ON WEAKLY REFLECTIVE SUBMANIFOLDS
IN HILBERT SPACES

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Abstract. This is a report on the author’s talk given at Yuzawa 2018. We introduce and study infinite dimensional versions of weakly reflective submanifolds.

INTRODUCTION

A submanifold $M$ of a Riemannian manifold $\hat{M}$ is called \textit{weakly reflective} ([4]) if for each $\xi \in T_p^\perp M$ at each $p \in M$, there is an isometry $\nu_\xi$ of $\hat{M}$ which satisfies $\nu_\xi(p) = p$, $(d\nu_\xi)_p \xi = -\xi$ and $\nu_\xi(M) = M$. Such $\nu_\xi$ is called a \textit{reflection} of $M$ with respect to $\xi$. The following relation is known.

\begin{align*}
\text{reflective ([7])} & \Rightarrow \text{totally geodesic} \Rightarrow \text{austere ([2])} \Rightarrow \text{minimal}
\end{align*}

In connection with submanifold geometry of orbits of Lie group actions, weakly reflective submanifolds are known as an important class of minimal submanifolds.

In this article we introduce and study infinite dimensional versions of weakly reflective submanifolds.

1. WEAKLY REFLECTIVE PF SUBMANIFOLDS

Let $V$ be a separable Hilbert space. Recall that an immersed submanifold $\mathcal{M}$ of finite codimension in $V$ is called \textit{Proper Fredholm} (PF) ([10]) if the restriction of the end point map $T^\perp \mathcal{M} \to V, (p, v) \mapsto p + v$ to a normal disk bundle of any finite radius is proper and Fredholm.

Now we define PF submanifolds to be weakly reflective similarly to the above.

\textbf{Definition.} Let $\mathcal{M}$ be a PF submanifold of $V$. $\mathcal{M}$ is called \textit{weakly reflective} if for each $\xi \in T_p^\perp \mathcal{M}$ at each $p \in \mathcal{M}$, there is an isometry $\nu_\xi$ of $V$ which satisfies $\nu_\xi(p) = p$, $(d\nu_\xi)_p \xi = -\xi$ and $\nu_\xi(\mathcal{M}) = \mathcal{M}$.

In the case of PF submanifolds, the following relation holds.

\begin{align*}
\text{reflective} \Leftrightarrow \text{totally geodesic} \Leftrightarrow \text{austere} \Leftrightarrow \text{weakly reflective} \Rightarrow \text{minimal' ([5], [3])}
\end{align*}

\textbf{Remark.} We can also extend the notions of reflective, totally geodesic, and austere submanifolds respectively to the infinite dimension by a similar way. There are mainly two ways ([5], [3]) to define ‘minimal’ PF submanifolds. Although we do not know whether austere PF submanifolds are ‘minimal’ or not in the general infinite dimensional case, it is true in an important case that PF submanifolds are inverse images of closed submanifolds by the parallel transport map (see the next section).

The author was partly supported by the Grant-in-Aid for JSPS Research Fellow (No.18J14857).
2. *P(G, H)*-actions and Parallel Transport Maps

Let *G* be a connected compact Lie group with Lie algebra *g*. Fix an Ad(*G*)-invariant inner product of *g*, and equip the corresponding bi-invariant Riemannian metric with *G*. Denote by *V* := *H*([0, 1], *g*) a Hilbert space of Sobolev *H*0-paths in *g* parametrized by *t* ∈ [0, 1]. Also denote by *G* := *H*([0, 1], *G*) a Hilbert Lie group of Sobolev *H*1-paths in *G* parametrized by *t* ∈ [0, 1]. Let 0 denote the constant path which values at 0 ∈ *g*. Define a *G*-action on *V*0 by the left gauge transformations: 

\[ g \ast u := gug^{-1} - g'g^{-1} \text{ for } (g, u) \in \mathcal{G} \times V_0. \]

Let *H* be a closed subgroup of *G* × *G*. *H* acts on *G* by (b1, b2) · a := b1ab2−1 for (a, (b1, b2)) ∈ *G* × *H*. Set *P*(G, *H*) := \{g ∈ *G* | (g(0), g(1)) ∈ *H*\}. The induced action of *P*(G, *H*) on *V*0 is called a *P*(G, *H*)-action ([11]). Note that *P*(G, *H*) is an inverse image of *H* by a Lie group homomorphism *Ψ* : *G* → *G* × *G*, *g* ↦ (g(0), g(1)).

The parallel transport map ([5]) \( \Phi : V_0 \to G \) is defined by \( \Phi(u) := E_u(1) \) for *u* ∈ *V*0, where \( E_u \in P(G, e \times G) \) is defined by a unique solution to the linear ordinary differential equation \( E_u' = u \). It follows ([11]) that \( \Phi(g \ast u) = \Psi(g) \ast \Phi(u) \) and \( P(G, H) \ast u = \Phi^{-1}(H \cdot \Phi(u)) \) for \( (g, u) \in G \times V_0 \). It is known ([12]) that for a closed submanifold *N* of *G*, *Φ*−1(*N*) is a PF submanifold of *V*0. In particular, each orbit of a *P*(G, *H*)-action is a PF submanifold of *V*0.

