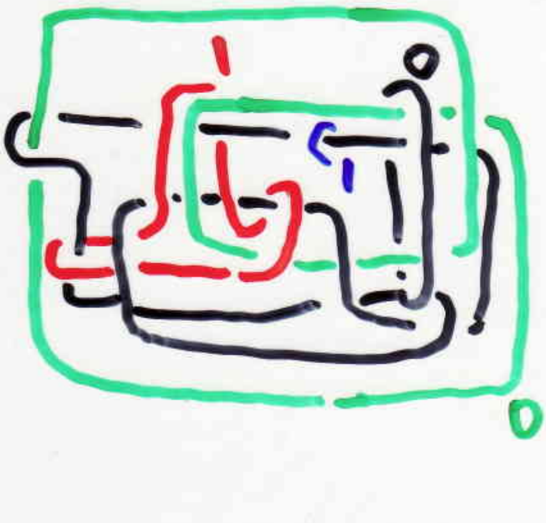
iso.



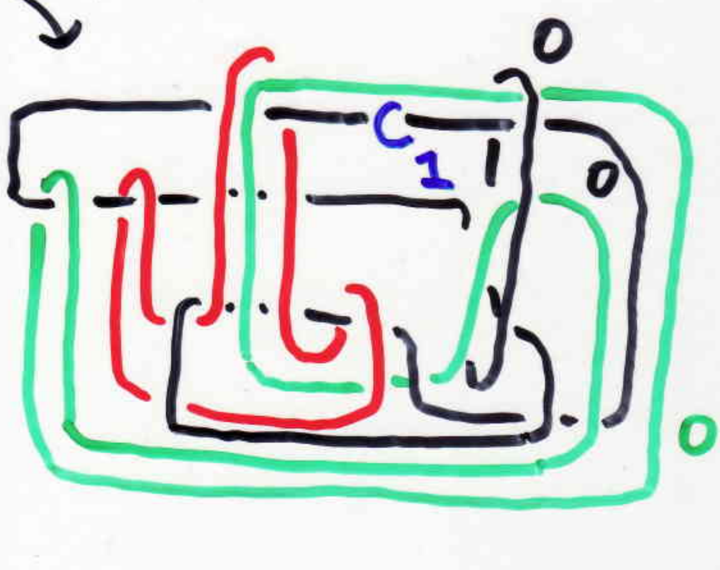
iso.

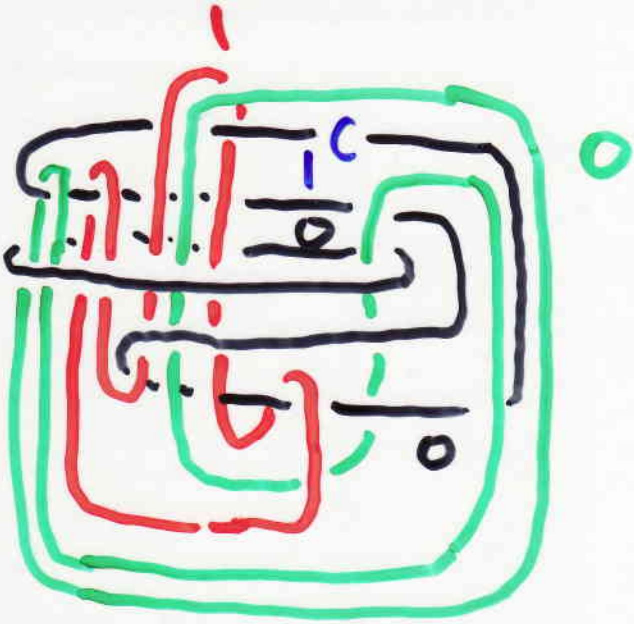


iso.

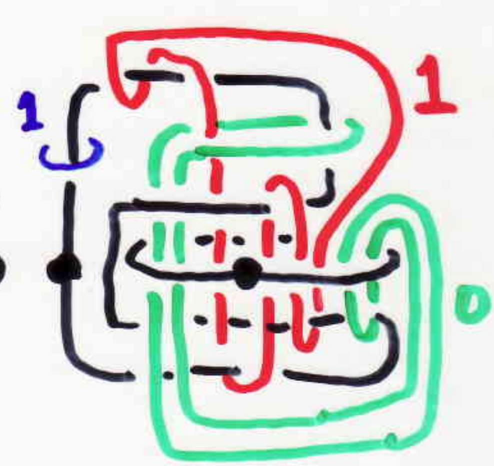


iso.



