A Basic Introduction to Model Theory

Akito Tsuboi

University of Tsukuba

Logic Summer School at Waseda
September 5, 2016

Outline

1. What is Model Theory?
2. Languages, Structures and Models
3. Compactness Theorem
4. Large and Small Models

Lecture 1

$M \models \phi$

Model Theory

Equations

- model theory = universal algebra + logic
- model theory = algebraic geometry − fields
- model theory = ordinary mathematics + compactness

Model Theory

Equations

- model theory = universal algebra + logic
- model theory = algebraic geometry − fields
- model theory = ordinary mathematics + compactness

Model Theory

Equations

- model theory = universal algebra + logic
- model theory = algebraic geometry − fields
- model theory = ordinary mathematics + compactness

Compactness

Compactness is the property of "being of finite character."

- In topology, compactness means that every open cover has a finite subcover.
- In model theory, compactness means that if a theory is contradictory then some finite sub-theory is contradictory.

In other words, if every finite part of a theory has a model then the whole theory has a model.

Compactness

Compactness is the property of "being of finite character."

- In topology, compactness means that every open cover has a finite subcover.
- In model theory, compactness means that if a theory is contradictory then some finite sub-theory is contradictory.

  In other words, if every finite part of a theory has a model then the whole theory has a model.

Compactness

Compactness is the property of "being of finite character."

- In topology, compactness means that every open cover has a finite subcover.
- In model theory, compactness means that if a theory is contradictory then some finite sub-theory is contradictory.

  In other words, if every finite part of a theory has a model then the whole theory has a model.

Language

Formal Language

A language is a set consisting of

constant symbols + function symbols + predicate symbols.

Formula

Let L be a language. An L-formula is a formal 'statement' constructed from L, using (individual) variables $x, y, z \ldots$ and logical symbols.

Logical symbols are: $\wedge$ (and), $\vee$ (or), $\neg$ (not) $\rightarrow$ (implies), $\forall$ (all elements) and $\exists$ (some elements).

Language

Formal Language

A language is a set consisting of

constant symbols + function symbols + predicate symbols.

Formula

Let L be a language. An L-formula is a formal 'statement' constructed from L, using (individual) variables $x, y, z \ldots$ and logical symbols.

Logical symbols are: $\wedge$ (and), $\vee$ (or), $\neg$ (not) $\rightarrow$ (implies), $\forall$ (all elements) and $\exists$ (some elements).

Examples

Example (Language)

- The language L_o of ordered sets is $\{* < *\}$.
- The language L_{gp} of groups is $\{e, * \cdot *, *^{-1}\}$.
- The language L_K of K-vector spaces is $\{\vec{0}, * + *, -*\} \cup \{F_a(*) : a \in K\}$.

Example (Formula)

If $L = \{c, F(*), P(*, *)\}$, the following are examples of L-formulas:

$$P(c, x), \;\; P(F(x), F(y)), \;\; \forall x[P(x, y) \to \exists z P(z, F(F(x)))], \;\; \ldots$$

The first two, which do not contain logical symbols, are called *atomic*.

In the third formula, y is free.

Examples

Example (Language)

- The language L_o of ordered sets is $\{* < *\}$.
- The language L_{gp} of groups is $\{e, * \cdot *, *^{-1}\}$.
- The language L_K of K-vector spaces is $\{\vec{0}, * + *, -*\} \cup \{F_a(*) : a \in K\}$.

Example (Formula)

If $L = \{c, F(*), P(*, *)\}$, the following are examples of L-formulas:

$$P(c, x),\ \ P(F(x), F(y)),\ \forall x[P(x, y) \to \exists z P(z, F(F(x)))],\ \ldots$$

The first two, which do not contain logical symbols, are called *atomic.*

In the third formula, y is free.

Mathematical Structures

Examples of mathematical structures are:

- $(\mathbb{N}, <)$,
- $(\mathbb{N}, 0, 1, +, \cdot)$,
- $(\mathbb{Z}, 0, 1, +\cdot)$,
- $(\mathbb{R}, 0, 1, +, \cdot)$,
- $(\mathbb{C}, 0, 1, +, \cdot)$,
- $(\mathbb{Q}, <)$,
- $(GL(2, \mathbb{R}), \cdot), \ldots$.

For a language L, which is a set of symbols, we can define the notion of L-structures so that each of the examples above becomes a structure in our sense.

Structures

L-structure

Let $L = \{c, F, P\}$. An L-structure $\mathfrak{M}$ consists of:

- the universe M, and
- the interpretation ι of symbols in L such that
  - $\iota(c)$ is an element in M,
  - $\iota(F)$ is a function $M^n \to M$ (n is the arity of F),
  - $\iota(P)$ is a subset of M^m (m is the arity of P).

c is a mere symbol, and in the structure $\mathfrak{M}$, c is interpreted as an element $\iota(c) \in M$.