Let *K* be a closed subgroup of *G* with Lie algebra *t*. Denote by *g* = *t* ⊕ *m* the orthogonal direct sum decomposition. We restrict the Ad(*G*)-invariant inner product of *g* to *m*, and define the induced *G*-invariant Riemannian metric on a homogeneous space *G*/K so that the projection \( π : G \to G/K \) is a Riemannian submersion. Define a map \( Φ_K : V_0 \to G/K \) by \( Φ_K := π \circ Φ \). \( Φ_K \) is called the parallel transport map over *G*/K. Note that if *K* = \{e\}, then \( Φ_K = Φ \).

3. Main Results

**Theorem 1.** Suppose that an orbit \( H \cdot e \) through \( e \in G \) satisfies \( (H \cdot e)^{-1} = H \cdot e \). Then \( (H \cdot e) \) is a weakly reflective submanifold of *G*, (2) \( P(G, H) \ast 0 \) is a weakly reflective PF submanifold of *V*0.

**Corollary.** Each fiber of \( Φ_K \) is a weakly reflective PF submanifold of *V*0.

We denote by \( (G \times G)_a \) the isotropy subgroup of *G* × *G* at \( a \in G \) with respect to the action \( G \times G \cdot G \), (b1, b2) · a := b1ab2−1. Also denote by \( G_{aK} \) the isotropy subgroup of *G* at \( aK \in G/K \) with respect to the action \( G \cdot G/K \), \( b \cdot aK := baK \).

**Theorem 2.** (1) Let *N* be a weakly reflective submanifold of *G*. Suppose that for each \( (a, ξ) \in T^1 N \), a reflection \( ν_ξ \) of *N* with respect to \( ξ \) belongs to \( (G \times G)_a \). Then \( Φ^{-1}(N) \) is a weakly reflective PF submanifold of *V*0.

(2) Let *N* be a weakly reflective submanifold of *G*/K. Suppose that for each \( (aK, ξ) \in T^1 N \), a reflection \( ν_ξ \) of *N* with respect to \( ξ \) belongs to \( G_{aK} \). Then \( Φ_K^{-1}(N) \) is a weakly reflective PF submanifold of *V*0.

**Example.** (1) If \( H \cdot a \) \((a \in G)\) is a singular orbit of an *H*-action with cohomogeneity 1, then an orbit \( P(G, H) \ast u \) through \( u \in Φ^{-1}(a) \) is a weakly reflective PF submanifold of *V*0. (2) Let \( K' \) be a closed subgroup of *G*. If \( K' \cdot aK \) \((aK \in G/K)\) is a singular orbit of the *K*-action with cohomogeneity 1, then the orbit \( P(G, K' \times K) \ast u \) through \( u \in Φ^{-1}(a) \) is a weakly reflective PF submanifold of *V*0.
Example. Let $G$ be a connected compact semisimple Lie group. Let $K, K'$ be two symmetric subgroups of $G$ whose involutions commute. Consider an action $(K' \times K) \curvearrowright G$, $(b_1, b_2) \cdot a := b_1 a b_2^{-1}$. If an orbit $(K' \times K) \cdot a$ through $a \in G$ satisfies the condition in [8, Theorems 4] or [8, Theorem 5], the orbit $P(G, K' \times K) * u$ through $u \in \Phi^{-1}(a)$ is a weakly reflective PF submanifold of $V$.

**Theorem 3.** Suppose that $G$ is semisimple and $(G, K)$ is an effective symmetric pair. If $N$ is a weakly reflective submanifold of $G/K$, then $\Phi^{-1}(N)$ is a weakly reflective PF submanifold of $V$.

Example. Let $(U, L)$ be a compact Riemannian symmetric pair. Suppose $L$ is connected. Denote by $u = l \oplus p$ the canonical decomposition, and $\text{Ad} : L \to SO(p)$ the isotropy representation. If an orbit $\text{Ad}(L) \cdot x$ through $x \in p$ is a weakly reflective submanifold of the hypersphere $S'(||x||)$ in $p$, then the orbit $P(SO(p), \text{Ad}(L) \times SO(p)_x) * \hat{0}$ is a weakly reflective PF submanifold of the Hilbert space $H^0([0, 1], \mathfrak{so}(p))$. Thus in particular we can construct weakly reflective PF submanifolds from examples of weakly reflective submanifolds given in [4, Theorem 4].

Example. By [1, Section 6], an orbit $P(SO(7), G_2 \times (SO(3) \times SO(4))) * \hat{0}$ is a weakly reflective PF submanifold of the Hilbert space $H^0([0, 1], \mathfrak{so}(7))$.

**Remark.** It is known that any homogeneous minimal submanifold in a finite dimensional Euclidean space must be totally geodesic ([9]). The above results and examples show that in infinite dimensional Hilbert spaces there are so many homogeneous minimal submanifolds which are not totally geodesic.

**References**


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