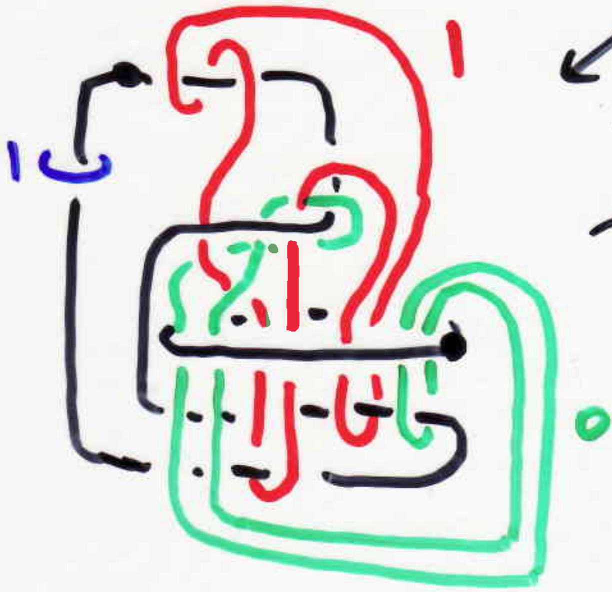


iso.

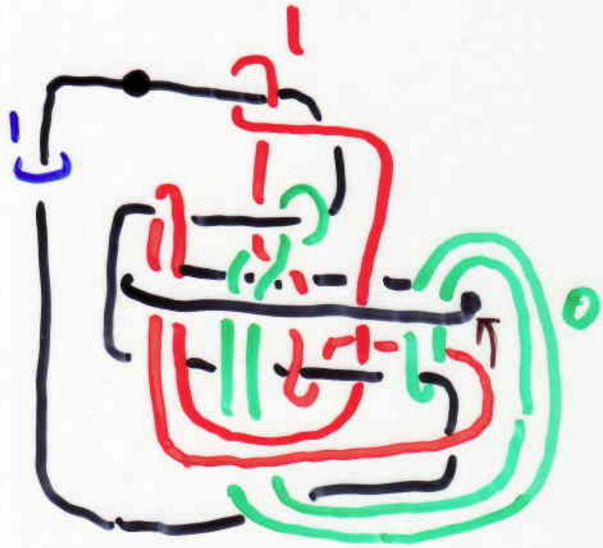


The upside-down finishes.

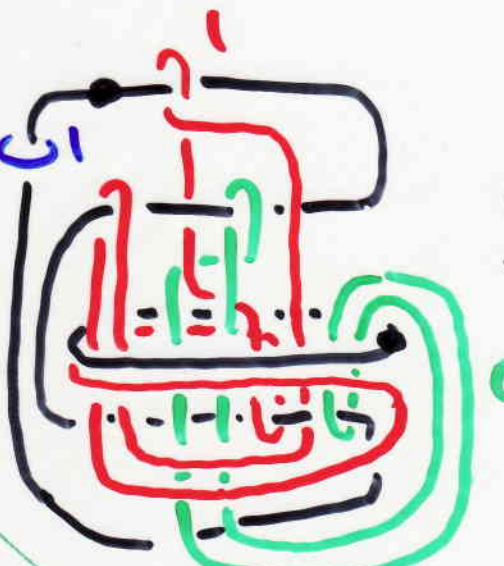
iso.



