Maximal antipodal subgroups of the compact Lie group $G_2$ of exceptional type

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1 Maximal antipodal sets of a symmetric space

Let $M$ be a connected compact Riemannian symmetric space. And $I(M)_0$ the identity connected component of the isometry group. For $x \in M$, the geodesic symmetry at $x$ is denoted as $s_x$.

Definition (Chen-Nagano [1])

1. An antipodal set in $M$ is defined to be a subset $A$ of $M$ such that $s_x y = y$ for all $x, y \in A$.
2. The 2-number $\sharp_2 M$ of $M$ is defined to be the supremum of cardinality $\sharp A$ of an antipodal set $A$ in $M$.
3. A great antipodal set $A_2$ in $M$ is defined to be an antipodal set in $M$ such that $\sharp A_2 = \sharp_2 M$.
4. An antipodal set $A$ in $M$ is said to be maximal iff $A' = A$ for all antipodal subset $A'$ in $M$ such that $A' \supseteq A$.
5. Two antipodal sets $A, A'$ in $M$ are said to be congruent iff $\alpha A = A'$ for some $\alpha \in I(M)_0$.

2 Poles and polars of a symmetric space

For $x \in M$, put $F(s_x, M) := \{ y \in M \mid s_x y = y \}$. Then $F(s_x, M) \setminus \{x\} = \{ o_i \mid 1 \leq i \leq a \} \cup (\cup_{j=1}^b M_j^+)$ as a disjoint union of some poles (i.e., zero-dimensional connected components) $\{ o_i \mid 1 \leq i \leq a \}$ and polars (i.e., positive-dimensional connected components) $M_j^+$ $(1 \leq j \leq b)$ for some non-negative integers $a, b$, where $a = 0$ or $b = 0$ means that $\{ o_i \mid 1 \leq i \leq a \}$ or $\cup_{j=1}^b M_j^+$ is an empty set, respectively.
**Lemma 1.** For \( x \in M \), if \( b = 1 \) and \( a = 0 \) or 1, then the assignment

\[
A_1 \mapsto A'_1 := \{ x \} \cup \{ o_i \mid 1 \leq i \leq a \} \cup A_1
\]

from the set of all maximal antipodal sets in \( M_1^+ \) to that in \( M \) induces a surjection between their congruent class.

**Proof.** Let \( A \) be a maximal antipodal set in \( M \) containing \( x \). Then \( A_1 := A \setminus \{ x, o_i \mid 1 \leq i \leq a \} \subseteq F(s_x, M) \setminus \{ x, o_i \mid 1 \leq i \leq a \} = M_1^+ \) as a maximal antipodal set in \( M_1^+ \) such that \( A'_1 = A \). \( \square \)

3 Maximal antipodal subgroups of a Lie group

Let \( M \) be a connected compact Lie group being a Riemannian symmetric space by a bi-invariant metric on \( M \). Then any two conjugate subgroups of \( M \) are congruent in \( M \), and vice versa if \( M \) is a simple Lie group.

**Remark** (Chen-Nagano[1, Remarks 1.2, 1.3]). Any maximal antipodal set \( A \) in \( M \) containing the unit element \( e \) is a discrete abelian subgroup of \( M \), which is isomorphic to \( \mathbb{Z}^2 \) with \( 2^t < \infty \).

4 Connected Lie group \( G_2 \) of exceptional type

Let \( G_2 \) be a connected compact simple Lie group of type \( G_2 \). And \( S^2 \cdot S^2 \) the quotient space \( (S^2 \times S^2)/\mathbb{Z}_2 \) of \( S^2 \times S^2 \) by a natural action of \( \mathbb{Z}_2 := \{ \pm(1,1) \} \) on \( S^2 \times S^2 \) [1, 3.8]. Then the following theorem 1 was given by Nagano without proof.

