大学院生向け講義 Generic Structure について

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E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, preprint, 1988.

Hrushovski's pseudoplane

Hrushovski constructed an ω-categorical (merely) stable pseudoplane, which gives a negative answer to the following Lachlan's conjectures:

- (C3) There exists no ω-categorical pseudoplane.
- (C1) If T is stable and ω-categorical then T is totally transcendental.

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Many interesting examples were constructed using Hrushovski's method.

- An almost strongly minimal set interpreting two algebraically closed fields of different characteristics (Hrushovski).
- An almost strongly minimal non-desarguesian projective plane (Baldwin)
- Ikeda's minimal structure,
- Herwig's structure of weight omega, pause
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Outline

My talk today is based on:

References

1. Baldwin, John T.; Shi, Niandong, Stable generic structures, Ann. Pure Appl. Logic 79, No.1, 1-35 (1996).
2. Wilfrid Hodges, Model Theory (Encyclopedia of Mathematics and its Applications), Cambridge University Press, 2008
3. E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, unpublished notes, 1988.
4. Frank O. Wagner, Relational structures and dimensions, in Automorphisms of First-Order Structures (Oxford Logic Guides), Oxford Univ Pr on Demand, 1994

Graph

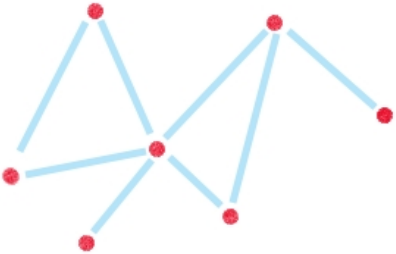
R is a binary relation symbol.

Definition

An R-structure G is said to be a graph if

- R is symmetric.
  $G \models \forall x \forall y [R(x, y) \to R(y, x)]$.
- R is irreflexive.
  $G \models \forall x [\neg R(x, x)]$.

Garph – Picture



A graph is something like this.

Remark

- There is an edge between vertices $a, b \in G$ if $R(a, b)$ holds in G.
- Our graph is an undirected graph.

Subgraphs

Let $G = (G, R^G)$ and $H = (H, R^H)$ be two graphs.

Subgraph

H is a subgraph of G if $H \subset G$ and $R^H \subset R^G$.

Full Subgraph

H is a full subgraph of G if $H \subset G$ and $R^H = R^G \cap H^2$.

Subgraphs

Let $\boldsymbol{G} = (G, R^G)$ and $\boldsymbol{H} = (H, R^H)$ be two graphs.

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Full Subgraph

$\boldsymbol{H}$ is a full subgraph of $\boldsymbol{G}$ if $H \subset G$ and $R^H = R^G \cap H^2$.

Notation

We simply write $H \subset G$ if H is a full subgraph.

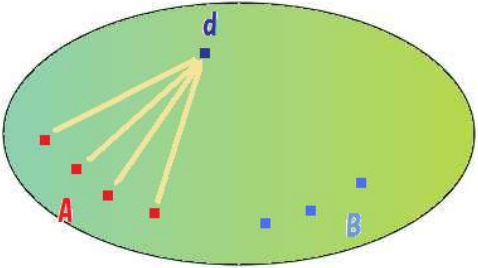
Random Graph

Definition

A graph $G = (G, R)$ is said to be a random graph if the following are satisfied

- For any two disjoint finite subsets $A, B \subset G$, there is $d \in G$ such that $G \models \bigwedge_{a \in A} R(a, d) \wedge \bigwedge_{b \in B} \neg R(b, d)$.

Random Graph – Picture



Remark

- If a random graph G exists, then it is an infinite graph:
  Suppose that G has n elements $a_1, ..., a_n$. Then there is $d \in G$ such that $\bigwedge R(a_i, d)$. By the irreflexiveness, $d \notin \{a_1, ..., a_n\}$.
- The axiom T_{RG} of random graphs can be expressed by an infinite set of first order sentences.

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- The axiom T_{RG} of random graphs can be expressed by an infinite set of first order sentences.

Existence

Theorem

A random graph exists.

A Proof

Proof.

