Singularities of solutions to the Cauchy problem for second-order hyperbolic operators with the coefficients of their principal parts depending only on the time variable

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1. Introduction

Let \( x = (x_0, x'') = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \), and denote by \( \xi = (\xi_0, \xi'') = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1} \) their dual variables. The \( x_0 \) variable plays the role of the time variable. We consider second-order hyperbolic operators with symbols

\[
P(x, \xi) = p(x_0, \xi) + \sum_{j=0}^{n} b_j(x) \xi_j + c(x),
\]

where

\[
p(x_0, \xi) = \xi_0^2 + \sum_{|\alpha|=2, \alpha_0\leq 1} a_\alpha(x_0) \xi^\alpha.
\]

We assume the following conditions:

(A) the \( a_\alpha(x_0) \) are real analytic on \([0, \infty)\) and \( b_j(x), c(x) \in C^\infty(\mathbb{R}_+^{n+1}) \) (\( 0 \leq j \leq n \)).

Here \( \mathbb{R}_+^{n+1} = \{ x \in \mathbb{R}^{n+1}; x_0 > 0 \} \). We consider the following Cauchy problem:

\[
\text{(CP)} \quad \begin{cases} P(x, D)u(x) = f(x) & \text{in } (0, \infty) \times \mathbb{R}^n, \\ D^j_0 u(x)|_{x_0=0} = u_j & \text{in } \mathbb{R}^n \quad (j = 0, 1), \end{cases}
\]

where \( f \in C([0, \infty); \mathcal{D}'(\mathbb{R}^n)) \) and \( u_j \in \mathcal{D}'(\mathbb{R}^n) \) (\( j = 0, 1 \)). We may assume by coordinate transformation

\[
a_\alpha(x_0) \equiv 0 \text{ if } |\alpha| = 2 \text{ and } \alpha_0 = 1.
\]

So \( P(x, \xi) \) can be written as follows:

\[
P(x, \xi) = \xi_0^2 - a(x_0, \xi') + b_0(x) \xi_0 + b(x, \xi''),
\]

\[
a(x_0, \xi'') = \sum_{j,k=1}^{n} a_{j,k}(x_0) \xi_j \xi_k, \quad b(x, \xi'') = \sum_{j=1}^{n} b_j(x) \xi_j, \quad a_{j,k}(x_0) = a_{k,j}(x_0).
\]

We assume the following conditions:

(H) \( a(x_0, \xi'') \geq 0 \) for \( (x_0, \xi'') \in [0, \infty) \times \mathbb{R}^n \).
where \( \vartheta \), \( \xi \) and \( V \) case (CP) is defined analytically to \( \Omega \), and define
\[
\partial_{x_0} a(x_0, \xi') \neq 0 \quad \text{for} \quad \xi' \neq 0.
\]
Let \( \Omega \) be a neighborhood of \( (0, T) \) For any \( x, \xi' \in R^n \); \( \xi_1 = \cdots = \xi_{n'} = 0 \), since the case \( V = R^n \) is trivial. Then by (F) we have
\[
a(x_0, \xi'') = a(x_0, \xi') \neq 0 \quad \text{in} \quad x_0 \quad \text{for} \quad \xi' \neq 0, \quad b(x, \xi'') = b(x, \xi'),
\]
where \( \xi' = (\xi_1, \cdots, \xi_{n'}) \). From (A) we have the following:
(i) For \( T > 0 \) there is \( k_T \in N \) such that \( \sum_{j=0}^{k_T} \partial_{x_0}^j a(x_0, \xi') \neq 0 \) for \( (x_0, \xi') \in [0, T] \times S^{n'-1} \), where \( S^{n'-1} \) denotes the \((n'-1)\) dimensional unit sphere.
(ii) There are \( r \in N \), real analytic functions \( \lambda_j(x_0) \) and \( v_{j,k}(x_0) \) \((1 \leq j \leq r, 1 \leq k \leq n')\) defined on \([0, \infty)\) such that \( \lambda_j(x_0) \neq 0, a(x_0, \xi') = \sum_{j=1}^{r} \lambda_j(x_0) \zeta_j(x_0, \xi')^2 \), where \( \zeta_j(x_0, \xi') = \sum_{k=1}^{n'} v_{j,k}(x_0) \xi_k \).

Let \( \Omega \) be a neighborhood of \([0, \infty)\) in \( C \) such that the \( a_{j,k}(x_0) \) can be extended analytically to \( \Omega \), and define \( R(\xi') = \{(\Re \lambda); \lambda \in \Omega \quad \text{and} \quad a(\lambda, \xi') = 0\} \) for \( \xi' \in R^n \setminus \{0\} \), where \( a_+ = \max\{a, 0\} \). We assume
(L) For any \( T > 0 \) and \( x'' \in R^n \), there is \( C > 0 \) such that
\[
\min_{t \in R(\xi')} |x_0 - t| |\partial_{x_0} a(x_0, \xi')| \leq C \sqrt{a(x_0, \xi')} \quad \text{for} \quad (x_0, \xi') \in [0, T] \times (R^n \setminus \{0\}),
\]
where \( \min_{t \in R(\xi')} |x_0 - t| = 1 \) if \( R(\xi') = \emptyset \).

