Model theory of a quantum 2-torus

Masanori Itai
joint work with Boris Zilber

Dept of Math Sci, Tokai University

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References

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Hilbert 12th Problem

Let K denote either

1. $\mathbb{Q}$, or
2. an imaginary quadratic extension of $\mathbb{Q}$, or
3. a real quadratic extenstion of $\mathbb{Q}$.

Problem (Hilbert 12th)
Describe K^{ab}, the maximal abelian extension of K.

Kronecker-Weber (KW) theorem

$$\mathbb{Q}^{ab} = \mathbb{Q}(\text{all roots of unity})$$

Complex multiplication (CM) case

Let $K = \mathbb{Q}(\sqrt{-d})$. Then

$$K^{ab} = K(t(E_{K,tors}), j(E_K))$$

- E_K is the elliptic curve with complex multiplication by O_K,
- t is a canonical coordinate of $E_K/\mathbf{Aut}E_K \simeq \mathbb{P}^1$,
- $j(E_K)$ is the absolute invariant.

Real multiplication (RM) case

Let $K = \mathbb{Q}(\sqrt{d})$. Then

$$K^{ab} = K(\text{Stark's numbers})$$

(Stark's conjectures, not yet proven)

Manin’s RM Program

Use two-dimensional quantum tori corresponding to real quadratic irrationalities as a replacement of elliptic curves with CM.

Model Theorests may make some contributions

Construct *quantum tori* by model theoretic tools so that we can study their algebro-geometric structures.

Quantum Tori

Quantum tori are geometric objects associated with non-commutative algebras $\mathcal{A}_q$ of unitary operations with q generating multiplicative groups.

When q is a root of unity, we have a quantum torus which is a Zariski structure (Zilber's result).

Example 1: Noncummutative Geometry, Connes

Consider the algebra generated by P, Q satisfying the Heisenberg commutation relation

$$QP - PQ = i\hbar,$$

where $\hbar = h/2\pi$ and h is Planck's constant. This algebra is usually represented by actions on various Hilbert spaces and its generalizations (known also as rigged Hilbert spaces). This results in calculations in terms of inner products, eigenvectors and eigenvalues of certain operators expressed in terms of P and Q. See the page 39 of Connes' book.

Example 2: Manin's quantum plane

Manin's quantum plane is the following skew polynomial ring in two indeterminates;

$$\mathcal{O}_q(k^2) = k\langle x, y | xy = qyx \rangle$$

where k is a field and q is a constant. Generalizing this definition to algebraic tori we obtain the notion of *quantum torus of rank n* as the k-algebra $\mathcal{O}_q((k^\times)^n)$ with generators $x_1^\pm, \cdots, x_n^\pm$ with the relation

$$x_i x_j = q x_j x_i.$$

Main theorems of [IZxx]

Two main theorems proved in [IZxx] are;

1. The theory of quantum line-bundles is superstable.
2. With the pairing function, within $(\Gamma, \cdot, 1, q)$ we can define $(\Gamma, \oplus, \otimes, 1, q)$ and $(\Gamma, \oplus, \otimes, 1, q) \simeq (\mathbb{Z}, +, \cdot, 0, 1)$. Hence the theory of the quantum 2-torus $(\mathbf{U}, \mathbf{V}, \mathbb{F}^*, \Gamma)$ with the pairing function is undecidable and unstable.

In this talk I give a brief overview of [IZxx].

Quantum tori over $\mathbb{C}$

First we give the description of a quantum torus defined over the complex numbers $\mathbb{C}$.

Quantum tori over $\mathbb{C}$, cnt'd

Consider a $\mathbb{C}$-algebra $\mathcal{A}_q^2$ generated by operators U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV$$

where $q = e^{2\pi i h}$ with $h \in \mathbb{R}$. Let $\Gamma_q = q^{\mathbb{Z}}$ be a multiplicative subgroup of $\mathbb{C}^*$.