1. Let G_0 be a one-point graph.
2. Inductively define $G_0 \subset G_1 \subset G_2 \cdots$ such that
   - for any $A, B \subset G_n$ ($A \cap B = \emptyset$) there is $d \in G_{n+1}$ such that $G_{n+1} \models \bigwedge_{a \in A} R(a, d) \wedge \bigwedge_{b \in B} \neg R(b, d)$.
3. $G = \bigcup_{n \in \omega} G_n$ is a (countable) random graph.

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Properties of Random Graphs

Theorem

A random graph embeds every finite graph (as a full subgraph).

Proof.

1. Let G be a random graph and H a finite graph.
2. Let $H = H_0 \cup \{e\}$. We can assume $H_0 \subset G$.
3. Let $A = \{a \in H_0 : R(a, e)\}$ and $B = \{b \in H_0 : R(b, e)\}$.
4. By T_{RG}, we can find $d \in G$ such that $G \models \bigwedge_{a \in A} R(a, d) \wedge \bigwedge_{b \in B} \neg R(b, d)$.
5. Then $H = H_0 \cup \{e\} \cong H_0 \cup \{d\} \subset G$.

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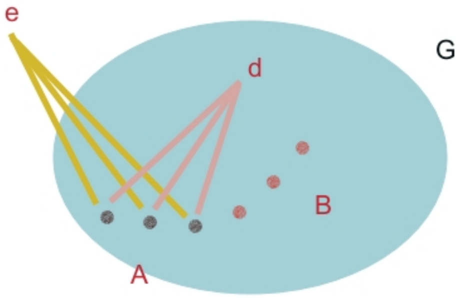
Embedding – Picture



A similar argument shows

Theorem

A random graph embeds every countable graph.

Theorem

T_{RG} is complete and ω-categorical.
In other words, any two countable random graphs are isomorphic.

Proof.

1. Use a back-and-forth argument.
2. Let $G = \{g_i : i \in \omega\}$ and $H = \{h_i : i \in \omega\}$ be two random graphs.
3. Construct finite partial isomporphisms $\sigma_i : G \to H$ such that
   - $\emptyset = \sigma_0 \subset \sigma_1 \subset \sigma_2 \cdots$,
   - $g_0, ..., g_j \in \mathrm{dom}(\sigma_i) \quad (j < i)$,
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4. $\sigma = \bigcup_{i \in \omega} \sigma_i$ is an isomorphism between G and H.

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□

Limit of Finite Graphs

A random graph can be considered as a limit of finite graphs.
Let K be the class of all (isomorphism types of) finite graphs.
A random graph G clearly has the following two properties:

1. Any finite $X \subset G$ is a member of K.
2. If $A \subset B \in K$ and $A \subset G$ then there is a copy $B' \subset G$ such that $B \cong_A B'$.

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Remark

Let us consider the following graphs G_n (finite random graph):

- $|G_n| = \{1, ..., n\}$ (verteces).
- Add edges between them at random.
  $\mathrm{Prob}(R(l, m)) = p = const, (l < m \leq n)$.

Then, for any R-sentence φ,

$$\lim_{n \to \infty} (\mathrm{Prob}(G_n \models \varphi)) = 1 \iff T_{RG} \models \varphi.$$

In particular,

$$\lim_{n \to \infty} (\mathrm{Prob}(G_n \models \varphi)) = 0 \ or \ 1,$$

(Fagin)

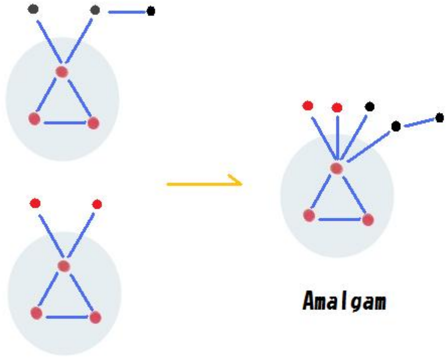
Now we work in a more general setting.

