Fixed point sets of isometries and the intersection of real forms in a Hermitian symmetric space of compact type

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The 17th International Workshop on Differential Geometry
NIMS
September 30 – October 2, 2013
Joint with Hiroyuki Tasaki (University of Tsukuba) and Osamu Ikawa (Kyoto Institute of Technology)

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1. Introduction and known results

In $S^2$ any two distinct great circles intersect two antipodal points. If at least one of circles is a small circle, the intersection could be empty. $S^2$ is considered as $\mathbb{CP}^1$, which is a Hermitian symmetric space of compact type, and canonically embedded $\mathbb{RP}^1$ is a great circle, which is a real form in $\mathbb{CP}^1$, that is, the fixed point set of an involutive anti-holomorphic isometry. In general, we obtained the following.

**Theorem 1.1 (T.-Tasaki 2012)** Let $M$ be a Hermitian symmetric space of compact type and let $L_1$ and $L_2$ be real forms in $M$. If $L_1 \cap L_2$ is discrete, $L_1 \cap L_2$ is an antipodal set.

Here an antipodal set is a subset on which the geodesic symmetry at each point acts trivially.
We say \( L_1 \) is congruent to \( L_2 \) if \( L_1 \) is transformed to \( L_2 \) by an element in \( I_0(M) \), the identity component of the isometry group of \( M \). If \( M = G_k(\mathbb{C}^n) \), any real form is

\[
G_k(\mathbb{R}^n), \ G_l(\mathbb{H}^m) \text{ if } k = 2l, n = 2m, \text{ or } U(k) \text{ if } n = 2k.
\]

So in general, \( L_1 \) and \( L_2 \) are not necessarily congruent. But if they are congruent, we obtained the following.

**Theorem 1.2 (T.-Tasaki 2012)** If \( L_1 \) and \( L_2 \) are real forms in \( M \) which are congruent and \( L_1 \cap L_2 \) is discrete, then \( L_1 \cap L_2 \) is a great antipodal set of \( L_1 \) and \( L_2 \).

Here a great antipodal set is an antipodal set with maximal cardinality. The maximal cardinality of the antipodal sets in a compact Riemannian symmetric space \( M \) is called the 2-number of \( M \) denoted by \( \#_2M \).
For a point $o$ in a compact Riemannian symmetric space, each connected component of the fixed point set of the geodesic symmetry $s_o$ at $o$ is called a polar. By making use of polars, we obtained the following.

**Theorem 1.3 (T.-Tasaki 2012)** Let $M$ be an irreducible Hermitian symmetric space of compact type and let $L_1$ and $L_2$ be real forms in $M$. Assume that $\#_2 L_1 \leq \#_2 L_2$. If $L_1 \cap L_2$ is discrete, then $L_1 \cap L_2$ is a great antipodal set of $L_1$ except for the case where

$$M = G_{2m}(\mathbb{C}^{4m}) \ (m \geq 2), \ L_1 \cong G_m(\mathbb{H}^{2m}), \ L_2 \cong U(2m).$$

In this case we have

$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$ 

On the other hand, we have another approach to investigate the intersection of real forms. Let $L_1 = F(\tau_1, M)$ and $L_2 = F(\tau_2, M)$ be real forms, then
\[ L_1 \cap L_1 = F(\tau_1, M) \cap F(\tau_2, M) \subset F(\tau_2\tau_1^{-1}, M), \]

where \( \tau_2\tau_1^{-1} \) is a holomorphic isometry of \( M \). If \( F(\tau_2\tau_1^{-1}, M) \) is discrete, \( L_1 \cap L_2 \) is discrete. Moreover, if \( F(\tau_2\tau_1^{-1}, M) \) is antipodal, \( L_1 \cap L_2 \) is antipodal. Conversely, if \( L_1 \cap L_2 \) is discrete, is \( F(\tau_2\tau_1^{-1}, M) \) discrete? If \( L_1 \cap L_2 \) is antipodal, is \( F(\tau_2\tau_1^{-1}, M) \) antipodal? We will refer to these problems in Section 3.