Structures

L-structure

Let $L = \{c, F, P\}$. An L-structure $\mathfrak{M}$ consists of:

- the universe M, and
- the interpretation ι of symbols in L such that
  - $\iota(c)$ is an element in M,
  - $\iota(F)$ is a function $M^n \to M$ (n is the arity of F),
  - $\iota(P)$ is a subset of M^m (m is the arity of P).

c is a mere symbol, and in the structure $\mathfrak{M}$, c is interpreted as an element $\iota(c) \in M$.

An L-structure $\mathfrak{M}$ has the form:

$$(M, \iota(c), \iota(F), \iota(P)).$$

- $\iota(X)$, $X \in L$, is sometimes simply written as X^M.
- So $\mathfrak{M}$ has the form

$$(M, c^M, F^M, P^M).$$

- We sometimes say M is an L-structure, if there is no confusion.

An L-structure $\mathfrak{M}$ has the form:

$$(M, \iota(c), \iota(F), \iota(P)).$$

- $\iota(X)$, $X \in L$, is sometimes simply written as X^M.
- So $\mathfrak{M}$ has the form

$$(M, c^M, F^M, P^M).$$

- We sometimes say M is an L-structure, if there is no confusion.

An L-structure $\mathfrak{M}$ has the form:

$$(M, \iota(c), \iota(F), \iota(P)).$$

- $\iota(X)$, $X \in L$, is sometimes simply written as X^M.
- So $\mathfrak{M}$ has the form

$$(M, c^M, F^M, P^M).$$

- We sometimes say M is an L-structure, if there is no confusion.

Let $\mathfrak{M}$ be an L-structure. We write $\mathfrak{M} \models *$, if $*$ is true in M.

Example

- $(\mathbb{R}, \mathbf{0}, \mathbf{1}, +, \cdot, <) \models \forall x(\mathbf{0} < x \rightarrow \exists y(x = y \cdot y \wedge \neg(y = \mathbf{0})))$;
  A positive element is a square.
- $(\mathbb{N}, +, \cdot) \models \forall x \exists y_0, y_1, y_2, y_3(x = y_0 \cdot y_0 + x = y_1 \cdot y_1 + x = y_2 \cdot y_2 + x = y_3 \cdot y_3)$.
  Four Square Theorem

We say $\mathfrak{M}$ is a model of T if $\mathfrak{M} \models T$.

Example

Let R be a binary predicate symbol. An undirected graph G is considered as an R-structure satisfying

$$G \models \forall x, y(R(x, y) \rightarrow R(y, x)) \ \wedge \ \forall x(\neg R(x, x)).$$

Definable Sets

Definition

A subset A of M is called a definable set if there is an L-formula $\varphi(x, \bar{y})$ and $\bar{b} \in M$ (tuples from M, called parameters) such that

$$A = \{a \in M : M \models \varphi(a, \bar{b})\}.$$

Definable sets of M^n is defined similarly.

A as above is sometimes called $\bar{b}$-definable.

Example

- $\mathbf{2}\mathbb{Z}$ (the even numbers) is a definable subset of $(\mathbb{Z}, \mathbf{0}, +)$, because

$$\mathbf{2}\mathbb{Z} = \{a \in \mathbb{Z} : \mathbb{Z} \models \exists x(a = x + x)\}.$$

- Contrary to this, $\mathbf{2}\mathbb{Z}$ is not a definable subset of $(\mathbb{Z}, \mathbf{0}, <)$.

Undefinability of $\mathbf{2}\mathbb{Z}$ in $(\mathbb{Z}, \mathbf{0}, <)$ is shown by using compactness.

Remark

If M is a countable (infinite) structure, there are $2^{\aleph_0}$-many subsets of M.
But there are only countably many formulas (with parameters from M). So there are only countably many definable sets of M.

In general, if M has the cardinality κ, there are only κ-many definable subsets of M.