Class K

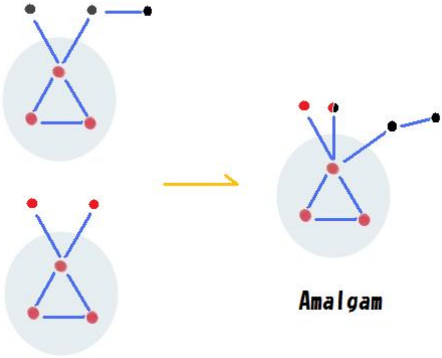
Let L be a (finite) relational language.
Let K be a class of (isomorphism types of) finite L-structures.
We assume the following:

- $\emptyset \in K$
- K is closed under substructures.
- AP (Amalgamation Property):
  Suppose that $A \subset B_1 \in K$ and $A \subset B_2 \in K$. Then there is $\exists C \in K$ such that
  - $A \subset C$,
  - $\exists B_1', B_2' \subset C$ s.t. $B_1' \cong_A B_1$, $B_2' \cong_A B_2$

Free Amalgamation – Picture



Amalgamation – Picture



Free amalgam of B_1, B_2 over A will be denoted by

$$B_1 \oplus_A B_2$$

Sometimes the free amalgama is written as $B_1 \otimes_A B_2$ or $B_1 \coprod_A B_2$, ...

The domain of $B_1 \oplus_A B_2$ is the disjoint union of B_1 and B_2 over A, and the relation on $B_1 \oplus_A B_2$ is the union of those on B_1 and B_2.

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Examples of K

Example

Let K_g be the class of all finite graphs. Then K_g clearly has the AP.

Example

Let K_{tfg} be the class of all triangle free finite graphs. Then K_{tfg} has the AP.
A triangle consists of three points a, b, c such that
$R(a, b) \wedge R(b, c) \wedge R(c, a)$.

Fraïssé Limit

Let K be a class of (isomorhism types of) finite L-structures. We always assume the following:

- $\emptyset \in K$
- K is closed under substructures.

Theorem

Suppose that $\boldsymbol{K}$ has the AP. Then there is a countable $\boldsymbol{L}$-structure $\boldsymbol{M}$ with the following properties:

1. *Any finite $\boldsymbol{X} \subset \boldsymbol{M}$ is a member of $\boldsymbol{K}$.*
2. *If $\boldsymbol{A} \subset \boldsymbol{B} \in \boldsymbol{K}$ and $\boldsymbol{A} \subset \boldsymbol{M}$ then there is a copy $\boldsymbol{B}' \subset \boldsymbol{M}$ such that $\boldsymbol{B} \cong_{\boldsymbol{A}} \boldsymbol{B}'$.*

A countable $\boldsymbol{L}$-strucute having the properties 1 and 2 will be called a Fraïssé Limit of $\boldsymbol{K}$.

Fraïssé Limit is universal and homogeneous.

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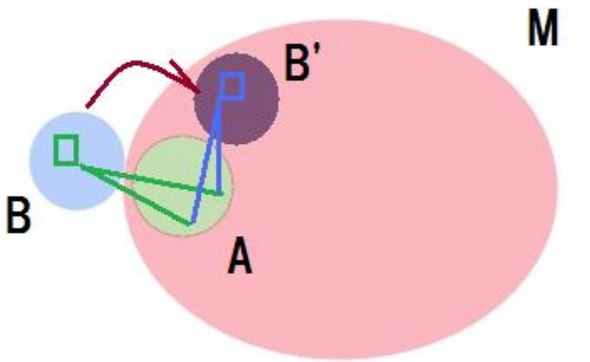
Property 2 – Picture



A Proof

Proof.

1. Let (A_i, B_i) $(i \in \omega)$ be an enumeration of all the pairs (A, B) with $A \subset B \in K$. (We assume any such pair appears infinitely many times.)
2. Using AP we can construct a sequence of finite L-structures $M_0 \subset M_1 \subset \cdots$ such that for any i
   - $M_i \in K$,
   - $A_i \cong A \subset M_i \Rightarrow \exists B$ s.t. $B_i \cong_A B \subset M_{i+1}$.
3. Then $M = \bigcup_{i \in \omega} M_i$ has the required properties.

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Uniqueness

Theorem

For given $\mathbf{K}$*, a Fraïssé Limit is unique up to isomorphism.*

Proof.

