Around the Kueker conjecture

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Contents

What is the Kueker conjecture?

Throughout T is a countable, complete theory with infinite models.

Definition. T is said to be λ-*categorical* if any two models of T of λ power are isomorphic.

Definition. A model M of T is said to be κ-*saturated* if for every subset A of M which $|A| < \kappa$ and type $p \in S(A)$, M realize p.

Fact. Every uncountable model of a categorical theory in some infinite power is $\aleph_0$-saturated.

Proof.

1. Countable categorical case.
2. Uncountable categorical case.

$\square$

Kueker Conjecture

A theory which every uncountable model is $\aleph_0$-saturated is categorical in some infinite power.

Definition. T is said to be *Kueker* if it is not $\aleph_0$-categorical and every uncountable model is $\aleph_0$-saturated.

Example(Position of Kueker theory).

	uncountable model is $\aleph_0$-saturated	$\aleph_0$-categorical	$\aleph_1$-categorical
$\mathrm{Th}(2^{<\omega})$	×	×	×
DLO, E^2_∞	◯	◯	×
$Vec_{<\omega}$	◯	◯	◯
ACF_p	◯	×	◯
Kueker	◯	×	?

Restated Kueker Conjecture

Kueker theory is $\aleph_1$-categorical.

Definition. Let $\varphi = \varphi(x) \in L(A)$.

1. φ is *almost complete over A* if there is unique non-algebraic type over A containing it i.e. φ is non-algebraic and for any formula $\psi = \psi(x)$ over A, $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ is algebraic.
2. A almost complete formula φ is *trivial* if the witness is isolated.
3. A almost complete formula φ over A is *non-trivial* if it is not trivial.

Definition. Let $p \in S(A)$. The *multiplicity* of p, denote $\mathrm{mult}(p)$, is the number of extensions of p over $\mathrm{acl}^{eq}(A)$, that is

$$\mathrm{mult}(p) := |\{q \in S(\mathrm{acl}^{eq}(A)) \mid p \subset q\}|.$$

Note. If $\mathrm{mult}(p) \geq \aleph_0$, then $\mathrm{mult}(p) = 2^{\aleph_0}$.

Definition. Let $A \subset B$.

1. B is *atomic over* A if for every $b \in B$, $\mathrm{tp}(a/A)$ is isolated.
2. B is *almost atomic over* A if for every $b \in B$ and $a \in A$, there is $a' \in A$ such that $\mathrm{tp}(b/aa')$ is isolated.

Definition. Let $\mathcal{C}$ be infinite set.

1. A type is *$\mathcal{C}$-coheir* if it is finitely satisfiable in $\mathcal{C}$.
2. A sequence $I = (a_i)_{i<\lambda}$ is *$\mathcal{C}$-coheir sequence* over A if for all $i < \lambda$, $\mathrm{tp}(a_i/Aa_{<i})$ is $\mathcal{C}$-coheir.

Remark. Since $\mathcal{C}$ is infinite, $\mathcal{C}$-coheir sequence is extensible.

General Results

Fact. Assume T is Keker theory.

1. T has no an uncountable model which is atomic over a finite set.
2. T is small.
3. The prime model of T over a finite set is minimal.
4. Non-trivial almost complete formulas are dense in the prime model.
5. Every isolated type over a finite set has finite multiplicity.

Previous Research

additional property	result	main researcher
ω-stable	◯	Lachlan
superstable	◯	Buechler
stable	◯	Hrushovski
simple	?	Shami
O-minimal	◯	Marker?
VC-minimal	◯	Guingona
NIP	?	Tanović
built-in Skolem functions	◯	Hrushovski
interpret a linear ordering	◯	Hrushovski
$\mathrm{dcl}(\emptyset)$ is infinite	?	Tanović
$\mathrm{acl}(\emptyset)$ is infinite	?	

Tanović's argument in dcl($\emptyset$)-infinite

Lemma. Let $\mathrm{tp}(a/A)$ be dcl($\emptyset$)-coheir, $\mathrm{tp}(b/A)$ be isolated type, B be almost atomic over A.