It is known that a Hermitian symmetric space \( M \) of compact type is realized as an adjoint orbit of a compact semisimple Lie group \( G \):

\[ M = \text{Ad}(G)J \subset \mathfrak{g}, \]

where \( J \) satisfies \( (\text{ad}J)^3 = -\text{ad}J \). By making use of the realization we obtained a necessary and sufficient condition for the intersection of two real forms is discrete. When the intersection is discrete, it is an orbit of a certain Weyl group, which will be mentioned in Section 4.
2. Basic notions

Let $M$ be a compact Riemannian symmetric space. A subset $S \subset M$ is called an antipodal set if

$$s_x(y) = y \quad \text{for any } x, y \in S,$$

where $s_x$ denotes the geodesic symmetry at $x$.

The 2-number $\#_2 M$ of $M$ is defined by

$$\#_2 M = \max\{\#S \mid S \subset M : \text{antipodal set}\}.$$

An antipodal set $S$ is called great if $\#S = \#_2 M$. These notions were introduced by Chen-Nagano.

In general, great antipodal sets are not necessarily congruent to each other but for symmetric $R$-spaces we have the following.

**Theorem 2.1 (T.-Tasaki 2013)** Let $M$ be a symmetric $R$-space.

1. Any antipodal set is included in a great antipodal set.

2. Any two great antipodal sets are congruent.
Here a symmetric $R$-space is a compact Riemannian symmetric space which can be realized as a linear isotropy orbit of a Riemannian symmetric space of compact type. A Hermitian symmetric space of compact type is a symmetric $R$-space.

**Example** $\mathbb{CP}^n$

$e_1, \ldots, e_{n+1}$: a unitary basis of $\mathbb{C}^{n+1}$

$o = \langle e_1 \rangle_\mathbb{C} \in \mathbb{CP}^n$

$s_o$ is induced from the reflection $\rho_o : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$

$$\rho_o = \begin{cases} 
\text{Id} & \text{on } \langle e_1 \rangle_\mathbb{C} \\
-\text{Id} & \text{on } \langle e_2, \ldots, e_{n+1} \rangle_\mathbb{C} 
\end{cases}$$

$\{\langle e_1 \rangle_\mathbb{C}, \ldots, \langle e_{n+1} \rangle_\mathbb{C} \}$ is a great antipodal set and $\#_2 \mathbb{CP}^n = n + 1$. 

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Let $M$ be a Hermitian symmetric space of compact type. Let $\tau$ be an involutive anti-holomorphic isometry of $M$. Then the fixed point set $F(\tau, M)$ is called a **real form** in $M$. It is known that a real form is connected totally geodesic compact Lagrangian submanifold. A real form in a Hermitian symmetric space of compact type is a symmetric $R$-space, and vice versa (Takeuchi).

The classification of real forms in an irreducible Hermitian symmetric space of compact type was given by Leung and Takeuchi. As for the non-irreducible case we have the following.

**Theorem 2.2 (T.-Tasaki)** A real form in a Hermitian symmetric space $M$ of compact type is a product of real forms in irreducible factors of $M$ and diagonal real forms determined from irreducible factors of $M$.

Here a diagonal real form is defined as follows.
Let $\tau$ be an anti-holomorphic isometry of $M$. A map

$$M \times M \ni (x, y) \mapsto (\tau^{-1}(y), \tau(x)) \in M \times M$$

is an involutive anti-holomorphic isometry of $M \times M$. The real form determined by the map is

$$D_{\tau}(M) := \{(x, \tau(x)) \mid x \in M\},$$

which is called a **diagonal real form** determined from $M$.

The existence of the intersection of two real forms follows the next proposition.

