Equivariant definable Morse functions in definably complete structures

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  We say that $\mathcal{N}$ is *definably complete* if every nonempty definable subset A of R, $\sup A, \inf A \in R \cup \{\pm\infty\}$. Every o-minimal expansion of R is definably complete.

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  A weakly o-minimal structure is not always definably complete. For example $(\mathbb{R}_{alg}, +, \cdot, <, (-\pi, \pi) \cap \mathbb{R}_{alg})$ is weakly o-minimal but not definably complete.
  If R is the field $\mathbb{R}$ of real numbers, then every expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ is definably complete.

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  In this presentation everything is considered in $\mathcal{M}$ and every definable map is continuous unless otherwise stated.

Polynomially bounded and exponential o-minimal structures

- An o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of $\mathcal{R}$ is *polynomially bounded* if for every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ definable in $\mathcal{M}$, there exist an integer $N \in \mathbb{N}$ and a real number x_0 such that $|f(x)| < x^N$ for all $x > x_0$.

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  We say that $\mathcal{M}$ is *exponential* if the exponential function $\mathbb{R} \longrightarrow \mathbb{R}, x \mapsto e^x$ is definable in $\mathcal{M}$.

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Theorem (Miller 1994)

Every o-minimal expansion of $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of $\mathcal{R}$ is either polynomially bounded or exponential.

Definable C^r diffeomorphisms

- Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be definable open sets and $2 \leq r \leq \infty$. A C^r map $f : U \to V$ is a *definable C^r map* if the graph of f is definable.

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  In this presentation, we assume that $2 \leq r \leq \infty$ unless otherwise stated.

Definition

A Hausdorff space X is an *n-dimensional definable C^r manifold* if there exist a **finite** open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X, **finite** open sets $\{V_\lambda\}_{\lambda \in \Lambda}$ of $\mathbb{R}^n$, and **finite** homeomorphisms $\{\phi_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that for any λ, ν with $U_\lambda \cap U_\nu \neq \emptyset$, $\phi_\lambda(U_\lambda \cap U_\nu)$ is definable and $\phi_\nu \circ \phi_\lambda^{-1} : \phi_\lambda(U_\lambda \cap U_\nu) \to \phi_\nu(U_\lambda \cap U_\nu)$ is a definable C^r diffeomorphism.

- This pair $(U_\lambda, \phi_\lambda)$ of sets and homeomorphisms is called a *definable C^r coordinate system*.

Definable C^r manifolds and affine definable C^r manifolds

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- This pair $(U_\lambda, \phi_\lambda)$ of sets and homeomorphisms is called a *definable C^r coordinate system*.
- A definable C^r manifold is *affine* if it is definably C^r diffeomorphic to a definable C^r submanifold of some $\mathbb{R}^n$.

Nash manifolds and affine Nash manifolds

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Theorem (Shiota (1986))

Any compact C^∞ manifold of positive dimension admits uncountably many nonaffine Nash manifold structures.

Examples of definable C^∞ manifolds

Example

(1) The n*-dimensional unit sphere* $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$ *is an* n*-dimensional definable* C^∞ *manifold.*
(2) $T^2 = S^1 \times S^1$ *is a* 2*-dimensional definable* C^∞ *manifold.*
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- The above examples are affine Nash manifolds.
  If $\mathcal{M}$ is exponential, then the graph of the exponential function e^x is a definable C^∞ submanifold of $\mathbb{R}^2$.

Theorem ((2005))

For every o-minimal expansion $\mathcal{M}$ of $\mathcal{R}$, every definable C^r manifold is affine when $0 \leq r < \infty$.

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Imbeddings of definable C^∞ manifolds

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- The assumption that $\mathcal{M}$ is exponential is necessary for Fischer's theorem.

Critical points and critical values

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  We say that a point $p \in X$ is a *critical point* of f if the differential of f at p is zero.
  If p is a critical point of f, then $f(p)$ is called a *critical value* of f.