1. $\mathrm{tp}(a/A) \vdash \mathrm{tp}(a/Ab)$, in particular $\mathrm{tp}(b/Aa)$ is isolated.
2. $\mathrm{tp}(a/A) \vdash \mathrm{tp}(a/B)$, in particular B is almost atomic over Aa.

Lemma. Let $I = (a_i)_{i<\omega}$ and $J = (b_i)_{i<\omega_1}$ be dcl($\emptyset$)-coheir sequence.

1. There is an atomic model over I.
2. There is an almost atomic model over J.

Generalization to acl($\emptyset$)-infinite

Lemma. Let tp(a/A) be acl($\emptyset$)-coheir, tp(b/A) be isolated type, B be almost atomic over A.

1. tp(b/Aa) is isolated.
2. B is almost atomic over Aa.

Lemma. Let $I = (a_i)_{i<\omega}$ and $J = (b_i)_{i<\omega_1}$ be acl($\emptyset$)-coheir sequence.

1. There is an atomic model over I.
2. There is an almost atomic model over J.

Suitability of acl($\emptyset$)-infinite

Lemma. Let M be countable model and $p \in S(T)$. Assume that for all $N \succneqq M$, N realize p. Then there is $\psi(x, a) \in L(M)$ such that

- ψ^M is infinite,
- $\psi^M \subset \mathrm{acl}(a)$.

Propsition. If T is Kueker theory then there is finite set A such that T_A is Kueker theory with acl($\emptyset$)-infinite.

Corollary. If Kueker theory with acl($\emptyset$)-infinite is $\aleph_1$-categorical, then Kueker conjecture is hold.

Definition.

1. $\mathrm{acl}(\emptyset)$ is *dense* if for any non-algebraic formula $\varphi(x)$, there is $a \in \mathrm{acl}(\emptyset)$ such that $\models \varphi(a)$.
2. $\mathrm{acl}(\emptyset)$ is *rare* if for all $n < \omega$, the set of algebraic points $\mathrm{acl}_n(\emptyset)$ of degree n is finite, where

$$\mathrm{acl}_n(\emptyset) := \{a \in \mathrm{acl}(\emptyset) \mid |a^{\mathrm{Aut}(\mathcal{M})}| = n\}.$$

Theorem. Kueker conjecture in $\mathrm{acl}(\emptyset)$ is non-rare dense is hold.

Proof.

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Proof.

- By density of $\mathrm{acl}(\emptyset)$, T has $\exists^\infty$-elimination and for any infinite definable set D, $\mathcal{M} = \mathrm{acl}(D^{\mathcal{M}})$.

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Proof.

- By density of $\mathrm{acl}(\emptyset)$, T has $\exists^\infty$-elimination and for any infinite definable set D, $\mathcal{M} = \mathrm{acl}(D^{\mathcal{M}})$.
- Take $\mathrm{acl}_n(\emptyset)$-coheir sequence J of length ω_1 where n is witness of rarity of $\mathrm{acl}(\emptyset)$.

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- By density of $\mathrm{acl}(\emptyset)$, T has $\exists^{\infty}$-elimination and for any infinite definable set D, $\mathcal{M} = \mathrm{acl}(D^{\mathcal{M}})$.
- Take $\mathrm{acl}_n(\emptyset)$-coheir sequence J of length ω_1 where n is witness of rarity of $\mathrm{acl}(\emptyset)$.
- By existence of almost atomic model over J, for any type $\mathrm{tp}(a/A)$ over a finite set there are $a_{i_0}, \ldots, a_{i_{n-1}} \in J'(\equiv J)$ such that $\mathrm{tp}(a/Aa_{i_{<n}})$ is isolated.

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- Then $\mathrm{mult}(a/Aa_{i_{<n}}) < \omega$, so $\mathrm{mult}(a/A) < \omega$.