**Proposition (Cheng 2002)** Let $M$ be a compact Kähler manifold with positive holomorphic sectional curvature. If $L_1$ and $L_2$ are totally geodesic compact Lagrangian submanifolds in $M$, then $L_1 \cap L_2 \neq \emptyset$. 
3. Fixed point sets of isometries of a Hermitian symmetric space of compact type

It is known that a Hermitian symmetric space $M$ of compact type is realized as an adjoint orbit

$$M = \text{Ad}(G)J \subset \mathfrak{g},$$

where $G$ is a connected compact semisimple Lie group, $\mathfrak{g}$ is the Lie algebra of $G$ and $J \in \mathfrak{g} - \{0\}$ satisfies $(\text{ad}J)^3 = -\text{ad}J$. Let $K$ be the isotropy subgroup at $J$. Then the Lie algebra $\mathfrak{k}$ of $K$ is

$$\mathfrak{k} = \{X \in \mathfrak{g} | [J, X] = 0\} = \text{Ker} \text{ ad}J.$$

Let

$$\mathfrak{m} = \{[J, X] | X \in \mathfrak{g}\} = \text{Im} \text{ ad}J,$$

then we have an orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$ 

$\text{ad}J$ is a complex structure of $\mathfrak{m}$ which can be identified with the tangent space of $M$ at $J$. 
The action of $G$ on $M$ coincides with the action of $I_0(M)$ on $M$, where $I_0(M)$ denotes the identity component of the isometry group $I(M)$ of $M$.

Let $A(M)$ denote the group of the holomorphic isometries and $A_0(M)$ denote the identity component of $A(M)$. Then it is known that $I_0(M) = A_0(M)$. Moreover, if $M$ is irreducible,

$$I(M)/A(M) \cong \mathbb{Z}_2$$

and

$$A(M) = A_0(M)$$

except for the cases

$$M = Q_{2m}(\mathbb{C}) \ (m \geq 2), \ G_m(\mathbb{C}^{2m}) \ (m \geq 2)$$

where

$$A(M)/A_0(M) \cong \mathbb{Z}_2$$

(Murakami, Takeuchi).
Theorem 3.1 (Sánchez 1997, T.-Tasaki 2013)
Let $M = \text{Ad}(G)J$ be a Hermitian symmetric space of compact type. Then, a great antipodal set of $M$ is represented as

$$M \cap \mathfrak{t}$$

for a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$.

If $g \in G$ satisfies

$$\dim\{X \in \mathfrak{g} \mid \text{Ad}(g)X = X\} = \text{rank}(G),$$

$g$ is called a regular element.

Theorem 3.2 (T.-Tasaki) Let $M$ be a Hermitian symmetric space of compact type and let $g \in A_0(M)$.

(1) The fixed point set $F(g, M)$ is discrete if and only if $g$ is a regular element.
(2) If $F(g, M)$ is discrete, $F(g, M)$ is a great antipodal set of $M$.

If we take a maximal abelian subalgebra $t$ of $t$ with $J \in t$, then $t$ is also a maximal abelian subalgebra of $g$.

By using root systems we have the following lemma.

**Lemma 3.3** $g \in \exp t$ is a regular element if and only if 

$$F(\text{Ad}(g), g) = t.$$ 

Hence if $g \in \exp t$,

$$F(g, M) = F(\text{Ad}(g), g) \cap M = t \cap M,$$

which is a great antipodal set by Theorem 3.1.
Next, we consider the case where $g \in A(M) - A_0(M)$. The complex hyperquadric $M = Q_{2m}(\mathbb{C})$ ($m \geq 2$) can be considered as the oriented Grassmann manifold $\tilde{G}_2(\mathbb{R}^{2m+2})$. Then

$$A(M) - A_0(M) = \{g \in O(2m + 2) \mid \det g = -1\}.$$ 

If $g \in A(M) - A_0(M)$,

$$g \sim \begin{bmatrix} R(\theta_1) \\ \vdots \\ R(\theta_m) \end{bmatrix},$$

where $R(\theta_i) = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{bmatrix}$ ($1 \leq i \leq m$).
Theorem 3.4 (T.-Tasaki) Let $M = \tilde{G}_2(\mathbb{R}^{2m+2})$ ($m \geq 2$) and let $g \in A(M) - A_0(M)$.

(1) $F(g, \tilde{G}_2(\mathbb{R}^{2m+2}))$ is discrete if and only if $R(\theta_i) \neq R(\theta_j)$ for any $i$ and $j$ with $i \neq j$.