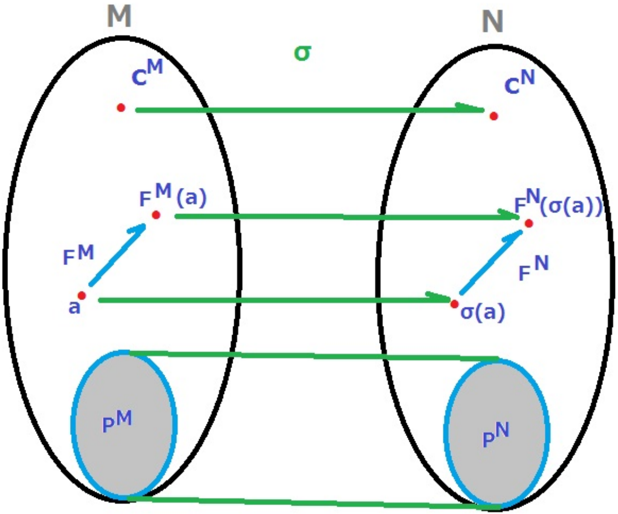
Definable Sets and Automorphisms

Definition (Isomorphism)

Let M and N be $\{c, F, P\}$-structures. A bijection $\sigma : M \to N$ is called an isomorphism of M and N, if it satisfies:

- $\sigma(c^M) = c^N$;
- $\sigma(P^M) = P^N$;
- $\sigma(F^M(\bar{a})) = F^N(\sigma(\bar{b}))$.

Isomorphism preserves formulas. $M \models \varphi(a) \Rightarrow N \models \varphi(\sigma(a))$.


M
N
σ
c^M
c^N
$F^M(a)$
$F^N(\sigma(a))$
F^M
F^N
a
σ(a)
P^M
P^N

Automorphisms fix definable sets

Let A be a definable set of M, defined by a formula with parameters $\bar{b}$. Let $\sigma \in Aut(M/\bar{b})$ be an automorphism of M fixing $\bar{b}$ point-wise. Then

$$\sigma(A) = A.$$

Proof.

$$\begin{aligned} a \in A \quad & \iff \quad M \models \varphi(a, \bar{b}) \\ & \iff \quad M \models \varphi(\sigma a, \sigma \bar{b}) \\ & \iff \quad M \models \varphi(\sigma a, \bar{b}) \\ & \iff \quad \sigma a \in A. \end{aligned}$$

□

Automorphisms fix definable sets

Let A be a definable set of M, defined by a formula with parameters $\bar{b}$. Let $\sigma \in Aut(M/\bar{b})$ be an automorphism of M fixing $\bar{b}$ point-wise. Then

$$\sigma(A) = A.$$

Proof.

$$\begin{aligned} a \in A \quad &\iff \quad M \models \varphi(a, \bar{b}) \\ &\iff \quad M \models \varphi(\sigma a, \sigma \bar{b}) \\ &\iff \quad M \models \varphi(\sigma a, \bar{b}) \\ &\iff \quad \sigma a \in A. \end{aligned}$$

□

Lecture 2

Compactness Theorem

Theorem

Let T be a set of L-sentences. The following two conditions on T are equivalent:

1. *T has a model;*
2. *Every finite subset of T has a model. (T is finitely satisfiable.)*

The implication $1 \Rightarrow 2$ is trivial. So we assume 2 and prove 1.
For simplicity, we assume L is countable.

Compactness Theorem

Theorem

Let T be a set of L-sentences. The following two conditions on T are equivalent:

1. *T has a model;*
2. *Every finite subset of T has a model. (T is finitely satisfiable.)*

The implication $\mathbf{1} \Rightarrow \mathbf{2}$ is trivial. So we assume $\mathbf{2}$ and prove $\mathbf{1}$.
For simplicity, we assume L is countable.

Several different proofs of Compactness Theorem are known:

1. Proof using Completeness Theorem,
2. Proof using Ultraproduct, definition
3. Others.

Sketch of Proof of Compactness

1. Let

$$L^* = L \cup \{c_0, c_1, \dots\},$$

$$T' = T \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i) : i \in \omega\},$$

where $\varphi_i(x)$'s enumerate all the L^*-formulas.

2. T' is finitely satisfiable.
3. By Zorn's lemma, we can choose a set $T^* \supset T'$ of L^*-sentences such that (i) T^* is finitely satisfiable and (ii) maximal among such.
4. Using T^*, we define an L^*-structure, which is a model of $T^* \supset T$.

Main Lemma

Definition

We say T (a set of L-sentences) has the witnessing property if

(*) for any L-formula $\varphi(x)$, there is a constant $c \in L$ such that '$\exists x \varphi(x) \to \varphi(c)$' $\in T$.

Lemma (Main Lemma)

Let T^* have the following properties:

1. *Every finite subset of T^* has a model;*
2. *T^* has the witnessing property;*
3. *T^* is complete, i.e., for all φ, $\varphi \in T^*$ or $\neg\varphi \in T^*$.*

Then T^* has a model M^* whose universe is (essentially) the set of all closed terms of L^*.