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- Then $\mathrm{mult}(a/Aa_{i_{<n}}) < \omega$, so $\mathrm{mult}(a/A) < \omega$.
- Consequently, there is a minimal set.

$\square$

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Fact. Every (uncountable) model of an $\aleph_0$-categorical theory is $\aleph_0$-saturated.

Proof.

- $\aleph_0$-categorical theory has a saturated model in countable cardinal.
- Let M be a uncountable model and $p(x)$ be a complete type over finite set A of M.
- Take countable model $M_0 \preceq M$ containing A.
- By $\aleph_0$-categoricity, there is $a \in M_0 \subset M$ such that $a \models p(x)$, so M is $\aleph_0$-saturated.

$\square$(back)

Fact. Every uncountable model of an $\aleph_1$-categorical theory is $\aleph_0$-saturated.

Proof.

- Uncountable categorical theory is ω-stable.
- ω-stable theory has a saturated model in every infinite cardinal.
- [Morley's uncountable categorical theorem]
  Uncountable categorical theory is λ-categorical in every uncountable cardinal λ.

$\square$(back)

The theory E^2_∞ is the set of the following formulas over the language $L = \{E\}$ where E is 2-ary relation symbol:

- E is a equivalence relation with two infinite E-classes.

Remark.

$$\begin{array}{ccc} \{M \mid M \models E^2_\infty\}/\cong & \xrightarrow{\approx} & \{(\lambda, \kappa) \mid \lambda, \kappa \text{ is cardinal}, \omega \leq \lambda \leq \kappa\} \\ M & \mapsto & (|a/E|, |b/E|) \text{ where } M \models \neg E(a, b) \\ & & \text{and } |a/E| \leq |b/E|. \end{array}$$

(back)

Fact 1. T has no an uncountable model which is atomic over a finite set.

Proof.

- If there is an uncountable model which is atomic over a finite set, say A.
- Since uncountable model is $\aleph_0$-saturated, every type over A is isolated.
- Then T_A is $\aleph_0$-categorical, hence T is so.

□(back)

Fact 2. T is small.

Proof.

- Take Ehrenfeucht-Mostowski model $EM(\aleph_1)$ of size $\aleph_1$.
- Since uncountable model is $\aleph_0$-saturated, $EM(\aleph_1)$ is so.
- $|S(T)| \leq |\{\mathrm{tp}(a) \mid a \in EM(\aleph_1)\}| \leq \aleph_0$.

$\square$(back)

Fact 3. The prime model of T over a finite set is minimal.

Proof.

- Suppose M_0 is prime over finite set and not minimal.
- Then there is $M_1 \succneqq M_0$ such that $M_1 \simeq M_0$.
- Repeat ω_1 times, put $M := \bigcup_{i<\omega_1} M_i$.
- Then M is uncountable model which is atomic over a finite set.

□(back)

Fact 4. Non-trivial almost complete formulas are dence in the prime model.

Proof.

- Let $\varphi(x, a)$ be non-algebraic, M be prime over a and take finite approximation of $M = \bigcup_{i<\omega} A_i$ with $a \in A_0$.
- By smallness, we can get the following a sequence $(\psi_i(x))_{i<\omega}$:
  - (i) $\psi_i \in L(a_i)$,
  - (ii) $\models \forall x(\psi_{i+1} \rightarrow \psi_i) \wedge \forall x(\psi_i \rightarrow \varphi)$,
  - (iii) ψ_i is almost complete over A_i.
- If all ψ_i is trivial, then M is not minimal.

$\square$ (back)

Fact 5. Every isolated type over a finite set has finite multiplicity.

Proof.

- Claim. Isolated type over finite set which has infinite multiplicity is extensible.
- Let p be a isolated type over finite set A which has infinite multiplicity.
- Take a prime model M over A and its finite approximation $M = \bigcup_{i<\omega} A_i$ where $A_0 = A$.
- By claim, take a suitable sequence of types $(p_i)_{i<\omega}$, and $a \models \bigcup_{i<\omega} p_i$.
- Then Ma is atomic over A, so M is not minimal.

□(back)