(2) When $F(g, \tilde{G}_2(\mathbb{R}^{2m+2}))$ is discrete, $F(g, \tilde{G}_2(\mathbb{R}^{2m+2}))$ is an antipodal set with

$$\#F(g, \tilde{G}_2(\mathbb{R}^{2m+2})) = 2m < 2m + 2 = \#2\tilde{G}_2(\mathbb{R}^{2m+2})$$

Since we do not know $A(M) - A_0(M)$ explicitly when

$$M = G_m(\mathbb{C}^{2m}) \ (m \geq 2),$$

the case of $M = G_m(\mathbb{C}^{2m})$ is unsolved.

When $M = G_k(\mathbb{C}^n)$, the complex Grassmann manifold, we obtain a refinement of Theorem 3.2.
Theorem 3.5 (T.-Tasaki)  Let $M = G_k(\mathbb{C}^n)$ and let $g \in U(n)$.

(1) $F(g, M)$ is discrete if and only if the multiplicity of each eigenvalue of $g$ is 1.

(2) $F(\tau, M) \cap F(g\tau g^{-1}, M)$ is discrete if and only if $F(g\tau g^{-1}\tau^{-1}, M)$ is discrete.

(3) When $F(\tau, M) \cap F(g\tau g^{-1}, M)$ is discrete, we have

$$F(\tau, M) \cap F(g\tau g^{-1}, M) = F(g\tau g^{-1}\tau^{-1}, M)$$

and it is a great antipodal set of $M$. 
4. The intersection of two real forms in a Hermitian symmetric space of compact type

Let $M = \text{Ad}(G)J \subset \mathfrak{g}$ be the canonical embedding of a Hermitian symmetric space $M$ of compact type. Let $L = F(\tau, M)$ be a real form in $M$ which contains $J$, where $\tau$ is an involutive anti-holomorphic isometry of $M$.

$$I_\tau : G \to G'; \ g \mapsto \tau g \tau^{-1}$$

is an involutive automorphism of $G$. Then $(G, F(I_\tau, G))$ is a compact symmetric pair.

The differential $dI_\tau : \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism of $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$$

be the direct sum decompositition where $\mathfrak{l}$ is $(+1)$-eigenspace of $dI_\tau$ and $\mathfrak{p}$ is $(-1)$-engenspace of $dI_\tau$. 
Let $K$ be the isotoropy subgroup at $J$. Then the Lie algebra $\mathfrak{k}$ of $K$ is

$$\mathfrak{k} = \text{Ker } \text{ad}J$$

and set

$$m = \text{Im } \text{ad}J,$$

then

$$g = \mathfrak{k} \oplus m$$

is the canonical decomposition corresponding to $M = G/K$. Then $J \in \mathfrak{k} \cap p$. We choose a maximal abelian subspace $a \subset p$ so that $J \in a$. Let $R$ denote the restricted root system of $(G, F(I_\tau, G))$ with respect to $a$.

Now we investigate $L \cap gL$ for $g \in G$. 

Since we have a decomposition
\[ G = F(I_\tau, G)(\exp a)F(I_\tau, G), \]
there exit \( b_1, b_2 \in F(I_\tau, G) \) and \( a \in \exp \alpha \) such that \( g = b_1ab_2 \). Since \( L = F(I_\tau, G)J \),
\[ L \cap gL = L \cap b_1ab_2L = L \cap b_1aL = b_1(L \cap aL). \]
Hence, it is enough to consider the case where \( g = a = \exp H \) for \( H \in \alpha \) in order to investigate \( L \cap gL \).

\( H \in \alpha \) is called a **regular** element if \( \exp H \) is a regular element in \( G \).

**Theorem 4.1 (Ikawa-T.-Tasaki)**

(1) \( L \cap aL \) for \( a = \exp H \) is discrete if and only if \( H \) is a regular element.