Use a back-and-forth argument. □

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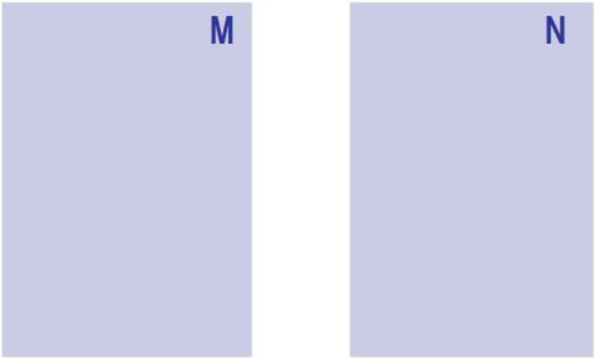
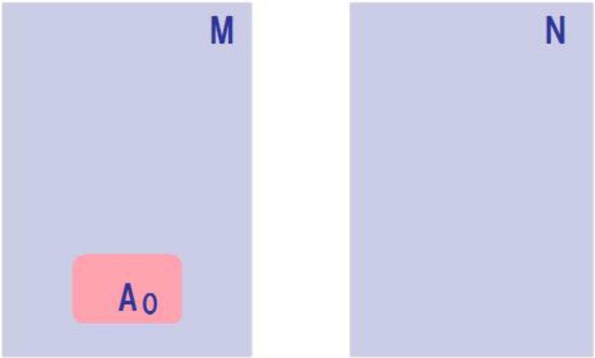
Back-and-forth



Back-and-forth



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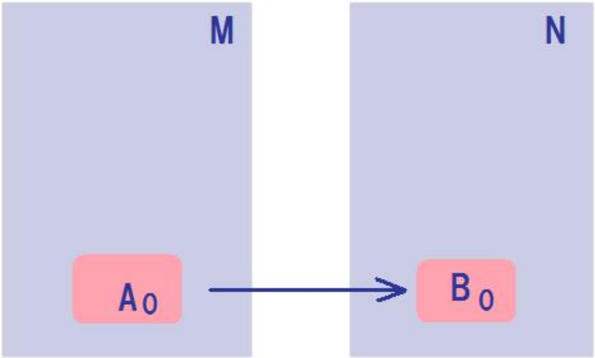

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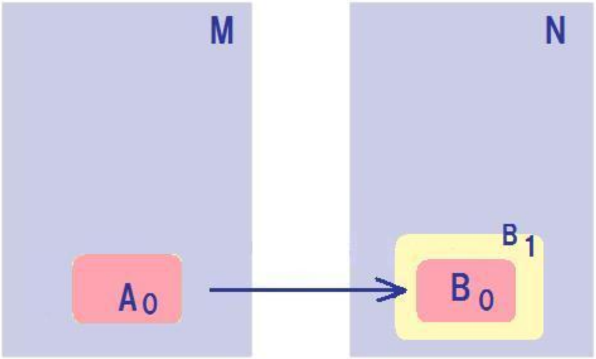

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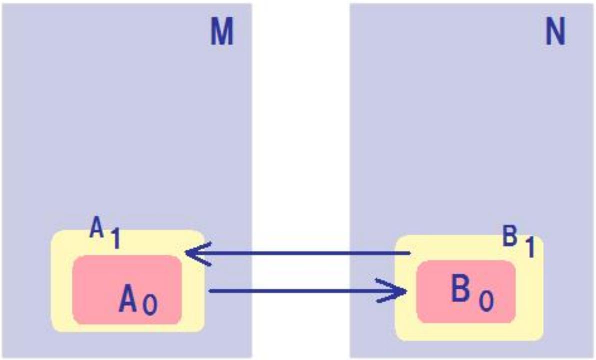

Example

Let K_g be the class of all finite graphs. Then a (countable) random graph is a K_g-Fraïssé Limit.

Example

Let K_{tfg} be the class of all triangle free finite graphs. Then there is a unique K_{tfg}-Fraïssé Limit.

Hrushovski Amalgamation

K with Predimension

As before,

- $L = \{P_1, ..., P_m\}$ is a (finite) relational language.

For simplicity, we only consider L-structures with

- $P_i(x_1, x_2, ..., x_{n_i}) \to \bigwedge_{j \neq k} x_j \neq x_k$
- $P_i(x_1, x_2, ..., x_{n_i}) \to P_i(x_{\sigma(1)}, ..., x_{\sigma(n_i)})$, where σ is a permutation of $\{1, ..., n_i\}$.

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Let $L = \{P_1, ..., P_n\}$.
Let $\alpha_1, ..., \alpha_n$ be positive real numbers. Mainly $0 < \alpha_i < 1$.

Definition

For a finite L-structure A, the predimension of A (with respect to $\alpha_1, ..., \alpha_n$) is defined by:

$$\delta(A) = \sum |A| - \alpha_i |P_i(A)|,$$

where $P_i(A)$ is the set of all n_i-element subsets $B \subset A$ satisfying P_i.