**Theorem 1** (Nagano [2, p.66]). Put \( M := G_2 \). Then \( F(s_e, G_2)\setminus \{ e \} = M_1^+ \cong G_2/\text{SO}(4) \). For \( o \in M_1^+ \), \( F(s_o, M_1^+)\setminus \{ o \} = M_{1,1}^+ \cong S^2 \cdot S^2 \).

**Lemma 2.** Put \( M := S^2 \cdot S^2 \ni [\vec{x}, \vec{y}] := \{ \pm(\vec{x}, \vec{y}) \} \) and \( x_{\pm i} := [\vec{e}_i, \pm\vec{e}_i] \) \((i = 1, 2, 3)\) for an arbitrary orthonormal frame \( \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \} \) of \( \mathbb{R}^3 \). Then any maximal antipodal set in \( M \) is congruent to \( A := \{ x_{\pm i} \mid i = 1, 2, 3 \} \).

**Proof.** \( F(s_{x_1}, M) \setminus \{ x_1 \} = \{ x_{-1} \} \cup M_1^+ \), \( M_1^+ := (S^2 \cap \vec{e}_i^+)^2 / \mathbb{Z}_2 \). By virtue of Lemma 1 \((a = 1)\), any maximal antipodal set in \( M \) is congruent to \( A_1 := \{ x_{\pm 1} \} \cup A_1 \) for some maximal antipodal set \( A_1 \) containing \( x_2 \) in \( M_1^+ \). Then \( A_1 \setminus \{ x_2 \} \subseteq \{ x_{-2} \} \cup (S^2 \cap \vec{e}_1^+ \cap \vec{e}_2^+)^2 / \mathbb{Z}_2 = \{ x_{-2} \cdot x_{\pm 3} \} \), so that \( A'_1 \subseteq A \) which is antipodal. Since \( A'_1 \) is maximal, \( A'_1 = A \). \( \square \)

By virtue of Lemma 1, the following result is then obtained.
Theorem 2 ([6]). Let $A$ be the maximal antipodal set in $(S^2 \times S^2)/\mathbb{Z}_2$ defined in Lemma 2. Moreover, let $\varphi : (S^2 \times S^2)/\mathbb{Z}_2 \to M^+_{1,1}$ be an isometry giving an isometry $(S^2 \times S^2)/\mathbb{Z}_2 \cong M^+_{1,1}$ mentioned in Theorem 1. Put $B := \varphi(A)$, $B' := \{0\} \cup B$ and $B'' := \{e, o\} \cup B$. Then

(1) Any maximal antipodal set in $M^+_{1,1}$ is congruent to $B'$; and

(2) Any maximal antipodal subgroup of $G_2$ is conjugate to $B''$.

5 Explicit description of $G_2$

The explicit description of $G_2$ is given after Yokota as follows: Let $H$ be the quaternions with the unit element 1 and the Hamilton’s triple $i, j, k$ with the conjugation $b := b_0 - b_1 i - b_2 j - b_3 k$ ($b = b_0 + b_1 i + b_2 j + b_3 k \in H$). By Cayley-Dickson process, the octanions are given as $O := H \times H$ with the $R$-bilinear product $xy := (m n - b a, a m + b n)$ for $x = (m, a)$ and $y = (n, b) \in O$. By the octanionic conjugation $\bar{x} := (\bar{m}, -a) \in O$, a positive-definite $R$-bilinear inner product is defined as $(x \mid y) := (x \bar{y} + y \bar{x})/2 \in R$. Put

$$G := \{\alpha \in \text{GL}_R(O) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$$

as the automorphism group of the $R$-algebra $O$. Then $a1 = 1$, $\overline{\alpha x} = \alpha \bar{x}$ and $(\alpha x \mid \alpha y) = (x \mid y)$ for $\alpha \in G$ and $x, y \in O$. Moreover, put $\text{Im}O := \{x \in O \mid \bar{x} = -x\} \cong \mathbb{R}^7$, $S^6 := \{x \in \text{Im}O \mid (x \mid x) = 1\} \ni (i, 0)$ and $H := \{\alpha \in G \mid \alpha(i, 0) = (i, 0)\}$.