Definition of M^*

Using T^*, we define an L^*-structure M^* by the following:

- CT =the set of all closed L^*-terms. (A closed term is a term without a variable.
- For $s, t \in CT$, $s \sim t \iff s = t$ belongs to T^*. (It will be shown that $\sim$ is an equivalence relation on CT.)
- $M^* = CT/\sim = \{[t] : t \in CT\}$.
  - $c^{M^*} := [c]$, where c is a constant symbol in L^*;
  - $F^{M^*}([t_1], \ldots, [t_m]) := [F(t_1, \ldots, t_m)]$, where F is an m-ary function symbol in L^*;
  - $P^M = \{([t_1], \ldots, [t_n]) : P(t_1, \ldots, t_n) \in T^*\}$, where P is an n-ary predicate symbol in L^*.

Proof of Main Lemma

Claim

For all L-formulas $\varphi(x_1, \ldots, x_n)$ and $t_1, \ldots, t_n \in CT$,

$$M^* \models \varphi([t_1], \ldots, [t_n]) \iff \varphi(t_1, \ldots, t_n) \in T^*.$$

Proof by induction on the number k of logical symbols in φ.

$\boxed{k = 0}$ φ is an atomic formula in this case. The equivalence is rather clear from the definition of the interpretation.

$$M^* \models P([t]) \iff ([t]) \in P^{M^*} \iff P(t) \in T^*.$$

$$M^* \models F([t]) = [u] \Leftrightarrow F^{M^*}([t]) = [u] \Leftrightarrow [F(t)] = [u] \Leftrightarrow F(t) = u \in T^*.$$

$\boxed{k+1}$ Case 1: $\varphi = \psi \wedge \theta$.

$$
\begin{aligned}
& M^* \models (\psi \wedge \theta)([t_1], \ldots, [t_n]) \\
\Longleftrightarrow\ & M^* \models \psi([t_1], \ldots, [t_n]) \text{ and } M^* \models \theta([t_1], \ldots, [t_n]) \\
\Longleftrightarrow\ & \psi(t_1, \ldots, t_n) \in T^* \text{ and } \theta(t_1, \ldots, t_n) \in T^* \\
\Longleftrightarrow\ & \psi(t_1, \ldots, t_n) \wedge \theta(t_1, \ldots, t_n) \in T^*.
\end{aligned}
$$

Case 2: $\varphi(x_1, \ldots, x_n) = \exists y \psi(x, x_1, \ldots, x_n)$.

$$
\begin{aligned}
& M^* \models \exists x \psi(x, [t_1], \ldots, [t_n]) \\
\Longleftrightarrow\ & M^* \models \psi([s], [t_1], \ldots, [t_n]), \text{ for some } s \in CT \\
\Longleftrightarrow\ & \psi(s, t_1, \ldots, t_n) \in T^*, \text{ for some } s \in CT \\
\Longleftrightarrow\ & \exists x \psi(s, t_1, \ldots, t_n) \in T^*.
\end{aligned}
$$

$\Leftarrow$ of the last line is the most essential part, and it follows from the fact $T^* \supset T'$.

Strategy of Proof

Extend T to T^* so that T^* satisfies the conditions in Main Lemma.

- $L^* = L \cup \{c_0, c_1, \ldots\}$,
- $T' = T \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i) : i \in \omega\}$. T' clearly has the witnessing property.

Strategy of Proof

Extend T to T^* so that T^* satisfies the conditions in Main Lemma.

- $L^* = L \cup \{c_0, c_1, \ldots\}$,
- $T' = T \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i) : i \in \omega\}$. T' clearly has the witnessing property.

Proof of Compactness

- $L^* = L \cup \{c_0, c_1, \dots\}$,
- $T' = T \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i) : i \in \omega\}$.

Claim 1

Every finite subset F of T' has a model.

Proof: Consider the simplest case. Let F have the form $\{\psi_i\}_{i<k} \cup \{\exists x \varphi_0(x) \to \varphi_0(c_0)\}$, where ψ_i's are in T.
Since T is finitely satisfiable, there is a model $M \models \{\psi_i\}_{i<k}$.
If $\varphi_0(x)$ has a solution in M, then let c_0^M be one of such solutions.
Then M becomes a model of F.

Proof of Compactness

- $L^* = L \cup \{c_0, c_1, \ldots\}$,
- $T' = T \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i) : i \in \omega\}$.

Claim 1

Every finite subset F of T' has a model.

Proof: Consider the simplest case. Let F have the form $\{\psi_i\}_{i<k} \cup \{\exists x \varphi_0(x) \to \varphi_0(c_0)\}$, where ψ_i's are in T.
Since T is finitely satisfiable, there is a model $M \models \{\psi_i\}_{i<k}$.
If $\varphi_0(x)$ has a solution in M, then let c_0^M be one of such solutions.
Then M becomes a model of F.