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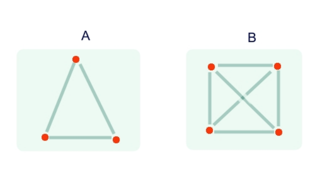

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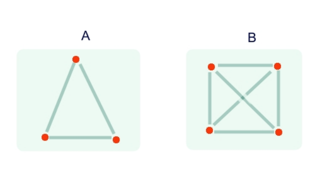

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Relative Predimension

Let A and B be subsets of a larger finite L-structure.

Definition

$$\delta(A/B) = \delta(AB) - \delta(B),$$

where AB denotes $A \cup B$.

Notice that $\delta(A/B) = \delta(A \setminus B/B)$.

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where AB denotes $A \cup B$.

Notice that $\delta(A/B) = \delta(A \setminus B/B)$.

From now on we assume $L = \{R\}$. This is for simplicity only.

Lemma

Let $A \cap B = \emptyset$.

1. $\delta(A/B) = \delta(A) - \alpha|R(A, B)|$,
   where $R(A, B)$ *denotes the set of all edges between* A *and* B.
2. *(Monotonicity)*
   $B_0 \subset B \Rightarrow \delta(A/B) \le \delta(A/B_0)$.

From this, we know that if $A \cap B = A \cap C$ and $B \subset C$ then $\delta(A/B) \ge \delta(A/C)$.

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$\delta(A/B) = \delta(A) - \alpha|R(A,B)|$

Proof.

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1. $B_0 \subset B$ implies $R(A, B_0) \subset R(A, B)$.
2. So $|R(A, B_0)| \leq |R(A, B)|$.
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Strong Subset

Definition

Let $\boldsymbol{A} \subset \boldsymbol{B}$ be finite $\boldsymbol{L}$-structures. We write $\boldsymbol{A} \leq \boldsymbol{B}$ if

$$\boldsymbol{A} \subset \boldsymbol{C} \subset \boldsymbol{B} \Rightarrow \delta(\boldsymbol{C}/\boldsymbol{A}) \geq \boldsymbol{0} \quad (\forall \boldsymbol{C}).$$

If $\boldsymbol{A} \leq \boldsymbol{B}$, we say (i) $\boldsymbol{A}$ is a strong subset of $\boldsymbol{B}$ or (ii) $\boldsymbol{A}$ is closed in $\boldsymbol{B}$.

Strong Subset

Definition

Let $A \subset B$ be finite L-structures. We write $A \leq B$ if

$$A \subset C \subset B \Rightarrow \delta(C/A) \geq 0 \quad (\forall C).$$

If $A \leq B$, we say (i) A is a strong subset of B or (ii) A is closed in B.

K_α

$L = \{R\}$. $\delta(A) = |A| - \alpha|R(A)|$.

$$K_\alpha = \{A : \emptyset \leq A\}.$$

Clearly K_α is closed under substructures. We consider K_α with $\leq$ (strong subset relation).

Properties of $(K_\alpha, \leq)$

Lemma

1. *$\leq$ is an order on K_α.*
2. $\emptyset \leq A$
3. $A \leq B, C \subset B \Rightarrow A \cap C \leq C.$
4. *In particular,* $A \leq B, A \subset C \subset B \Rightarrow A \leq C.$

$\leq$ is an order on K_α.

Proof.

1. It suffices to prove transitivity.
2. Let $A_0 \leq A_1 \leq A_2$ and $A_0 \subset X \subset A_2$.
3. $\delta(X/A_0) = \delta(X/X \cap A_1) + \delta(X \cap A_1/A_0)$
4. $\geq \delta(X/X \cap A_1)$
5. $\geq \delta(X/A_1)$ (Monotonicity)
6. ≥ 0.
7. So $A_0 \leq A_2$.

□

$A \le B, C \subset B \Rightarrow A \cap C \le C$.

Proof.

1. Assume $A \le B$, $C \subset B$.
2. Let $A \cap C \subset X \subset C$.
3. $\delta(X/A \cap C) = \delta(X \setminus A/A \cap C)$
4. $\ge \delta(X \setminus A)$
5. ≥ 0.
6. This shows $A \cap C \le C$.