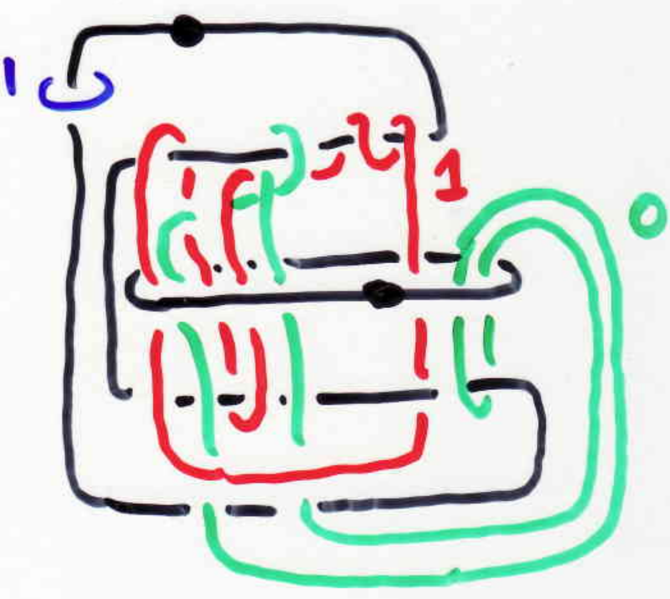
iso.

