On Small Models

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Outline

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Notations

1. T is a countable theory formulated in the language L.
2. We work in a very saturated model $\mathcal{M}$. Models are elementary submodels of $\mathcal{M}$.
3. $a, b, \ldots$: finite tuples from $\mathcal{M}$.
4. $A, B, \ldots$: (small) subsets of $\mathcal{M}$.
5. φ, ψ,...: L-formulas (possibly with parameters).
6. p, q, ... : complete types.

Definition

1. A finite tuple $\boldsymbol{a}$ is said to be almost atomic over $\boldsymbol{A}$, if for each finite set $\boldsymbol{A_0} \subset \boldsymbol{A}$, there is a finite $\boldsymbol{A_1} \subset \boldsymbol{A}$ with $\boldsymbol{A_0} \subset \boldsymbol{A_1}$ such that $\mathbf{tp}(\boldsymbol{a}/\boldsymbol{A_1})$ is an isolate type.
2. A set $\boldsymbol{B}$ is said to be almost atomic over $\boldsymbol{A}$, if every finite tuple $\boldsymbol{a}$ from $\boldsymbol{B}$ is almost atmic over $\boldsymbol{A}$.

Remark

There is a stronger version of "begin almost atomic":

- a *is almost atomic over* A *(in a stronger sense)* $\iff$ $\exists A_0 \subset_{fin} A \, \forall A_1 \subset_{fin} A$ *with* $A_0 \subset A_1$, $\mathrm{tp}(a/A_1)$ *is an isolated type.*

If T *is superstable, these two notions are equivalent. (Use the open mapping theorem)*

The following lemma is essentially due to Tanović.

Proposition

Let A be a countable set. Then there is a countable model $M \supset A$ that is almost atomic over A.

For proving Propositon above, we can use

Lemma

Let $B \supset A$ be a countable set almost atomic over A. Let $\varphi(x)$ be a consistent $L(B)$-formula. Then there is an element b satisfying $\varphi(x)$ such that Bb is almost atomic over A.

Proof.

First choose an increasing sequence $\{B_i : i \in \omega\}$ of finite subsets of B such that $B = \bigcup_{i \in \omega} B_i$. We assume B_0 contains the parameters of φ. Then choose an increasing sequence $\{A_i : i \in \omega\}$ of finite subsets of A such that $\mathrm{tp}(B_i/A_i)$ is isolated. Next we find isolated types $p_i \in S(A_i B_i)$ such that (i) $\varphi(x) \in p_0$ and (ii) $p_i \subset p_j$ if $i < j$. Let $b \models \bigcup_{i \in \omega} p_i$.

Claim Bb is almost atomic over A.

Let $X \subset B$ be a finite set. Choose $i \in \omega$ such that $Xb \subset A_i B_i b$. Then, by the transitivity, $\mathrm{tp}(A_i B_i b/A_i)$ is isolated. So $\mathrm{tp}(Xb/A_i)$ is isolated. □

Definition

1. Let $r(x) \subset p(x)$ be two complete types. We say that $p(x)$ is isolated modulo $r(x)$, if there is $\varphi(x) \in p(x)$ such that $r(x) \cup \{\varphi(x)\} \vdash p(x)$.
2. We say that $p(x) \in S(A)$ is $*$-isolated if there is an increasing sequence $\{A_n\}_{n \in \omega}$ of finite subsets of A such that every $p|A_n$ is isolated modulo $p|A_0$.
3. d is $*$-isolated over A if $\mathbf{tp}(d/A)$ is $*$-isolated.

Fact

Let $\mathbf{r} \subset \mathbf{p} \subset \mathbf{q}$. Suppose that $\mathbf{q}$ is a nonforking extension of $\mathbf{p}$. If $\mathbf{q}$ is isolated modulo $\mathbf{r}$, then $\mathbf{p}$ is isolated modulo $\mathbf{r}$.

Proof.

By the isolation modulo r, there is a formula $\varphi(x) \in q$ such that $r(x) \cup \{\varphi(x)\} \vdash q(x)$. By the open mapping theorem, the set $\{s|\mathrm{dom}(p) : \varphi(x) \in s(x)\}$ is an open set of $S(\mathrm{dom}(p))$. So there are $L(\mathrm{dom}(p)$-formulas $\theta_i(x)$ $(i \in I)$ such that $\{s|\mathrm{dom}(p) : \varphi(x) \in s(x) \in S(\mathrm{dom}(q))\} = \bigcup_{i \in I} U_{\theta_i}$.

$$\{s|\mathrm{dom}(p) : r(x) \cup \{\varphi(x)\} \subset s(x)\} = \bigcup_{i \in I} \{t(x) \in U_{\theta_i} : r(x) \subset t(x)\}$$

However the left hand side is the singleton $\{p\}$, so there is θ_i such that $\{p\} = \{t(x) \in U_{\theta_i} : r(x) \subset t(x)\}$. In other words, $p(x)$ is isolated modulo r. □

Lemma

Let $\mathbf{A} \subset \mathbf{B}$ be countable sets. Suppose that $\mathbf{B}$ is $*$-isolated over $\mathbf{A}$, i.e., every finite subset of $\mathbf{B}$ is $*$-isolated over $\mathbf{A}$. Let $\mathbf{r}(x) \in S(\bar{\mathbf{b}})$, where $\bar{\mathbf{b}}$ is a finite tuple from $\mathbf{B}$. Then there is $\mathbf{d}$ realizing $\mathbf{r}$ such that $\mathbf{Bd}$ is $*$-isolated over $\mathbf{A}$.

Proof.

A similar proof as before. □

Proposition

Let T be a small theory. Let A be a countable set. Then there is a saturated model $M \supset A$ such that M is $*$-isolated over A.

Proposition

Let $\boldsymbol{T}$ be a small theory with a non-isolated multidimensional type. Then $\boldsymbol{T}$ does not have the SB-property for countable models.

Proof.

Let M be a countable saturated model Let $p_a(x)$ be a non-isolated type witnessing the multidimensionality. We can assume that a and M are independent. Let N be a countable almost atomic model over Ma. Since M is saturated, M and N are bi-embeddable.

Claim $p_a(x)$ is not realized in N.

By way of contradiction, suppose that $d \in N$ realizes p_a. Since N is almost atomic over Ma, there is a finite set $M_0 \subset M$ such that $\mathrm{tp}(d/M_0a)$ is an isolated type. By the fact that M_0 and a are independent, using $p_a \perp \emptyset$, we deduce that $\mathrm{tp}(d/M_0a)$ does not fork over $\emptyset$. By the open mapping theorem, we have that $p_a = \mathrm{tp}(d/a)$ is an isolated type. This is a contradiction. $\square$

Definition

Let $\varphi(x)$ be a formula (with parameters) and let $p(x) \in S(A)$ be a type containing $\varphi(x)$. We say that (p, φ) is a strongly regular type if whenever $A \subset B$, $p^*(x) \in S(B)$ and $\varphi(x) \in p^*(x)$ then either (i) $p^*(x)$ is a nonforking extension of $p(x)$ or (ii) $p^*(x) \perp p(x)$.

Proposition

Let T be a multidimensional small theory witnessed by a non-isolated strongly regular type. Then there are infinitely many non-isomorphic countable universal models.

References

1. Tanović, On Kueker's conjecture, preprint.
2. Y. Tanaka, personal communication.
3. Nurmagambetov, Characterization of ω-stable theories with bounded dimension,