□

Amalgamation Property

Lemma

Let $A \leq B \in K_\alpha$ and $A \leq C \in K_\alpha$. Then $D = B \oplus_A C$ has the following properties:

1. $D \in K_\alpha$
2. $B \leq D$ *and* $C \leq D$.

$B \leq D$ and $C \leq D$.

Proof.

1. We want to show $B \leq D$.
2. Let $B \subset X \subset D$. We show $\delta(X/B) \geq 0$.
3. $\delta(X/B) = \delta(X \setminus B) - \alpha|R(X \setminus B, B)|$
4. $= \delta(X \setminus B) - \alpha|R(X \setminus B, A)|$ (by freeness)
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$D = B \oplus_A C \in K_\alpha$

1. Let $X \subset D$. We want to show $\delta(X) \geq 0$.
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$(K_\alpha, \leq)$-generic structure

Theorem

There is a countable structure $\boldsymbol{M}$ having the following properties:

1. *Any finite $\boldsymbol{X} \subset \boldsymbol{M}$ is a member of $\boldsymbol{K}_\alpha$.*
2. *If $\boldsymbol{A} \leq \boldsymbol{B} \in \boldsymbol{K}$ and $\boldsymbol{A} \leq \boldsymbol{M}$ then there is a copy $\boldsymbol{B}' \leq \boldsymbol{M}$ such that $\boldsymbol{B} \cong_{\boldsymbol{A}} \boldsymbol{B}'$.*

$\boldsymbol{A} \leq \boldsymbol{M}$ is an abbreviation of the statement $\boldsymbol{A} \leq \boldsymbol{F}$ $(\forall \boldsymbol{F} \subset_{\mathrm{fin}} \boldsymbol{M})$.

A countable structure $\boldsymbol{M}$ having the propeties 1 and will be referred as a $(K_\alpha, \leq)$-generic structure.

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The existence of $(\boldsymbol{K}_\alpha, \leq)$-generic structure can be shown exactly the same way as in the case of random graph. The point is that $(\boldsymbol{K}_\alpha, \leq)$ has the AP.

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Conclusion

Theorem

Let $(\boldsymbol{K}, \leq)$ *be a subclass of* $(\boldsymbol{K}_\alpha, \leq)$ *with the AP. Then there is a* $(\boldsymbol{K}, \leq)$*-generic structure.*

Conclusion

Or more generally:

Theorem

Let $(\mathbf{K}, \leq)$ be a class of finite $\mathbf{L}$-structures satisfying AP (+ some necessary conditions on $\leq$). Then there is a $(\mathbf{K}, \leq)$-generic structure $\boldsymbol{M}$ such that

1. *Any finite $\boldsymbol{X} \subset \boldsymbol{M}$ is a member of $\mathbf{K}$.*
2. *If $\boldsymbol{A} \leq \boldsymbol{B} \in \mathbf{K}$ and $\boldsymbol{A} \leq \boldsymbol{M}$ then there is a copy $\boldsymbol{B}' \leq \boldsymbol{M}$ such that $\boldsymbol{B} \cong_{\boldsymbol{A}} \boldsymbol{B}'$.*

Uniqueness

Closed Finite Sets

For showing the uniquenss of a (countable) $(\boldsymbol{K}, \leq)$-generic structure, we need to construct a tower of parital isomorphisms between closed finite subsets.

Again we assume $\boldsymbol{K} \subset \boldsymbol{K}_\alpha$.

Uniqueness

Closed Finite Sets

For showing the uniquenss of a (countable) $(K, \leq)$-generic structure, we need to construct a tower of parital isomorphisms between closed finite subsets.

Again we assume $K \subset K_\alpha$.

Let M be an L-structure such that every finite $A \subset M$ belongs to K.

Definition

Let $A \subset M$. A is called a closed subset of M if for any finite $B \subset M$,

$$A \cap B \leq B.$$

If A_0 is finite then A_0 is closed $\iff$ $A_0 \leq M$. So, even if A is infinite, we write $A \leq M$ if A is closed in M.