Claim 2

There is T^* (a set of L^*-sentences) extending

$$T' = T \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i) : i \in \omega\}$$

such that

1. T^* is finitely satisfiable, and
2. T^* is maximal among all such sets.

Proof: Simply use Zorn's lemma.

It is easy to see that T^* is complete.

Claim 2

There is T^* (a set of L^*-sentences) extending

$$T' = T \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i) : i \in \omega\}$$

such that

1. T^* is finitely satisfiable, and
2. T^* is maximal among all such sets.

Proof: Simply use Zorn's lemma.

It is easy to see that T^* is complete.

Remark

Construction of M^* is similar to that of a field extension $K[x]/I$, where I is a maximal ideal of $K[x]$.

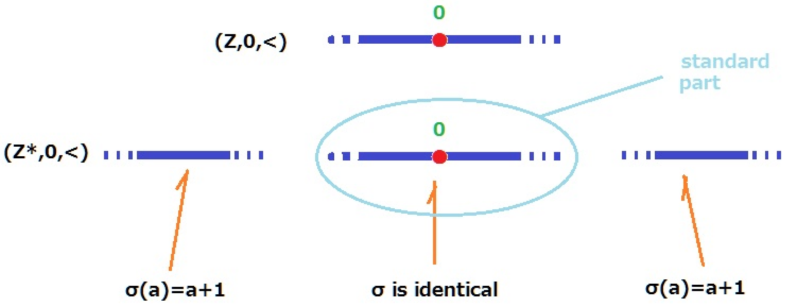
	M^*	$K[x]/I$
Preuniverse	All closed terms	All polynomials
$\sim$	$s = t$ modulo T^*	$s = t$ modulo I
Universe	(All closed terms)/$\sim$	(All polynomials)/$\sim$

Undefinability of $2\mathbb{Z}$ in $(\mathbb{Z}, 0, <)$

Example (Application of Compactness 1)

- In $\mathbb{Z} = (\mathbb{Z}, 0, <)$, every $n \in \mathbb{Z}$ is definable.
  For example 1 is the unique element satisfying
  $0 < x \wedge \neg\exists y(0 < y < x)$.
- Let $T = \{\varphi : \mathbb{Z} \models \varphi\} \cup \{0 < c, 1 < c, 2 < c, \ldots\}$.
- Every finite part of T has a model. So, by compactness, there is a model of T. Call it $\mathbb{Z}^*$.
- $\mathbb{Z}^* = (\mathbb{Z}, 0, <)$ + ‘copies of $(\mathbb{Z}, <)$’.
- Suppose, for a contradiction, $2\mathbb{Z}$ is definable by $\varphi(x)$. Then $\forall x(\varphi(x) \to \varphi(x+1)) \in T$, so it is true in $\mathbb{Z}^*$.
- However, the mapping $\sigma : \mathbb{Z}^* \to \mathbb{Z}^*$, $a \mapsto a$ (a standard) $a \mapsto a + 1$ (a non-standard), is an automorphism of $\mathbb{Z}^*$.
- This is a contradiction. (σ moves the set defined by φ.)

$\mathbb{Z}$ and $\mathbb{Z}^*$



Non-standard model of $\mathbb{R}$

Example (Application of Compactness 2)

1. We regard $\mathbb{R}$ as a $\{0, 1, +, \cdot, <, \dots\}$-structure.
2. Let
$$T = \{\varphi : \mathbb{R} \models \varphi\} \cup \{|c| < 1/n : n \in \mathbb{N}\},$$
where c is a new constant symbol.
3. Every finite subset of T has a model.
4. By compactness, there is a model $\mathbb{R}^*$ of T.
5. $\mathbb{R}^*$ is almost the same as $\mathbb{R}$, since every sentence true in $\mathbb{R}$ is true in $\mathbb{R}^*$. But $\mathbb{R}^* \ncong \mathbb{R}$, since $\mathbb{R}^*$ has an infinitesimal.
6. If every element in $\mathbb{R}$ is named by a constant symbol in the language, then we have $\mathbb{R} \subset \mathbb{R}^*$.