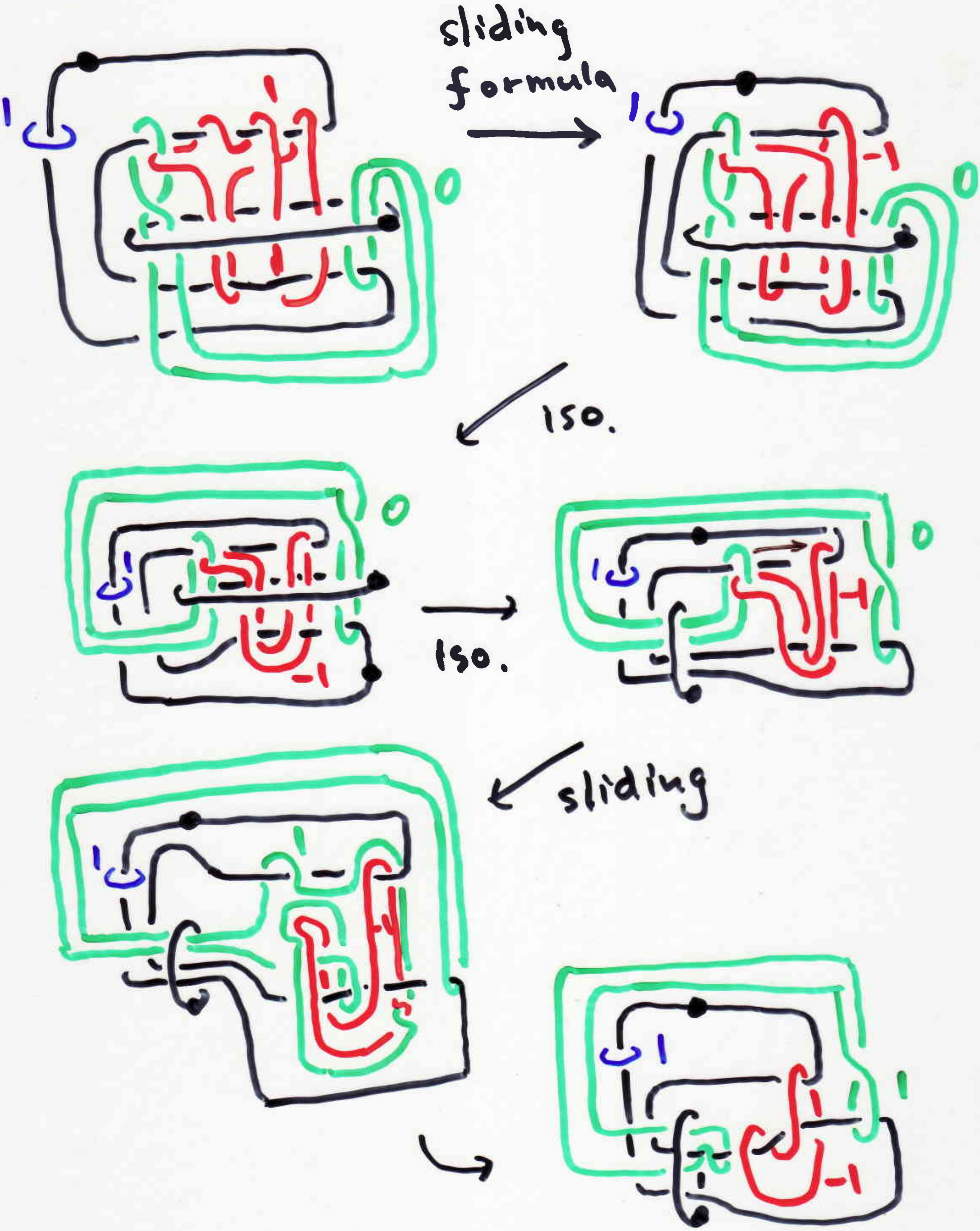


sliding