(L) is a so-called Levi condition. Put
\[
\Gamma(p(x_0, \cdot), \vartheta) = \{\xi \in R^{n+1}; \xi_0 > \sqrt{a(x_0, \xi')}\};
\]
\[
\Gamma^* = \{y \in R^{n+1}; y \cdot \xi \geq 0 \quad \text{for} \quad \xi \in \Gamma\},
\]
where \( \vartheta = (1, 0, \cdots, 0) \in R^{n+1} \). We define for \( x^0 \in \overline{R}_{n+1}^+ \)
\[
K_{x^0} = \{x(t); t \in [0, \infty); x(t) \in \Gamma(p(x_0(t), \cdot), \vartheta)^* \quad a.e. \quad t \quad \text{and} \quad x(0) = x^0\}
\]
satisfying \((d/dt)x(t) \in \Gamma(p(x_0(t), \cdot), \vartheta)^* \quad a.e. \quad t \quad \text{and} \quad x(0) = x^0\)
\[
( \subset \{x; x_j = x_j^0 \quad (n'+1 \leq j \leq n)\}).
\]

Concerning \( C^{\infty} \) well-posedness we have the following

**Theorem 1.** (CP) has a unique solution \( u \in C^2([0, \infty); D'(R^n)) \). Let \( x^0 \in \overline{R}_{n+1}^+ \).
If \( u \) satisfies (CP) and
\[
(supp f \cup \{0\} \times (supp u_0 \cup supp u_1)) \cap K_{x^0} = \emptyset,
\]

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then $x^0 \notin \text{supp } u$. Moreover, (CP) is $C^\infty$ well-posed.

**Remark.** We assume that (H), (F) and (A) are satisfied. Moreover, we assume that the $a_{j,k}(x_0)$ are polynomials of $x_0$, for example, when $n' \geq 3$. Then (CP) is $C^\infty$ well-posed if and only if (L) is satisfied.

For the proof of Theorem 1 we refer to [W].

### 2. Main results

**Definition 1.** Let $z^0 \equiv (x^0, \xi^0) \in \mathbb{R}_+^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$.

(i) The localization polynomial $p_{z^0}(X)$ at $z^0$ is defined by

$$
p(z^0 + sX) = s^{\sigma(z^0)}(p_{z^0}(X) + o(1)) \text{ as } s \to 0, \quad p_{z^0}(X) \neq 0 \text{ in } X \in \mathbb{R}^{2n+2}
$$

(ii) The generalized Hamilton flows $K_{z^0}^\pm$ are defined by

$$
K_{z^0}^\pm = \{z(t); \pm t \geq 0, \{z(t)\} \text{ is a Lipschitz continuous curve in } T^* \mathbb{R}_+^{n+1} \setminus 0 \text{ satisfying } (d/dt)z(t) \in \Gamma(p_{z^0}(\widetilde{\vartheta})^{\sigma}) \text{ a.e. } t \text{ and } z(0) = z^0\}.
$$

Here $\widetilde{\vartheta} \equiv (0, \vartheta) \in \mathbb{R}^{2n+2}$, $\Gamma^\sigma = \{X \in \mathbb{R}^{2n+2}; \sigma(Y, X) \geq 0 \text{ for any } Y \in \Gamma\}$ for $\Gamma \subset \mathbb{R}^{2n+2}$ and $\sigma$ denotes the symplectic form on $T^* \mathbb{R}_+^{n+1}$.

**Remark.** $p_{z^0}(X)$ is hyperbolic w.r.t. $\widetilde{\vartheta}$.

Let $z^0 \equiv (x^0, \xi^0) \in \mathbb{R}_+^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$. If $\xi^{0\vartheta} = 0$, then $K_{z^0}^\pm = (K_{z^0}^\pm \cap \mathbb{R}_+^{n+1}) \times \{\xi^0\}$. If $p(x_0^0, \xi_0^0, \xi^{0\vartheta}) \neq 0$, then $K_{z^0}^\pm = \{z^0\}$. Moreover, $K_{z^0}^\pm$ are the broken null bicharacteristics of $p$ in $T^* \mathbb{R}_+^{n+1} \setminus 0$ emanating from $z^0$ in the direction where $\pm x_0$ increase, if $\xi^{0\vartheta} \neq 0$ and $p(x_0^0, \xi_0^0, \xi^{0\vartheta}) = 0$. Assume that $\xi^{0\vartheta} \neq 0$ and $p(x_0^0, \xi_0^0, \xi^{0\vartheta}) = 0$.