Non-standard model of $\mathbb{R}$

Example (Application of Compactness 2)

1. We regard $\mathbb{R}$ as a $\{0, 1, +, \cdot, <, \ldots\}$-structure.
2. Let
$$T = \{\varphi : \mathbb{R} \models \varphi\} \cup \{|c| < 1/n : n \in \mathbb{N}\},$$
where c is a new constant symbol.
3. Every finite subset of T has a model.
4. By compactness, there is a model $\mathbb{R}^*$ of T.
5. $\mathbb{R}^*$ is almost the same as $\mathbb{R}$, since every sentence true in $\mathbb{R}$ is true in $\mathbb{R}^*$. But $\mathbb{R}^* \not\cong \mathbb{R}$, since $\mathbb{R}^*$ has an infinitesimal.
6. If every element in $\mathbb{R}$ is named by a constant symbol in the language, then we have $\mathbb{R} \subset \mathbb{R}^*$.

Non-standard model of $\mathbb{R}$

Example (Application of Compactness 2)

1. We regard $\mathbb{R}$ as a $\{0, 1, +, \cdot, <, \ldots\}$-structure.
2. Let
$$T = \{\varphi : \mathbb{R} \models \varphi\} \cup \{|c| < 1/n : n \in \mathbb{N}\},$$
where c is a new constant symbol.
3. Every finite subset of T has a model.
4. By compactness, there is a model $\mathbb{R}^*$ of T.
5. $\mathbb{R}^*$ is almost the same as $\mathbb{R}$, since every sentence true in $\mathbb{R}$ is true in $\mathbb{R}^*$. But $\mathbb{R}^* \ncong \mathbb{R}$, since $\mathbb{R}^*$ has an infinitesimal.
6. If every element in $\mathbb{R}$ is named by a constant symbol in the language, then we have $\mathbb{R} \subset \mathbb{R}^*$.

Non-standard model of $\mathbb{R}$

Example (Application of Compactness 2)

1. We regard $\mathbb{R}$ as a $\{0, 1, +, \cdot, <, \ldots\}$-structure.
2. Let
$$T = \{\varphi : \mathbb{R} \models \varphi\} \cup \{|c| < 1/n : n \in \mathbb{N}\},$$
where c is a new constant symbol.
3. Every finite subset of T has a model.
4. By compactness, there is a model $\mathbb{R}^*$ of T.
5. $\mathbb{R}^*$ is almost the same as $\mathbb{R}$, since every sentence true in $\mathbb{R}$ is true in $\mathbb{R}^*$. But $\mathbb{R}^* \ncong \mathbb{R}$, since $\mathbb{R}^*$ has an infinitesimal.
6. If every element in $\mathbb{R}$ is named by a constant symbol in the language, then we have $\mathbb{R} \subset \mathbb{R}^*$.

Non-standard model of $\mathbb{R}$

Example (Application of Compactness 2)

1. We regard $\mathbb{R}$ as a $\{0, 1, +, \cdot, <, \dots\}$-structure.
2. Let
$$T = \{\varphi : \mathbb{R} \models \varphi\} \cup \{|c| < 1/n : n \in \mathbb{N}\},$$
where c is a new constant symbol.
3. Every finite subset of T has a model.
4. By compactness, there is a model $\mathbb{R}^*$ of T.
5. $\mathbb{R}^*$ is almost the same as $\mathbb{R}$, since every sentence true in $\mathbb{R}$ is true in $\mathbb{R}^*$. But $\mathbb{R}^* \not\cong \mathbb{R}$, since $\mathbb{R}^*$ has an infinitesimal.
6. If every element in $\mathbb{R}$ is named by a constant symbol in the language, then we have $\mathbb{R} \subset \mathbb{R}^*$.

Non-standard model of $\mathbb{R}$

Example (Application of Compactness 2)

1. We regard $\mathbb{R}$ as a $\{0, 1, +, \cdot, <, \dots\}$-structure.
2. Let
$$T = \{\varphi : \mathbb{R} \models \varphi\} \cup \{|c| < 1/n : n \in \mathbb{N}\},$$
where c is a new constant symbol.
3. Every finite subset of T has a model.
4. By compactness, there is a model $\mathbb{R}^*$ of T.
5. $\mathbb{R}^*$ is almost the same as $\mathbb{R}$, since every sentence true in $\mathbb{R}$ is true in $\mathbb{R}^*$. But $\mathbb{R}^* \ncong \mathbb{R}$, since $\mathbb{R}^*$ has an infinitesimal.
6. If every element in $\mathbb{R}$ is named by a constant symbol in the language, then we have $\mathbb{R} \subset \mathbb{R}^*$.

Elementary Chain Theorem

Definition

We say $N \supset M$ is an elementary extension of M (in symbol $M \prec N$) if, for all $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(a_1, \ldots, a_n).$$

Example (Application of Main Lemma)

$M_0 \prec M_1 \prec \cdots \prec M_i \prec \ldots \ (i < \alpha) \implies M_i \prec \bigcup_{j<\alpha} M_j$.

Proof.

