Independent partitions and indiscernibility

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Outline

1. Simplicity and Independent Partitions,
   1. Definitions
   2. Examples

2. Ranks
   1. $D(\Sigma, \varphi, k)$
   2. $D(\Sigma, \varphi)$
   3. $D_{\text{inp}}$

3. Main Result
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- Formulas are denoted by $\varphi, \psi, \ldots$.
- $m, n, k, \ldots$ are natural numbers.
A **simple** theory is characterized as a theory in which the length of dividing sequence of types is bounded ($< \infty$).
A low theory is characterized by the following property: For each formula $\varphi(x, y)$ there is a number $n_\varphi \in \omega$ such that whenever 

\[ \{ \varphi(x, a_i) : i < m \} \] 

satisfies

1. $\{ \varphi(x, a_i) : i < m \}$ is consistent, and
2. $\varphi(x, a_i)$ divides over $A_i = \{ a_j : j < i \}$ ($i < m$),

then $m \leq n_\varphi$. 
Casanovas constructed a simple nonlow theory

\[ T_1 = Th(M, P, P_1, P_2, ..., Q, R). \]
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4. $Q$ is the set of all sequences $(A_1, A_2, ..., A_\omega)$, where $A_n$ is an $n$-element subset of $P_n$ and for some $a \in G$, $A_\omega \subset G$ is the set of all $g \in G$ directly connected to $a$. 
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5. $R \subset P \times Q$.
6. $R(a, (A_1, A_2, \ldots, A_\omega))$ if (i) $a \in P_n$ and $a \in A_n$ ($\exists n \in \omega$) or (ii) $a \notin \bigcup_n P_n$ and $a \in A_\omega$. 
This theory $T_1$ is not supersimple. $R(x, y)$ defines infinitely many mutually independent partitions in the following sense: If we enumerate $P_n$ as $P_n = \{a_{nm} : m \in \omega\}$, then
This theory $T_1$ is not supersimple. $R(x, y)$ defines infinitely many mutually independent partitions in the following sense: If we enumerate $P_n$ as $P_n = \{a_{nm} : m \in \omega\}$, then

- for each $\eta \in \omega^\omega$, $\{R(a_{n\eta(n)}, y) : n = 1, 2, ...\}$ is consistent, and
- for each $n = 1, 2, ..., \{R(a_{nm}, y) : m \in \omega\}$ is $(n + 1)$-inconsistent.
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However, for each $k \in \omega$, we can find a formula $\varphi(x, y)$ and parameter sets $A_i = \{a_{ij} : j \in \omega\}$ ($i < k$) defining $k$ independent partitions.
\[ D_{\text{inp}}(*, *) \]

**Definition**

\[ D_{\text{inp}}(\Sigma(x), \varphi(x, y)) \] is the first cardinal \( \kappa \) such that there are no \( \kappa \)-many independent partitions \( \{ \varphi(x, a_{ij}) : j \in \omega \} \) (\( i < \kappa \)) of \( \Sigma \).
Remark

- For $T_1$, $D_{\text{inp}}(x = x, R(y, x)) = \omega_1$.
- For $T_2$, for some $\varphi(x, y)$,
  $D_{\text{inp}}(x = x, \varphi(x, y)) = \omega$. 
So it is natural to ask whether there is a simple nonlow theory $T$ such that

$$D_{\text{inp}}(x = x, \varphi(x, y)) < \omega,$$

for any $\varphi$. 
First we recall definitions of basic ranks. Let $\Sigma(x)$ be a set of formulas and $\varphi(x, y)$ a formula. Let $k \in \omega$. 
Definition

\[ D(\Sigma(x), \varphi(x, y), k) \]

1. \( D(\Sigma(x), \varphi(x, y), k) \geq 0 \) if \( \Sigma(x) \) is consistent.
2. \( D(\Sigma(x), \varphi(x, y), k) \geq n + 1 \) if there is an indiscernible sequence \( \{b_i : i \in \omega\} \) over \( \text{dom}(\Sigma) \) such that \( D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi(x, y), k) \geq n \) for all \( i \in \omega \), and \( \{\varphi(x, b_i) : i \in \omega\} \) is \( k \)-inconsistent.
Independent partitions and indiscernibility

Definition

1. $D(\Sigma(x), \varphi(x, y)) \geq 0$ if $\Sigma(x)$ is consistent.
2. For a limit ordinal $\delta$, $D(\Sigma(x), \varphi(x, y)) \geq \delta$ if $D(\Sigma(x), \varphi(x, y)) \geq \alpha$ for all $\alpha < \delta$.
3. $D(\Sigma(x), \varphi(x, y)) \geq \alpha + 1$ if there is an indiscernible sequence $\{b_i : i \in \omega\}$ over $\text{dom}(\Sigma)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi(x, y)) \geq \alpha (i \in \omega)$, and $\{\varphi(x, b_i) : i \in \omega\}$ is inconsistent.
Fact

1. \( D(\Sigma(x), \varphi(x, y), k) \geq n \) if there is a tree \( A = \{a_\nu : \nu \in \omega^{\leq n}\} \) such that (1) \( \Sigma(x) \cup \{\varphi(x, a_\eta|_i) : 1 \leq i \leq n\} \) is consistent \((\forall \eta \in \omega^n)\), and (2) \( \{\varphi(x, a_\nu \neg|_i) : i \in \omega\} \) is \( k \)-inconsistent \((\forall \nu \in \omega^{<n})\).
Fact

$D(\Sigma(x), \varphi(x, y)) \geq n$ if there is a tree $A = \{a_\nu : \nu \in \omega^{\leq n}\}$ and numbers $k_0, \ldots, k_{n-1}$ such that (1) $\Sigma(x) \cup \{\varphi(x, a_{\eta|i}) : 1 \leq i \leq n\}$ is consistent $(\forall \eta \in \omega^n)$, and (2) $\{\varphi(x, a_{\nu\neg i}) : i \in \omega\}$ is $k_{lh(\nu)}$-inconsistent $(\forall \nu \in \omega^{<n})$. 
Theorem

Suppose that the size of independent partitions is bounded in $T$. Then the following are equivalent:

1. $T$ is simple.
2. $T$ is low.
Proposition

Suppose $D_{\text{inp}}(x = x, \varphi(x, y)) = k - 1 < \omega$ and $D(x = x, \varphi(x, y)) \geq \omega$. Then $T$ is not simple.
Proof.