The quantum 2-torus $T_q^2(\mathbb{C})$ associated with the algebra $\mathcal{A}_q^2$ and the group Γ_q is the 3-sorted structure $(\mathrm{U}_\phi, \mathrm{V}_\phi, \mathbb{C}^*)$ with the actions U and V satisfying $(\gamma \in \Gamma)$

$$\begin{array}{rcl} U & : & \mathrm{u}(\gamma u, v) \mapsto \gamma u \mathrm{u}(\gamma u, v) \\ V & : & \mathrm{u}(\gamma u, v) \mapsto v \mathrm{u}(q^{-1}\gamma u, v) \end{array} \tag{1}$$

and

$$\begin{array}{rcl} U & : & \mathrm{v}(\gamma v, u) \mapsto u \mathrm{v}(q\gamma v, u) \\ V & : & \mathrm{v}(\gamma v, u) \mapsto \gamma v \mathrm{v}(\gamma v, u) \end{array} \tag{2}$$

where $\mathrm{u}(\gamma u, v) \in \mathrm{U}_\phi, \mathrm{v}(\gamma v, u) \in \mathrm{V}_\phi$ and a function $\langle \cdot \mid \cdot \rangle$ called the pairing

$$\langle \cdot \mid \cdot \rangle : \mathrm{V}_\phi \times \mathrm{U}_\phi \to \Gamma$$

Intuitive Ideas

The intuitive ideas of $\mathbf{U}$, $\mathbf{V}$ and operations $\boldsymbol{U}$ and $\boldsymbol{V}$.

- Both $\mathbf{U}$ and $\mathbf{V}$ are two dimensional objects.
- Both $\mathbf{U}$ and $\mathbf{V}$ are bases for an ambient module which we do not give any formal description in the theory.
- The operator $\boldsymbol{U}$ moves each element (vector) of $\mathbf{U}$ on its fibre, say *vertically*. On the other hand the operator $\boldsymbol{V}$ moves each element of $\mathbf{U}$ to another element of $\mathbf{U}$, say *horizontally*.
- The operator $\boldsymbol{V}$ does the same actions on $\mathbf{U}$ and $\mathbf{V}$.
- The pairing function works as an inner product.

Γ-bundles

- Let $\phi : \mathbb{C}^*/\Gamma \to \mathbb{C}^*$ be a (non-definable) "choice function".
- Put $\Phi = \mathbf{ran}(\phi)$.
- We work with Φ^2.

Consider $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$.
Let

$$\begin{aligned} \mathbf{U}_\phi &:= \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \langle u, v\rangle \in \Phi^2, \gamma_1.\gamma_2 \in \Gamma\} \\ \mathbf{V}_\phi &:= \{\gamma_1 \cdot \mathbf{v}(\gamma_2 v, u) : \langle u, v\rangle \in \Phi^2, \gamma_1.\gamma_2 \in \Gamma\} \end{aligned} \tag{3}$$

$\mathbf{U}_\phi$, $\mathbf{V}_\phi$ are called Γ-bundles.

Γ-bundle over $(\mathbf{u}, \mathbf{v})$

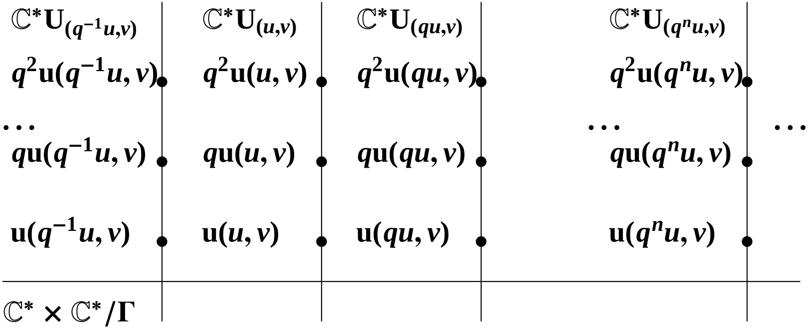

Figure: Γ-bundle over (u, v) inside an ambient $\mathbb{C}$-module

Line-bundles

Consider the following definable set $\mathbb{C}^*\mathbf{U}_\phi$.