Proposition 1. (1) $G$ acts transitively on $S^6$ such that $H \cong \text{SU}(3)$, so that $G/H \cong S^6$. As the result, $G$ is a connected and simply connected 14-dimensional compact Lie group.

(2) Take an isomorphism $f : \text{SU}(3) \to H$ given by (1). If $T^2$ is a maximal torus of $\text{SU}(3)$, then $G = \cup_{\alpha \in G} \alpha f(T^2) \alpha^{-1}$. As the result, rank $G = \text{rank } H = 2$.

Proof. (1) The first part was directly proved by Yokota [3, pp.250–251]. The last part follows from the first one.

(2) Since $G$ is connected, $G \subseteq \text{SO}(\text{Im}O) \cong \text{SO}(7)$. Since any element of $\text{SO}(7)$ admits a fixed-point in $S^6$, any $\alpha \in G$ admits some $p \in S^6$ such that $\alpha p = p$. By (1), $\beta p = (i, 0)$ for some $\beta \in G$. Then $(\beta \alpha \beta^{-1})(i, 0) = (i, 0)$. Hence, $\beta \alpha \beta^{-1} = f(A)$ for some $A \in \text{SU}(3)$. For some $B \in \text{SU}(3)$, $BAB^{-1} \in T^2$. Hence, $(f(B) \beta) \alpha (f(B) \beta)^{-1} \in f(T^2)$. □

Put $Sp(1) := \{q \in H \mid |q| = 1\}$, $\psi : Sp(1) \times Sp(1) \to \text{GL}_R(O)$;

$$\psi(p, q)(m, a) := (qm\overline{q}, pao\overline{q}).$$
Moreover, put \( e = \psi(1, 1), \gamma := \psi(1, -1), G^\gamma := \{ \alpha \in G \mid \alpha \gamma = \gamma \alpha \} \). An explicit description of the polar decomposition of the automorphism group of the real split octonions was given by Yokota [4], by which the following proposition 2 was also obtained (cf. [5, 1.3.3, 1.3.4] for a precise proof).

**Proposition 2**

1. \( \psi(\text{Sp}(1) \times \text{Sp}(1)) = G^\gamma \),
2. \( \ker \psi = \{ \pm (1, 1) \}, G^\gamma \cong \text{SO}(4) \).

**Corollary.** \( G \) is a connected, simply connected, compact, simple Lie group of type \( G_2 \) with \( z(G) = f e g \).

**Proof.**

1. \( z(G) = \{ e \} \) (Yokota, arXiv:0902.0431v1, Theorem 1.11.1): In fact, \( z(G) \subset z(G^\gamma) = z(\psi(\text{Sp}(1) \times \text{Sp}(1))) = \{ \psi(1, \pm 1) \} = \{ e, \gamma \} \) and \( \gamma \notin z(G) \) by \( \dim G^\gamma = 6 < 14 = \dim G \). (2) By the step (1) and Proposition 1 (2), \( G \) is semisimple of type \( A_1 \oplus A_1, A_2 \) or \( G_2 \) of dimension 6, 8 or 14. Hence, \( G \) is simple of type \( G_2 \) by Proposition 1 (1).