Fix \( m \in \omega \).

- By \( D(x = x, \varphi(x, y)) \geq \omega \), there is a set 
  \[ A = \{ a_\nu : \nu \in \omega^{\leq m} \} \]
  witnessing
  \[ D(x = x, \varphi(x, y)) \geq m. \]
Proof.

Fix $m \in \omega$.

- By $D(x = x, \varphi(x, y)) \geq \omega$, there is a set $A = \{a_\nu : \nu \in \omega^{\leq m}\}$ witnessing $D(x = x, \varphi(x, y)) \geq m$.

- We have
  1. $\{\varphi(x, a_{\eta|i}) : 1 \leq i \leq m\}$ is consistent ($\forall \eta \in \omega^m$),
  2. $\{\varphi(x, a_{\nu^{-i}}) : i \in \omega\}$ is $k_{lh(\nu)}$-inconsistent ($\forall \nu \in \omega^{<m}$).
We can assume that $A$ is an indiscernible tree.
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For $\nu \in \omega^m$, let $\nu^*$ be the sequence

$$\nu(0), 0^k, \nu(1), 0^k, \ldots, \nu(\text{lh}(\nu) - 1), 0^k.$$

For $\nu = \nu_0 \overline{m}$, let

$$a_{\nu}^* = a_{\nu_0 \overline{m} 0}, a_{\nu_0 \overline{m} 0^2}, \ldots, a_{\nu_0 \overline{m} 0^k}.$$
Let $\varphi^*(x, y_1, \ldots, y_k)$ be the formula $\varphi(x, y_1) \land \ldots \land \varphi(x, y_k)$.

Claim A \( \{\varphi^*(x, a^*_\nu m) : m \in \omega\} \) is $k$-inconsistent.
Let $\varphi^*(x, y_1, ..., y_k)$ be the formula $\varphi(x, y_1) \land \ldots \land \varphi(x, y_k)$.

**Claim A** $\{\varphi^*(x, a^*_\nu \_m) : m \in \omega\}$ is $k$-inconsistent.

Suppose this is not the case. Then there is $F = \{i_1, ..., i_k\} \subset \omega$ such that

$$\{\varphi^*(x, a^*_\nu \_i_1), ..., \varphi^*(x, a^*_\nu \_i_k)\}$$

is consistent.
By the definition of $\varphi^*$, in particular, the following set is consistent.

$$\{\varphi(x, a_{\nu_0}^{i_1 \ldots i_k \nu_{k-1}}), \ldots, \varphi(x, a_{\nu_0}^{i_k \ldots i_1 \nu_{k-1}})\}$$
By the definition of \( \varphi^* \), in particular, the following set is consistent.

\[
\{ \varphi(x, a_{\nu_0}^{i_1 \cdots i_k} 0^{k-1} \nu), ..., \varphi(x, a_{\nu_0}^{i_1 \cdots i_k} 0^{k-1} \nu) \}
\]

For each \( \nu \) of length \( k \), let \( \Gamma_\nu \) be the set:

\[
\{ \varphi(x, a_{\nu_0}^{i_1 \cdots i_k} 0^{k-1} \nu), ..., \varphi(x, a_{\nu_0}^{i_1 \cdots i_k} 0^{k-1} \nu) \}.
\]
Then each $\Gamma_\nu$ is consistent, by the indiscernibility of $A$. 
Then each $\Gamma_{v}$ is consistent, by the indiscernibility of $A$.

On the other hand, by our choice of the tree $A$, for each $l = 0, ..., k - 1$, the set

$$\{ \varphi(x, a^{*}_{v_{0} \sim i_{2} \sim 0 \sim i_{1}}) : i \in \omega \}$$

is inconsistent ($k_{lh(v_{0})+(1+l)}$-inconsistent).
Then each $\Gamma_\nu$ is consistent, by the indiscernibility of $A$.

On the other hand, by our choice of the tree $A$, for each $l = 0, \ldots, k - 1$, the set

$$\{\varphi(x, a^*_{\nu_0 \neg i_2 \neg 0^l \neg i}) : i \in \omega\}$$

is inconsistent ($k_{lh(\nu_0)} + (1 + l)$-inconsistent).

This yields $D_{\text{inp}}(x = x, \varphi(x, z)) \geq k$, a contradiction. (End of Proof of Claim)
By Claim A, the set \( \{ \varphi^*(x, a^*_\nu) : \nu \in \omega^m \} \) witnesses \( D(x = x, \varphi^*, k) \geq m \).
By Claim A, the set $\{\varphi^*(x, a^*_\nu) : \nu \in \omega^m\}$ witnesses $D(x = x, \varphi^*, k) \geq m$.

Since $m$ is arbitrary, we conclude $D(x = x, \varphi^*, k) = \infty$, which means that $T$ is not simple.