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Remark

1. M itself is a closed set.
2. $C_1 \leq M$ and $C_2 \leq M$ then $C_1 \cap C_2 \leq M$.
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Remark

4. *Suppose that there is no infinite sequence* $A_0 \subset A_1 \subset A_2 \cdots$ *of* K*-sets such that*

$$\delta(A_0) > \delta(A_1) > \cdots .$$

Then, for any $N \equiv M$*,*

$$A \subset_{\text{fin}} N \Rightarrow \overline{A} \subset_{\text{fin}} N.$$

Finite Closure Property

Example

1. For $\alpha \in Q$, K_α has the finite closure property.
2. Suppose that there is an increasing function $f : \omega \to \mathbb{R}$ such that
   - $\lim_{n\to\infty} f(n) = \infty$
   - $A \in K \Rightarrow \delta(A) \geq f(|A|)$.

   Then K has the finite closure property.

Conclusion

Theorem

Let $(\boldsymbol{K}, \leq)$ be a class of finite $\mathbf{L}$-structures satisfying AP+ Finite Closure Property. Then there is a unique $(\boldsymbol{K}, \leq)$-generic structure $\boldsymbol{M}$:

1. *Any finite $\boldsymbol{X} \subset \boldsymbol{M}$ is a member of $\boldsymbol{K}_\alpha$.*
2. *If $\boldsymbol{A} \leq \boldsymbol{B} \in \boldsymbol{K}$ and $\boldsymbol{A} \leq \boldsymbol{M}$ then there is a copy $\boldsymbol{B}' \leq \boldsymbol{M}$ such that $\boldsymbol{B} \cong_{\boldsymbol{A}} \boldsymbol{B}'$.*

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Let $(K, \leq)$ be a class of finite L-structures satisfying AP+ Finite Closure Property. Then there is a unique $(K, \leq)$-generic structure M:

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On Saturation of Generic Structures

A generic structure need not to be ω-saturated.
We assume that $(K, \leq)$ has AP and Finite Closure Propery.

Theorem

Let M be a $(K, \leq)$-generic structure. The following conditions are equivalent:

1. *M is ω-saturated.*
2. *For any $N \equiv M$, $A \leq N$, $A \leq B \in K$ and $n \in \omega$, there is $B' \leq_n N$ such that $B \cong_A B'$.*

$X \leq_n Y$ (n-closedness) is the statement that $X \leq XZ$ for any $Z \subset Y$ with $|Z| \leq n$.

1 implies that any N is n-generic.

Proof: 1 ⇒ 2.

1. Suppose that 2 is not the case.
2. For some $n \in \omega$, $A \leq B \in K$, The following set $\Gamma(X)$ is consistent with $T = Th(M)$.
   - $X \cong A$ is a closed set
   - $(X, Y) \cong (A, B) \Rightarrow Y$ is not n-closed, for any Y.
3. By saturation, there is $A' \subset M$ realizing Γ. But then M is not a generic structure.

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Proof: 1 ⇒ 2.

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2. For some $n \in \omega$, $A \leq B \in K$, The following set $\Gamma(X)$ is consistent with $T = Th(M)$.
   - $X \cong A$ is a closed set
   - $(X, Y) \cong (A, B) \Rightarrow Y$ is not n-closed, for any Y.
3. By saturation, there is $A' \subset M$ realizing Γ. But then M is not a generic structure.

□

2 implies that M is saturated.

Proof: 2 ⇒ 1.

1. Condition 2 implies that any ω-saturated model N of T has the following property:

$$A \leq B \in K,\ A \leq N \Rightarrow \exists B' \leq N, B \cong_A B'.$$

2. So any finite partial isomorphism σ between closed sets $A \leq M$ and $A_1 \leq N$ can be extended to $\sigma^* : M \to N$, $\sigma^*(M) \prec N$.
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Conclusion

If a generic structure M is ω-saturated, then any κ-saturated $N \equiv M$ has the following property:

$$A \leq N, A \leq B \in^{*} K, |B| < \kappa \Rightarrow \exists B' \leq N, B' \cong_{A} B.$$

Let M be a $(K, \leq)$-generic structure, where $(K, \leq)$ has the finite closure property. Let $N \equiv M$.

Definition (Dimension)

Let $A \subset N$ be a finite set.

$$d(A) = \inf\{\delta(B) : A \subset B \subset_{\text{fin}} N\} = \delta(\overline{A})$$

Definition (Relative Dimension)

$d(a/A) := d(aA) - d(A)$ (A is finite).

Lemma (Monotonicity)

1. $A \subset B \Rightarrow d(A) \leq d(B)$.
2. $a \subset b \Rightarrow d(a/A) \leq d(b/A)$.
3. $A \subset B \Rightarrow d(a/A) \leq d(a/B)$.