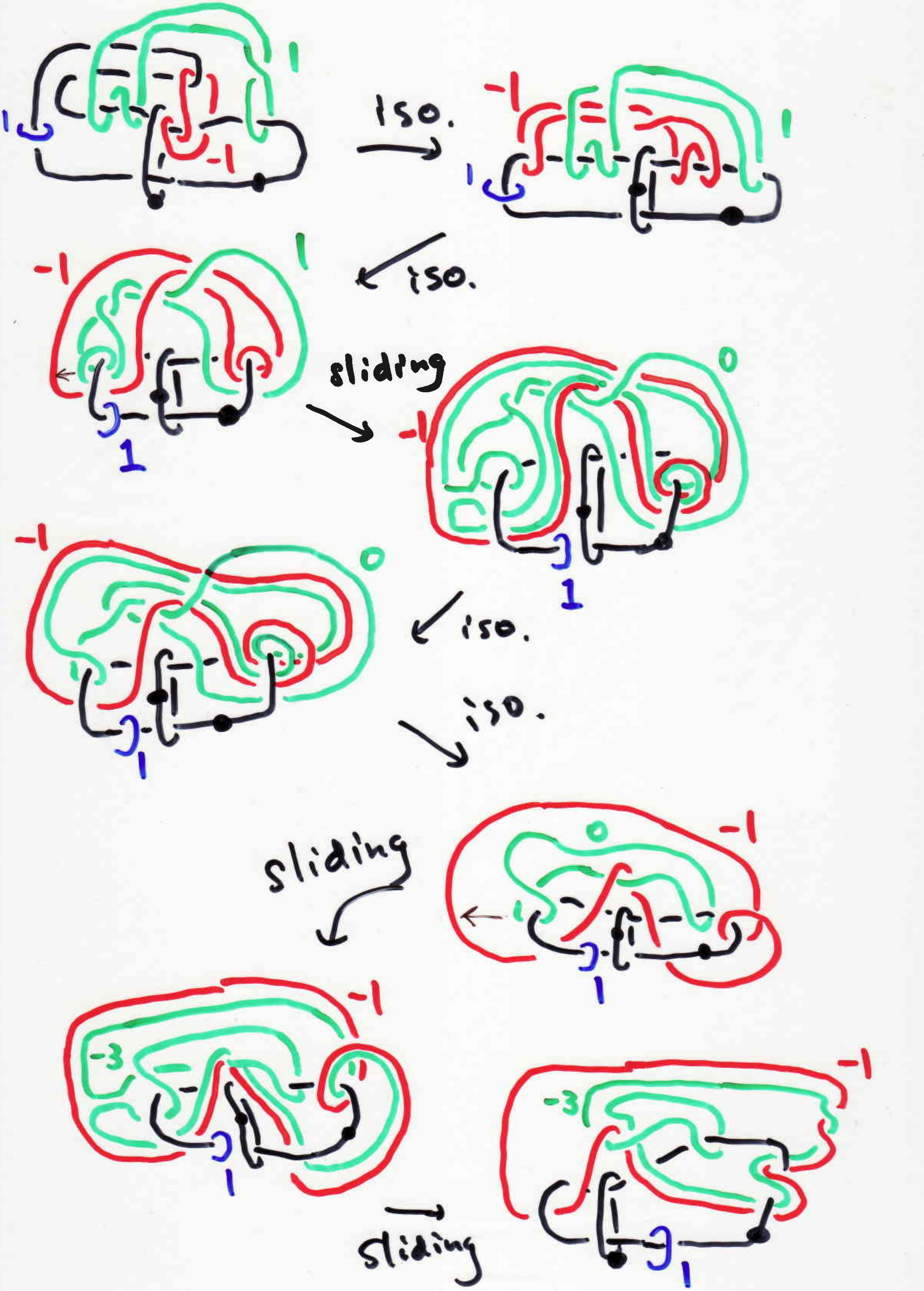


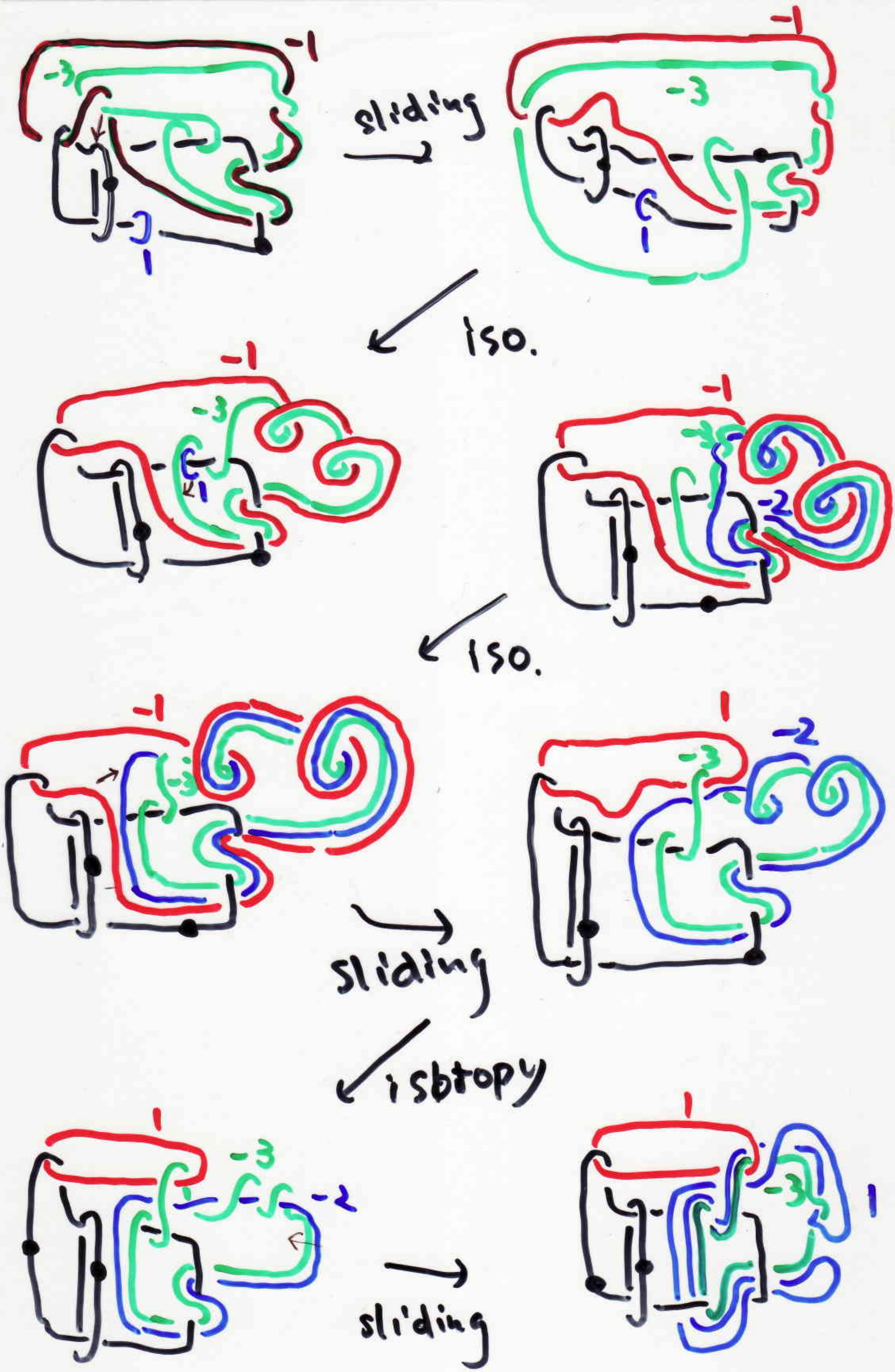
iso.