$$\mathbb{C}^*\mathbf{U}_\phi \quad := \quad \{x \cdot \mathbf{u}(\gamma u, v) : \langle u, v\rangle \in \Phi^2, x \in \mathbb{C}^*, \gamma \in \Gamma\} \tag{4}$$

Notice that we have

$$\mathbb{C}^*\mathbf{U}_\phi \simeq (\mathbb{C} \times \mathbf{U}_\phi)/E \tag{5}$$

where E is an equivalence relation identifying $\gamma \in \Gamma$ as an element of $\mathbb{C}^*$. We also consider the similar definable set $\mathbb{C}^*\mathbf{V}_\phi$. We call $\mathbb{C}^*\mathbf{U}_\phi$ and $\mathbb{C}^*\mathbf{V}_\phi$, **line-bundles** over $\mathbb{C}^*$.

Pairing function -1

Consider a function $\langle \cdot \mid \cdot \rangle$ called the *pairing function* which plays as an *inner product* of two $\mathbf{\Gamma}$-bundles $\mathbf{U}_\phi$ and $\mathbf{V}_\phi$:

$$\langle \cdot \mid \cdot \rangle : \left(\mathbf{V}_\phi \times \mathbf{U}_\phi\right) \cup \left(\mathbf{U}_\phi \times \mathbf{V}_\phi\right) \to \mathbf{\Gamma}. \tag{6}$$

Pairing function -2

We demand two operators U, V to behave like *unitary operators* with respect to the pairing function and the pairing function to have the *sesquilinear* property. These requirements forces us to postulate the following:

1. $\langle \mathbf{u}(u,v) | \mathbf{v}(v,u) \rangle = 1$,
2. for each $r, s \in \mathbb{Z}$, $\langle U^r V^s \mathbf{u}(u,v) | U^r V^s \mathbf{v}(v,u) \rangle = 1$,
3. for $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$,
$$\langle \gamma_1 \mathbf{u}(\gamma_2 u, v) | \gamma_3 \mathbf{v}(\gamma_4 v, u) \rangle = \langle \gamma_3 \mathbf{v}(\gamma_4 v, u) | \gamma_1 \mathbf{u}(\gamma_2 u, v) \rangle^{-1},$$
4. $\langle \gamma_1 \mathbf{u}(\gamma_2 u, v) | \gamma_3 \mathbf{v}(\gamma_4 v, u) \rangle = \gamma_1^{-1} \gamma_3 \langle \mathbf{u}(\gamma_2 u, v) | \mathbf{v}(\gamma_4 v, u) \rangle$, and
5. for $v' \notin \Gamma \cdot v$ or $u' \notin \Gamma \cdot u$, $\langle q^s \mathbf{v}(v', u) | q^r \mathbf{u}(u', v) \rangle$ is not defined.

Proposition

Given $q \in \mathbb{C}^*$ any two structures of the form $T_q^2(\mathbb{C})$ are isomorphic over $\mathbb{C}$. In other words, the isomorphism type of $T_q^2(\mathbb{C})$ does not depend on the system of representative Φ.

Corollary

Suppose $\mathbb{F}$ and $\mathbb{F}'$ are isomorphic algebraically closed fields of characteristic zero. Let $q \in \mathbb{F}$ and $q' \in \mathbb{F}'$ such that both q and q' are transcendental and $\Gamma = q^{\mathbb{Z}}$ and $\Gamma' = q'^{\mathbb{Z}}$ are elementarily equivalent infinite multiplicative subgroups. Then

$$T_q^2(\mathbb{F}) \simeq T_{q'}^2(\mathbb{F}')$$

as quantum 2-tori.

First order theory of $T_q^2(\mathbb{F})$

From now on, we consider the quantum 2-torus over an algebraically closed field $\mathbb{F}$ of characteristic zero and an infinite multiplicative cyclic subgroup $\mathbf{\Gamma}$ of $\mathbb{F}$ generated by q.