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Proof.

1. $d(A) = \inf\{\delta(X) : A \subset X\}$, $d(B) = \inf\{\delta(X) : B \subset X\}$.
2. Since $A \subset B$, $\{\delta(X) : A \subset X\} \supset \{\delta(X) : B \subset X\}$.
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Remark

By monotonicity, for not necessarily finite A, we can define

$$d(a/A) = \inf\{d(a/A_0) : A_0 \subset_{\text{fin}} A\}.$$

Lemma

Let A, B, C *be closed finite sets with* $A = B \cap C$. *Suppose* $d(B/C) = d(B/A)$. *Then* $BC = B \oplus_A C$ *and* BC *is closed.*

Proof.

1. $d(BC) = d(B/C) + d(C)$
2. $= d(B/A) + d(C)$
3. $= d(B) + d(C) - d(A)$
4. $= \delta(B) + \delta(C) - \delta(A)$
5. $\geq \delta(BC)$.
6. Since $d(BC) \leq \delta(BC)$, we have $d(BC) \leq \delta(BC)$, hence BC is closed.
7. By $\delta(B) + \delta(C) - \delta(A) = \delta(BC)$, $R(BC) \subset R(B) \cup R(C)$, so $BC = B \oplus_A C$.

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By an $\varepsilon - \delta$ type argument, for not necessarily finite A and $A_0 \subset A$, we have the following:

(*) Suppose

$$d(a/A_0) = d(a/A) \text{ and } \overline{A_0 a} \cap \overline{A} = \overline{A_0}.$$

Then

$$\overline{Aa} = \overline{A_0 a} \oplus_{\overline{A_0}} \overline{A} \leq N.$$

Theorem

Let M be an ω-saturated $(K, \leq)$-generic structure. Then $T = Th(M)$ is stable.

We work in a sufficiently saturated $N \equiv M$.

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Proof of Stability

Proof.

1. Let A be a closed subset of N with $|A| = 2^\omega$.
2. We show that $|S(A)| = 2^\omega$.
3. Let $\mathrm{tp}(a/A) \in S(A)$. We can choose a countable closed A_0 such that $d(a/A_0) = d(a/A)$ and $\overline{A_0 a} \cap A = A_0$.
4. Then $\overline{Aa} = \overline{A_0 a} \oplus_{A_0} A \leq N$.
5. So the information of $\mathrm{tp}(a/A)$ is completely included in $\mathrm{tp}(a/A_0)$.
6. So $|S(A)| = |A^\omega| \times |S(A_0)| = 2^\omega$.

□

Hrushovski’s pseudoplane

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1. What is pseudoplane?
2. What is K in this case?

Pseudoplane

A pseudoplane is a triple (P, L, I) with the following properties:

- Every line $l \in L$ has infinitely many points $p \in P$.
- (Its dual) Every point $p \in P$ lies on infinitely many lines $l \in L$.
- For any distinct points $p \neq q \in P$, at most finite number of lines $l \in L$ pass both p and q.
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For defining K (a class of finite graphs), Hrushovski defined a function $f : \omega \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

1. f increases very slowly.
2. $\lim f = \infty$
3. $f(4) > 4 - 4\alpha = \delta(\square)$.

K is the class of all finite graphs A such that, for every $A_0 \subset A$, $f(|A_0|) \leq \delta(A_0)$.
From 1, we have the (free) AP.
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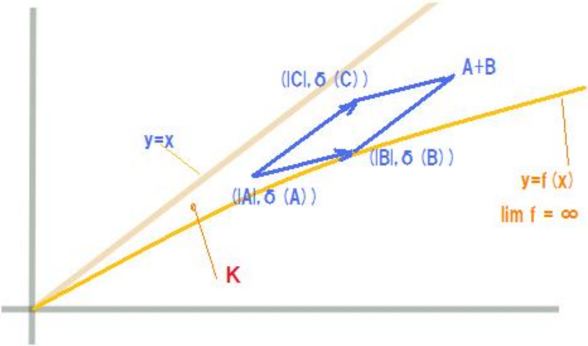
A+B
(|C|, δ (C))
y=x
(|B|, δ (B))
(|A|, δ (A))
y=f (x)
lim f = ∞
K

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