- Let p be a critical point of f and (U, u) a definable C^r coordinate system on X at p (i.e. U is a definable open subset of X containing p and u is a definable C^r diffeomorphism from U onto a definable open subset of $\mathbb{R}^n$ with $u(p) = 0$).

Nondegenerate critical points

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  The critical point p is *nondegenerate* if the Hessian matrix of $f \circ u^{-1}$ at 0 is nonsingular.
  Direct computations show that the notion of nondegeniricity does not depend on the choice of a local coordinate system.
  In the non-equivariant setting, Y. Peterzil and S. Starchenko (2007) introduced definable C^r Morse functions in an o-minimal expansion of the standard structure of a real closed field.

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  A pair (X, ϕ) consisting of a definable C^r manifold X and a group action $\phi : G \times X \to X$ is a *definable C^rG manifold* if ϕ is a definable C^r map.
  We simply write X to mean a definable C^rG manifold

- A *representation* of a definable C^r group G means a definable C^r group homomorphism from G to some orthogonal group $O_n(\mathbb{R})$ and the *representation space* of this representation is $\mathbb{R}^n$ with the orthogonal action induced from the representation.
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  A G invariant definable C^r submanifold of a representation space of G is called a *definable C^rG submanifold*.

Affine definable $C^r G$ manifolds

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  In this presentation, we always assume that every representation is orthogonal.
  A G invariant definable C^r submanifold of a representation space of G is called a *definable $C^r G$ submanifold*.
  A definable $C^r G$ manifold is *affine* if it is definably $C^r G$ diffeomorphic to some definable $C^r G$ submanifold of a representation space of G.

Imbeddings of compact definable $C^\infty G$ manifolds

Theorem ((1999))

Let G be a compact affine definable C^∞ group. Then every compact definable $C^\infty G$ manifold is affine.

Equivariant definable Morse functions

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  A closed definable C^rG submanifold Y of X is called a *critical manifold* (resp. a *nondegenerate critical manifold*) of f if each $p \in Y$ is a critical point (resp. a nondegenerate critical point) of f.
  We say that f is an *equivariant definable Morse function* if the critical locus of f is a finite union of nondegenerate critical manifolds of f without interior.

Theorem (A)

Let G be a compact affine definable C^r group and f an equivariant definable Morse function on a compact affine definable C^rG manifold X. If f has no critical value in $[a,b]$, then $f^a := f^{-1}((-\infty, a])$ is definably C^rG diffeomorphic to $f^b := f^{-1}((-\infty, b])$.

Definable C^r topology

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  For each $f \in Def^r(\mathbb{R}^n)$ and for each positive definable function $\epsilon : \mathbb{R}^n \to \mathbb{R}$, the ϵ-neighborhood $N(f;\epsilon)$ of f in $Def^r(\mathbb{R}^n)$ is defined by
  $\{h \in Def^r(\mathbb{R}^n) \mid |\partial^\alpha(h-f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \leq r\}$,
  where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, |\alpha| = \alpha_1 + \cdots + \alpha_n, \partial^\alpha F = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. We call the topology defined by these ϵ-neighborhoods the *definable C^r topology*.

Theorem (La Le Loi (2006))

Let X be a definable C^r submanifold of $\mathbb{R}^n$ and $2 \leq r < \infty$. Then the set of definable C^r functions on $\mathbb{R}^n$ which are Morse functions on X and have distinct critical values are open and dense in $Def^r(\mathbb{R}^n)$ with respect to the definable C^r topology.

Density theorem

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- Remark that the definable C^r topology and the C^r Whitney topology do not coincide in general when $0 \leq r < \infty$.
  If X is compact and $0 \leq r < \infty$, then these topologies of the set $Def^r(X)$ of definable C^r functions on X are the same (Shiota (1997)).