6 **Explicit description of polars in \( G_2 \)**

By Corollary, \( G \) is denoted also as \( G_2 \). By explicit description of polars in \( G_2 \), the results of Theorems 1 and 2 are directly examined as follows:

**Theorem 3** ([6]).

1. \( F(s_e, G) \setminus \{ e \} = M_1^+ = \{ g^\gamma g^{-1} \mid g \in G \} \cong G_2/\text{SO}(4) \).
2. For \( o := \gamma \in M_1^+ \), \( F(s_o, M_1^+) \setminus \{ o \} = M_{1,1}^+ \) and
   \[ M_{1,1}^+ = \{ \psi(p, q) \mid p^2 = q^2 = -1 \} \cong (S^2 \times S^2)/\mathbb{Z}_2. \]
3. Any maximal antipodal set in \( M_{1,1}^+ \) is congruent to
   \[ B := \{ \psi(p, \pm p) \mid p = i, j, k \}. \]
4. Any maximal antipodal set in \( M_1^+ \) is congruent to
   \[ B' := \{ \psi(1, -1) \} \cup B. \]
5. Any maximal antipodal subgroup of \( G_2 \) is conjugate to
   \[ B'' := \{ \psi(1, \pm 1) \} \cup B. \]

**Proof.**

1. Put \( T^2 := \{ A = \text{diag}(\alpha_1, \alpha_2, \alpha_3) \mid A \in \text{SU}(3) \} \), which is a maximal torus of \( \text{SU}(3) \). Then \( F(s_e, T^2) = \{ \text{diag}(\pm 1, \pm 1, \pm 1) \in T^2 \} = \)
\{e\} \cup \{A_i \text{diag}(1, -1, -1)A_i^{-1} \mid i = 1, 2, 3\} \text{ for some } A_i \in SU(3) \text{ (} i = 1, 2, 3). \\
By virtue of Proposition 1 (2), \gamma \in F(s_e, G) \setminus \{e\} = \cup_{g \in G} g f(F(s_e, T^2))g^{-1} \\
\setminus \{e\} = \cup_{g \in G} g \gamma g^{-1} \mid g \in G \} \cong G/G^\gamma, \text{ which is connected since } G \text{ is connected. Hence, } G_2/ \text{SO}(4) \cong F(s_e, G) \setminus \{e\} = M_1^+. \\
(2) F(s_\gamma, M_1^+) \setminus \{\gamma\} = M_1^+ \cap G^\gamma \setminus \{\gamma\} = \{\psi(p, q) \mid (p^2, q^2) = \pm(1, 1)\} \setminus \{e, \gamma\} \\
= \{\psi(p, q) \mid (p^2, q^2) = -(1, 1)\} \cong (S^2 \times S^2)/\mathbb{Z}_2, \text{ because of } e = \psi(1, 1), \\
\gamma = \psi(1, -1) \text{ and } \{p \in Sp(1) \mid p^2 = -1\} = \{p \in Sp(1) \mid p = -\bar{p}\} \\
= \{p = p_1 i + p_2 j + p_3 k \mid \sum_{i=1}^{3} p_i^2 = 1\}. \\
(3) \text{ follows from Lemma 2 because of (2). (4) \text{ (resp. (5)) follows from Lemma 1 with } a = 0 \text{ because of (3) (resp. (4)) and (1).} \quad \Box \\
\text{Remark. (1) The result } \sharp_2(S^2 \cdot S^2) = 6, \sharp_2G_2/ \text{SO}(4) = 7 \text{ and } \sharp_2G_2 = 8 \text{ of Chen-Nagano [1, Examples 3.13] is refined by Theorem 3 since } B \text{ (resp. } B' \text{ or } B'') \text{ is a great antipodal set in } S^2 \cdot S^2 \text{ (resp. } G_2/ \text{SO}(4) \text{ or } G_2) \text{ as unique maximal antipodal set up to congruence.} \\
(2) \text{ Lemma 1 provides a priori or clear-sighted geometric method to Theorems 2 and 3. Posteriorly or arithmetically, Theorem 3 (5) is verified by calculations of weights of } B'' \text{ on } O = R^8. \\
\text{References} \\
[5] I. Yokota, Realizations of involutive automorphisms } \sigma \text{ and } G_2^\gamma \text{ of exceptional linear Lie groups } G, \text{ Part I, } G = G_2, F_4 \text{ and } E_6, Tsukuba J.Math. 14-1 (1990), 185–223. \\