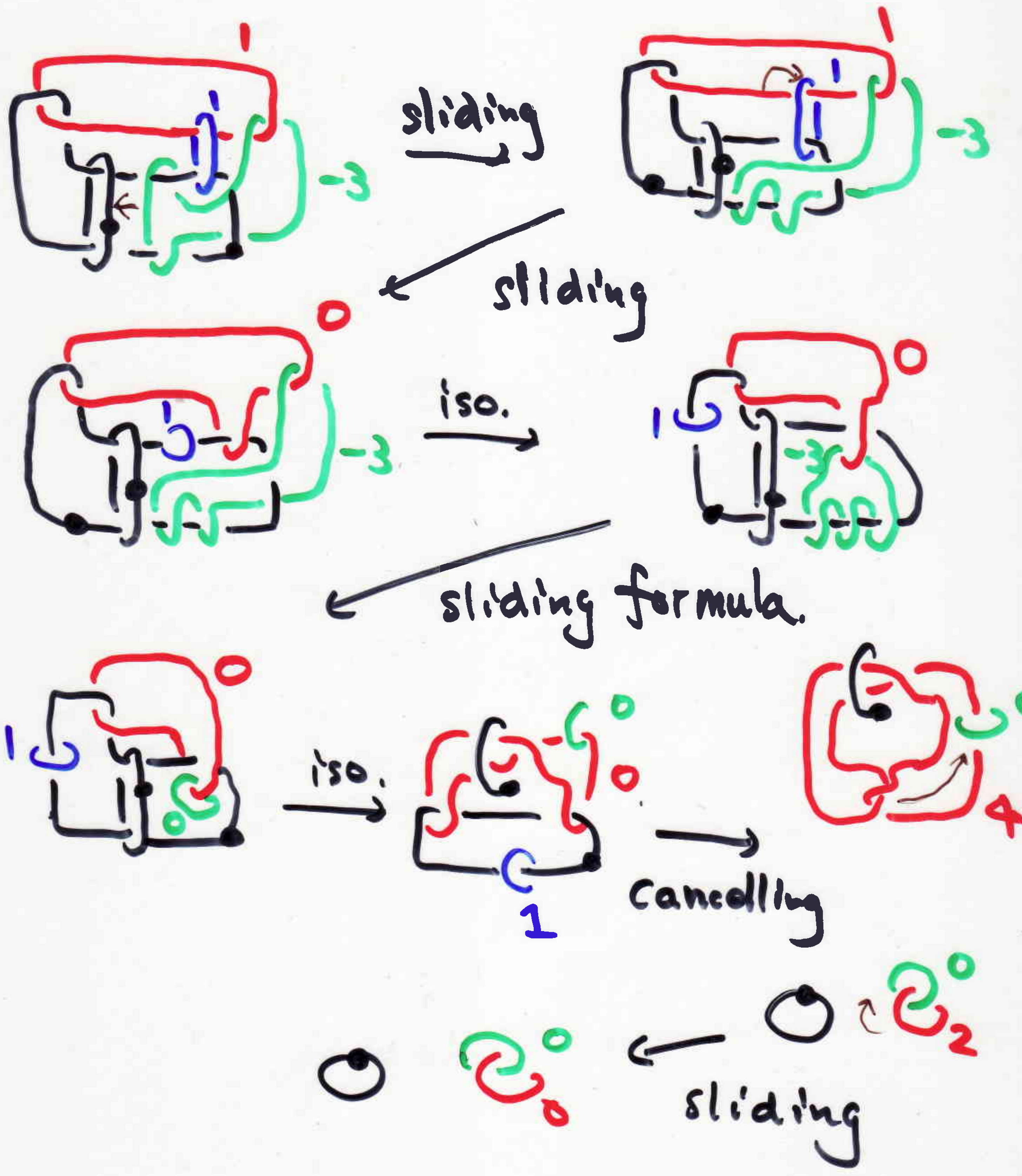




sliding formula
iso.
iso.
sliding


iso.
iso.
sliding
iso.
iso.
sliding
sliding


-1
-3
1
sliding
-1
-3
1
iso.
-1
-3
1
-1
-3
-2
iso.
-1
-3
1
-2
sliding
isotopy
1
-3
-2
1
-3
1
sliding



$$\parallel$$

$$S^3\times S^1 \# S^2\times S^2$$