By extending L, we can assume each element a in $\bigcup_{j<\alpha} M_j$ is named by a constant c_a in L. Let $T^* = \bigcup_{i<\alpha}\{\varphi : M_i \models \varphi\}$. Then T^* satisfies the three conditoins in Main Lemma. So, T^* has a model whose universe is the c_a's. □

Elementary Chain Theorem

Definition

We say $N \supset M$ is an elementary extension of M (in symbol $M \prec N$) if, for all $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(a_1, \ldots, a_n).$$

Example (Application of Main Lemma)

$M_0 \prec M_1 \prec \cdots \prec M_i \prec \ldots \ (i < \alpha) \implies M_i \prec \bigcup_{j<\alpha} M_j$.

Proof.

By extending L, we can assume each element a in $\bigcup_{j<\alpha} M_j$ is named by a constant c_a in L. Let $T^* = \bigcup_{i<\alpha}\{\varphi : M_i \models \varphi\}$. Then T^* satisfies the three conditoins in Main Lemma. So, T^* has a model whose universe is the c_a's. □

Elementary Chain Theorem

Definition

We say $N \supset M$ is an elementary extension of M (in symbol $M \prec N$) if, for all $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(a_1, \ldots, a_n).$$

Example (Application of Main Lemma)

$M_0 \prec M_1 \prec \cdots \prec M_i \prec \ldots \ (i < \alpha) \implies M_i \prec \bigcup_{j<\alpha} M_j$.

Proof.

By extending L, we can assume each element a in $\bigcup_{j<\alpha} M_j$ is named by a constant c_a in L. Let $T^* = \bigcup_{i<\alpha}\{\varphi : M_i \models \varphi\}$. Then T^* satisfies the three conditoins in Main Lemma. So, T^* has a model whose universe is the c_a's. □

Lecture 3

Theorem (Compactness Theorem)

Let T be a set of L-sentences. The following two conditions on T are equivalent:

1. *T has a model;*
2. *Every finite subset of T has a model.*

T having a model is called a theory.

Let $A \subset M$, where M is an L-structure.

$$L(A) := L \cup \{c_a : a \in A\}.$$

M naturally becomes an $L(A)$-structure, by letting

$$c_a{}^M = a \quad (a \in M)$$

Types

Definition

1. A set $\Sigma(x)$ of formulas (x free) is finitely satisfiable in M if whenever $F(x) \subset_{fin} \Sigma(x)$ then $M \models \exists x \bigwedge F(x)$.
2. $\Sigma(x)$ is realized in M if there is $a \in M$ that satisfies all formulas in $\Sigma(x)$.
3. For $A \subset M$, a set $\Sigma(x)$ of $L(A)$-formulas is called a type over A, if
   - $\Sigma(x)$ is finitely satisfiable in M, and
   - $\Sigma(x)$ is complete for $L(A)$-formulas.
     For all $\varphi(x)$ ($L(A)$-formula), $\varphi(x) \in \Sigma(x)$ or $(\neg\varphi(x)) \in \Sigma(x)$.

A finitely satisfiable set can be extended to a type. Use Zorn's lemma.

Remark

A finitely satisfiable set is not necessarily to be realized.

- *In* $\mathbb{N}$, $\Sigma(x) = \{0 < x, 1 < x, \ldots\}$ *is finitely satisfiable, but it does not have a solution in* $\mathbb{N}$.
- *In* $\overline{\mathbb{Q}}$ *(algebraic closure of* $\mathbb{Q}$*),*

$$\Sigma(x) = \{f(x) \neq 0 : f(x) \in \mathbb{Q}[x], f \not\equiv 0\}$$

is finitely satisfiable, but it is not realized in $\overline{\mathbb{Q}}$.

Elementary Extension

Definition

Let M be an L-structure and N be an extension of M (as an L-structure). We say N is an elementary extension of M, in symbol,

$$M \prec N$$

if $M \models \varphi$ iff $N \models \varphi$, for all $L(M)$-sentences φ.

Finite satisfiability is preserved under elementary extensions.

Lemma

Let $\Sigma(x)$ be finitely satisfiable in M. Then there is $M^* \succ M$ such that $\Sigma(x)$ is realized in M^*.

Proof.

Let

$$T := \{\varphi \text{ } (L(M)\text{-sentence}) : M \models \varphi\} \cup \{\psi(c) : \psi(x) \in \Sigma(x)\}.$$

Then every finite subset of T has a model. So, by compactness, T has a model M^*. Clearly c^{M^*} realizes $\Sigma(x)$. □

Lemma

Let $\Sigma(x)$ *be finitely satisfiable in* M. *Then there is* $M^* \succ M$ *such that* $\Sigma(x)$ *is realized in* M^*.