Density of equivariant Morse functions

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Theorem (B-(1))

Let G be a compact definable C^r group and X a compact affine definable $C^r G$ manifold.
(1) The set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^r_{inv}(X)$ of G invariant C^r functions on X with respect to the C^r Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^r_{inv}(X)$ of G invariant definable C^r functions with respect to the definable C^r topology.

Theorem (B-(2))

Let G be a compact definable C^∞ group and X a compact affine definable $C^\infty G$ manifold.
(2) If $\mathcal{M}$ is exponential, then the set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^\infty_{inv}(X)$ of G invariant C^∞ functions on X with respect to the C^r Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^\infty_{inv}(X)$ of G invariant definable C^∞ functions with respect to the definable C^r topology.

Definable G CW complexes

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A *definable set with a definable G action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \longrightarrow X$ such that θ is a definable map. This action is not necessarily linear (orthogonal).

Definable G CW complexes

Definable $G\ CW$ complexes are introduced in 2004.
Let G be a definable group. A *definable G set* is a G invariant definable subset of a representation space of G. This action is orthogonal.
A *definable set with a definable G action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \longrightarrow X$ such that θ is a definable map. This action is not necessarily linear (orthogonal).
A *definable G space* is a pair (X, θ) consisting of a definable space X and a group action $\theta : G \times X \longrightarrow X$ such that θ is a definable map.
We simply write X instead of (X, θ).

Definable $G\ CW$ complexes

Definition

Let G be a compact definable group.
(1) An open definable $G\ CW$ complex is a pair of $(X, \{c_i | i \in I\})$ consisting of a Hausdorff definable G space X and a finite family of open G cells $\{c_i | i \in I\}$ such that
(a) The underlying space $|X|$ of X is a definable set with a definable G action.
(b) The orbit space X/G is a definable subset of some $\mathbb{R}^n$.
(c) For each open G n-cell c_i, there exist a definable subgroup H_{c_i} of G and the characteristic map $f_{c_i} : G/H_{c_i} \times \Delta \to \overline{c_i} \subset X$ such that $f_{c_i}|G/H_{c_i} \times Int\ \Delta \to c_i$ is a definable G homeomorphism and the boundary ∂c_i is equal to $f_{c_i}(G/H_{c_i} \times \partial\Delta)$, where Δ is a subset of the standard compact n-simplex Δ^n obtained by removing some open lower dimensional faces of Δ^n, $\overline{c_i}$ denotes the closure of c_i in X, $Int\ \Delta$ means the interior of Δ and $\partial\Delta = \Delta - Int\ \Delta$.
(d) For each c_i, $\overline{c_i} - c_i$ is a finite union of open G cells.

Definable G CW complexes

Definition

(2) *An open definable G CW complex is called a complete definable G CW complex if every Δ is a standard compact simplex.*

Note that a complete definable G CW complex is a compact finite G CW complex.

Theorem (2004, 2008)

Let $\mathcal{M}$ be an o-minimal expansion of $\mathcal{R}$. Let G be a compact definable group and X a definable G manifold.

1. *X is definably G homeomorphic to an open definable G CW complex.*
2. *If X is compact, then X is definably G homeomorphic to a complete definable G CW complex. In particular, X is G homeomorphic to a finite G CW complex.*

Remark

In the above theorem, the assumption that $\mathcal{M}$ is o-minimal is necessary. If $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \mathbb{Z})$, then the above theorem does not hold even when G is the trivial group because a definable set $\mathbb{Z}$ is not homeomorhic to a finite union of open cells.

Theorem (C)

Let X be an n-dimensional compact definable C^r manifold having a definable Morse function $f : X \longrightarrow \mathbb{R}$ with only two critical points. Then X is definably homeomorphic to the n-dimensional unit sphere S^n. If $n \leq 6$, then X is definably C^r diffeomorphic to S^n. If $\mathcal{M}$ is exponential, then we can take $r = \infty$.