Language

The language $\mathcal{L}_q = \mathcal{L}_{T_q^2} = \{\mathbf{U}, \mathbf{V}, \mathbb{F}, \mathbf{\Gamma}, U, V, q, 0, 1, T_p\}$ has the following predicates and symbols;

- $\mathbf{U}, \mathbf{V}, \mathbb{F}, \mathbf{\Gamma}$ are unary predicates,
- U, V are 4-ary relations,
- q is a constant symbol,
- T_p is a ternary relation symbol corresponding to the pairing function.

The theory $T_q^2(\mathbb{C})$

The theory $T_q^2(\mathbb{F})$ is a set of first-order sentences describing the properties of $T_q^2(\mathbb{C})$ given in the previous slides.

Here we show that the theory of $(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$ is axiomatizable and superstable.

The predicate $\Gamma(x)$ describes the property of the set $q^{\mathbb{Z}}$ as a multiplicative subgroup with the following Lang-type property.

Lang-type property

Let K be an algebraically closed field, and A a commutative algebraic group over K and Γ a subgroup of A.

We say that (K, A, Γ) is of $\mathrm{Lang\text{-}type}$ if for every $n < \omega$ and every subvariety X (over K) of $A^n = A \times \cdots \times A$ (n times), $X \cap \Gamma^n$ is a finite union of cosets of subgroups of Γ^n.

Lang-type and One-basedness

The Lang-type property gives us :
Let K be an algebraically closed field, A a commutative algebraic group over K, and Γ a subgroup of A. Then (K, A, Γ) is of Lang-type if and only if $\mathrm{Th}(K, +, \cdot, \Gamma, a)_{a \in K}$ is stable and $\Gamma(x)$ is one-based.

Here $\Gamma(x)$ is *one based* means that for every n and every definable subset $X \subset \Gamma^n$, X is a finite boolean combination of cosets of definable subgroups of Γ^n.

Axiomatization of Γ

Axioms for Γ

A. 1 Γ satisfies the first order theory of a cyclic group with generator q,

A. 2 (Lang-type) for every n and every variety X of $(\mathbb{F}^*)^n$, $X \cap \Gamma^n$ is a finite union of cosets of definable subgroups of Γ^n.

$\mathbf{Th}(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$ is superstable

- Before discussing the theory $T_q^2(\mathbb{F})$, we consider the theory of $(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$.
- Recall that $\Gamma(x)$ is a unary predicate and q is a constant symbol. $\Gamma(x)$ describes the property of the set $q^{\mathbb{Z}}$ as a multiplicative subgroup.
- Due to the fact that the theory $(\mathbb{Z}, +, 0)$ is superstable, we see that the theory $(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$ is also superstable by counting types and the Lang-type property [A. 2].

From the superstability of the theory $(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$, we see that $\mathbf{Th}(T_q^2(\mathrm{U}, \mathbb{F}))$ is superstable.

Remark:
Notice that $\mathbf{Th}(T_q^2(\mathrm{U}, \mathbb{F}))$ does not mention the pairing function.

Non-tameness of pairing function

With the pairing function the ring of integers can be defined in Γ. In this regard it is similar to the theory of pseudo-exponentiation, the model theory of which can successfully be investigated “modulo arithmetic” .

Main theorems of [IZxx]

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2. With the pairing function, within $(\Gamma, \cdot, 1, q)$ we can define $(\Gamma, \oplus, \otimes, 1, q)$ and $(\Gamma, \oplus, \otimes, 1, q) \simeq (\mathbb{Z}, +, \cdot, 0, 1)$. Hence the theory of the quantum 2-torus $(\mathbf{U}, \mathbf{V}, \mathbb{F}^*, \Gamma)$ with the pairing function is undecidable and unstable.

This is just a beginning!

The role of q is not clear in $T_q(\mathbb{F})$!

1. When do we have $T_q(\mathbb{F}) \simeq T_{q'}(\mathbb{F})$?
2. Can we define a *Morita equivalence* among $T_q(\mathbb{F})$ for all q?
3. Is there any intereting structure on the set of all endomorphisms of $T_q(\mathbb{F})$?