Proof.

Let

$$T := \{\varphi \text{ } (L(M)\text{-sentence}) : M \models \varphi\} \cup \{\psi(c) : \psi(x) \in \Sigma(x)\}.$$

Then every finite subset of T has a model. So, by compactness, T has a model M^*. Clearly c^{M^*} realizes $\Sigma(x)$. □

By a repeated use of this lemma, we can prove the following.

Existence of Large Models

Corollary

Let M be an L-structure and let κ be an infinite cardinal. There is M^* such that

- $M^* \succ M$, *and*
- M^* *is κ-saturated.*

A structure M is called κ-saturated, if $A \subset M$ has the cardinality $< \kappa$ then every type over A is realized in M.

A κ-saturated structure M elementarily embeds every $N \equiv M$ of size $\leq \kappa$.

Definition

1. We say $\Sigma(x)$ is omitted in M, if it is not realized in M.
2. We say $\Sigma(x)$ is isolated in T, if there is no (consistent) formula $\varphi(x)$ such that (in any model $M \models T$) if a satisfies $\varphi(x)$ then a realizes $\Sigma(x)$.

Omitting Types Theorem

Theorem (Omitting Types Theorem)

Let T be a countable L-theory and $\Sigma(x)$ a set of L-formulas. Suppose that $\Sigma(x)$ is not isolated in T. Then there is a model $M \models T$ omitting Σ.

Countability is essential in this theorem.

Sketch of Proof of Omitting Types Theorem

1. We put $L^* = L \cup \{c_0, c_1, \ldots\}$.
2. Enumerate all L^*-formulas $\varphi(x)$ as:
$$\varphi_0(x), \varphi_1(x), \ldots$$
3. Letting $T_0 = T$, we shall define a theory T_i and a formula $\psi_i(x) \in \Sigma(x)$ inductively:
$$T_{i+1} = T_i \cup \{\exists x \varphi_i(x) \to \varphi_i(c_i)\} \cup \underline{\{\neg\psi_i(c_i)\}}$$
4. Using $T^* = \bigcup T_i$, we can define a model $M^* \models T^*$. (Exactly by the same argument as in Compactness Theorem)
5. Any element $a \in M^*$ has a form $[c_n]$ (for some n). So a satisfies $\neg\psi_n$ and hence it does not realize Σ.

Existence of Small Models

Let T be a complete theory formulated in a countable L.

Corollary

Suppose that T is small. Then there is a model $M \models T$ such that if $N \models T$ is another model, then M can be elementarily embedded into N.

Such an M is called a prime model of T.

Proof.

1. By (an extended version of) Omitting Types Theorem, there is a model $M \models T$ that omits all non-isolated types over $\emptyset$.
2. For each n,

$$\Sigma_n(x_0, \ldots, x_n) = \{\varphi(x_0, \ldots, x_n) : M \models \varphi(m_0, \ldots, m_n)\}$$

   is an isolated type.
3. Let $N \models T$.
4. Chose $\varphi(x_0)$ isolating $\Sigma_0(x_0)$. Since $\exists x_0 \varphi(x_0)$ is true in M, it is also true in N. So, we can choose $n_0 \in$ satisfying φ.
5. The mapping $m_0 \mapsto n_0$ is a partial elementary embedding.
6. By continuing this, we get an elementary embedding $m_i \to n_i$ $(i \in \mathbb{N})$.

□

Summary

1. Lecture 1: L-structures
2. Lecture 2: Compactness
3. Lecture 3: Large Models and Small Models
    - Existence of κ saturated models — Application of Compactness
    - Existence of prime models — Application of Omitting Types Theorem

References

1. Model Theory: Third Edition (Dover Books on Mathematics) , 2012/6/13 C.C. Chang, H. Jerome Keisler.
2. Akito Tsuboi's web page.

Definition

A complete theory T is called small if there are only countably many types over $\emptyset$.

previous page

Definition

Let I be a set. $U \subset \mathcal{P}(I)$ is called an ultrafilter over I if

1. U has the finite intersection property, and
2. U is maximal among such.

Sets in U can be considered as 'large' subsets of I.

Fact

Let $\{M_i : i \in I\}$ be a set of L-structures. The product $\prod_{i \in I} M_i$ naturally becomes an L-structure.

For $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod M_i$, $(a_i)_{i \in I} \sim_U (b_i)_{i \in I}$ iff $\{i \in I : a_i = b_i\} \in U$. Then the set

$$\prod_{i \in I} M_i / \sim_U$$

becomes an L-structure, and is called the ultraproduct of M_i's.

previous page