Theorem (C)

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Remark that if $n = 7$, then there exsits a C^∞ manifold which is homeomorphic to S^7, but not C^∞ diffeomorphic to S^7

Idea of Proof of Theorem A

Theorem (A)

Let G be a compact affine definable C^r group and f an equivariant definable Morse function on a compact definable C^rG manifold X. If f has no critical value in $[a, b]$, then $f^a := f^{-1}((-\infty, a])$ is definably C^rG diffeomorphic to $f^b := f^{-1}((-\infty, b])$.

Idea of Proof of Theorem A

Theorem (A)

Let G be a compact affine definable C^r group and f an equivariant definable Morse function on a compact definable $C^r G$ manifold X. If f has no critical value in $[a, b]$, then $f^a := f^{-1}((-\infty, a])$ is definably $C^r G$ diffeomorphic to $f^b := f^{-1}((-\infty, b])$.

We cannot apply the classical proof because the integration of a definable C^r function is not necessarily definable.

Idea of Proof of Theorem A and C

We prove the following theorem.

Theorem

Let X and Y be compact affine definable $C^r G$ manifolds possibly with boundary. Then X and Y are $C^1 G$ diffeomorphic if and only if they are definably $C^r G$ diffeomorphic. If $\mathcal{M}$ is exponential, then we can take $r = \infty$.

Idea of Proof of Theorem A and C

The following theorem is useful to prove the above theorem.

Theorem

Let G be a compact definable C^r group and X a compact affine definable $C^r G$ manifold and $1 \leq r < \infty$. Suppose that A, B are G invariant definable disjoint closed subsets of X. Then there exists a G invariant definable C^r function $f : X \to \mathbb{R}$ such that $f|A = 1$ and $f|B = 0$. If $\mathcal{M}$ is exponential, we can take $r = \infty$.

To prove this theorem, we need the assumption that $\mathcal{M}$ is exponential if $r = \infty$.

Idea of Proof of Theorem B

Theorem (B-(1))

Let G be a compact definable C^r group and X a compact affine definable $C^r G$ manifold.
(1) The set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^r_{inv}(X)$ of G invariant C^r functions on X with respect to the C^r Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^r_{inv}(X)$ of G invariant definable C^r functions with respect to the definable C^r topology.

Idea of Proof of Theorem B

Theorem (B-(2))

Let G be a compact definable C^∞ group and X a compact affine definable $C^\infty G$ manifold.
(2) If $\mathcal{M}$ is exponential, then the set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^\infty_{inv}(X)$ of G invariant C^∞ functions on X with respect to the C^r Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^\infty_{inv}(X)$ of G invariant definable C^∞ functions with respect to the definable C^r topology.

Idea of Proof of Theorem B

Let G be a compact definable C^r group. Let f be a map from a $C^r G$ manifold X to a representation Ω of G. Denote the Haar measure of G by dg and let $C^r(X, \Omega)$ denote the set of C^r maps from X to Ω. Define

$$A : C^r(X, \Omega) \to C^r(X, \Omega), A(f)(x) = \int_G g^{-1} f(gx) dg.$$

We call A the *averaging function*. In particular, if $G = \{g_1 \dots, g_n\}$, then $A(f)(x) = \frac{1}{n} \sum_{i=1}^{n} g_i^{-1} f(g_i x)$.

Idea of Proof of Theorem B

To prove Theorem B, we use the following proposition.

Proposition

Let G be a compact definable C^r group.
(1) $A(f)$ is equivariant, and $A(f) = f$ if f is equivariant.
(2) If $f \in C^r(X, \Omega)$, then $A(f) \in C^r(X, \Omega)$.
(3) If f is a polynomial map, then so is $A(f)$.
(4) If X is compact, then $A : C^r(X, \Omega) \longrightarrow C^r(X, \Omega)$ is continuous in the C^r Whitney topology.
(5) If G is a finite group, X is a definable C^rG manifold and f is a definable C^r map, then $A(f)$ is a definable C^rG map.

Questions

To extend in the real closed field case.

Thank you very much.