(2) If \( L \cap aL \) is discrete,
\[ L \cap aL = M \cap a = W(R)J \]
and it is a great antipodal set of \( L \). Here \( W(R) \) denotes the Weyl group of \( R \).
Next, we consider the case where two real forms $L_1$ and $L_2$ are not congruent. Hereafter we assume that $M$ is irreducible. Let

$$L_i = F(\tau_i, M) \ (i = 1, 2).$$

As mentioned before, each $\tau_i$ defines an involutive automorphism $I_{\tau_i}$ of $G$ and we obtain a compact symmetric pair $(G, F(I_{\tau_i}, G))$ and a direct sum decomposition

$$g = l_i \oplus p_i \ (i = 1, 2).$$

By the classification of real forms, it is possible to assume that

$$\tau_1 \tau_2 = \tau_2 \tau_1.$$

Then we have a direct sum decomposition

$$g = (l_1 \cap l_2) \oplus (p_1 \cap p_2) \oplus (l_1 \cap p_2) \oplus (l_2 \cap p_1).$$
We take a maximal abelian subspace $a$ in $p_1 \cap p_2$. Under this situation we obtain a “symmetric triad” $(\tilde{\Sigma}, \Sigma, W)$, which is introduced by Ikawa. $\Sigma$ is the restricted root system of $(l_1 \cap l_2) \oplus (p_1 \cap p_2)$ with respect to $a$. $W$ is a certain subset in $a$ invariant under $-\text{Id}$. $\tilde{\Sigma} = \Sigma \cup W$ which is an irreducible root system of $a$.

**Theorem 4.2 (Ikawa-T.-Tasaki)**

(1) $L_1 \cap aL_2$ for $a = \exp H$ is discrete if and only if $H$ is a regular element.

(2) If $L_1 \cap aL_2$ is discrete,

$$L_1 \cap aL_2 = M \cap a = W(\tilde{\Sigma})J = W(R_1)J \cap a = W(R_2)J \cap a.$$  

By the result, we obtain Theorem 1.3 again.
Moreover, by using the classification of irreducible root systems, we can show that an orbit of the Weyl group through $J$ is two-point homogeneous. Consequently, a great antipodal set of an irreducible Hermitian symmetric space of compact type and the intersection of two real forms in an irreducible Hermitian symmetric space of compact type are two-point homogeneous.
Example \( M = \mathbb{C}P^1, \; L = \mathbb{R}P^1 \)

\[
g = su(2) = \left\{ \begin{bmatrix} ix & y + iz \\ -y + iz & -ix \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}
\]

\[
\cong \{(x, y, z) \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3
\]

\[
J = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; (\text{ad}J)^3 = -\text{ad}J
\]

\[
M = \text{Ad}(SU(2))J = SU(2)/S(U(1) \times U(1)) = \mathbb{C}P^1 \cong S^2 \subset \mathbb{R}^3
\]

\[
\tau : su(2) \to su(2); \; X \mapsto -\bar{X}
\]

\[
\tau(J) = J, \; \tau(M) = M
\]

\[
F(\tau, M) = F(\tau, su(2)) \cap M \cong \{(x, 0, z) \mid x, z \in \mathbb{R}\} \cap S^2 \subset \mathbb{R}^3
\]

\[
= \{(\cos \theta, 0, \sin \theta) \mid \theta \in \mathbb{R}\} = S^1
\]
$I_{\tau} : SU(2) \rightarrow SU(2); \quad g \mapsto \tau g \tau^{-1} = \bar{g}$

$F(I_{\tau}, SU(2)) = SO(2)$

$g = l \oplus p$

$l = \left\{ \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \mid y \in \mathbb{R} \right\} = so(2)$

$p = \left\{ \begin{bmatrix} ix & iz \\ iz & -ix \end{bmatrix} \mid x, z \in \mathbb{R} \right\}$

$a = \left\{ \begin{bmatrix} ix & 0 \\ 0 & -ix \end{bmatrix} \mid x \in \mathbb{R} \right\} = \mathbb{R}J$

$\alpha = 4J, \quad R = \{ \pm \alpha \} = A_1, \quad W(R) = \{ \pm 1 \}$

For $H \in a$, if $\langle \alpha, H \rangle \in \pi \mathbb{Z}$, $S^1 = \text{Ad}(\exp H)S^1$

If $\langle \alpha, H \rangle \notin \pi \mathbb{Z}$, $S^1 \cap \text{Ad}(\exp H)S^1 = \{ \pm J \} = W(R)J$