![Diagram](diagram.png)

$K_{z^0}^\pm$ branch at every double characteristic point. Each segment is a null bicharacteristic. Each null bicharacteristic satisfies the following:

$$
\begin{cases}
(d/dx_0)x''(x_0) = (\mp \sqrt{a(x_0, \xi^0)} \xi^{0\vartheta} |_{\xi^{0\vartheta} = 0, \cdots, 0}) \\
\xi_0(x_0) = \pm a(x_0, \xi^{0\vartheta}), \quad \xi''(x_0) = \xi^{0\vartheta}
\end{cases}
$$
By continuity $K_{\pm}^0$ can be defined as sets in $\mathbb{R}_+^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ for $z^0 \in \mathbb{R}_+^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$.

**Definition 2.** Let $\delta > 0$ and $f \in C([0, \delta]; D'(\mathbb{R}^n))$. $WF_0(f) \subset T^*\mathbb{R}^n \setminus 0$ can be defined as follows: We say that $z^{0\nu} \equiv (x^{0\nu}, \xi^{0\nu}) \notin WF_0(f)$ if there are $\chi(x'', \xi'') \in S^0_{1,0}(\mathbb{R}^n)$, which is elliptic at $z^{0\nu}$, and $\delta' > 0$ such that $\chi(x'', D'')f \in C([0, \delta]; H^\infty(\mathbb{R}^n))$.

**Remark.** (i) The above definition is a variant of Chazarain’s definition. (ii) $z^{0\nu} \equiv (x^{0\nu}, \xi^{0\nu}) \notin WF_0(f)$ if and only if there are a neighborhood $U''$ of $x^{0\nu}$, a conic neighborhood $\Gamma''$ of $\xi^{0\nu}$ and $\delta' > 0$ such that for any $\varphi \in C^\infty_0(U'')$ there are $C_N > 0$ ($N \in \mathbb{N}$) satisfying

$$|\mathcal{F}_{x''} \varphi(x'')f(x)(\xi'')| \leq C_N |\xi''|^{-N}$$

for $N \in \mathbb{N}$, $x_0 \in [0, \delta']$ and $\xi'' \in \Gamma''$, where $\mathcal{F}_{x''}$ denotes the partial Fourier transformation with respect to $x''$.

Now we can state our main results.

**Theorem 2.** (I) Let $u \in D'(\mathbb{R}^{n+1}_+)$ satisfy, with $\delta > 0$, $u \in C^2([0, \delta]; D'(\mathbb{R}^n))$, and let $z^0 \equiv (x^0, \xi^0) \in WF(u)$, where $x_0 > 0$.

(i) When $0 < t < x^0_0$, $WF(u) \cap K^0_0 \cap \{x_0 = t\} \neq \emptyset$ if $WF(Pu) \cap K^0_0 \cap \{x_0 \geq t\} = \emptyset$.

(ii) When $t > x^0_0$, $WF(u) \cap K^0_+ \cap \{x_0 = t\} \neq \emptyset$ if $WF(Pu) \cap K^0_+ \cap \{x_0 \leq t\} = \emptyset$.

(iii) If $WF(Pu) \cap K^0_0 \cap \{x_0 > 0\} = \emptyset$, then

$$\left(\bigcup_{j=0}^{1} WF((D^0_0u)(0, x'')) \cup WF_0(Pu))
\cap \{(x'', \xi''); (0, x'', \xi_0, \xi'') \in K^0_0 \text{ for some } \xi_0 \in \mathbb{R}\} \neq \emptyset.\right.$$

(II) (i) $\bigcup_{k=0}^{2} WF_0(D^0_0u) = (\bigcup_{j=0}^{1} WF((D^0_0u)(0, x'')) \cup WF_0(Pu))$.

(ii) Assume that the $a_{j,k}(x_0)$ can be extended to $\mathbb{R}$ so that $a_{j,k}(x_0) \in C^2(\mathbb{R})$ and $a(x_0, \xi') \geq 0$ and that $Pu \in C^\infty(\mathbb{R}^{n+1}_+)$, for simplicity. If $t > 0$ and $(x^{0\nu}, \xi^{0\nu}) \in \bigcup_{j=0}^{1} WF((D^0_0u)(0, x''))$, then

$$WF(u) \cap \{(x, \xi); x_0 = t \text{ and } (x, \xi) \in K^0_{0, x^{0\nu}, \xi^{0\nu}} \text{ for some } \xi_0 \in \mathbb{R}\} \neq \emptyset.$$

Let us illustrate Theorem 2 with some figures. Assume that $Pu \in C^\infty(\mathbb{R}^{n+1}_+)$,
for simplicity, and that $z^0 \in WF(u)$. In the right figure the intersection $K^+_z \cap \{ x^0 = t_1 \}$ consists of 4 points. Then Theorem 2 insists that at least one point of these 4 points in the intersection must belong to $WF(u)$. Similarly, at least one point of 2 points of the intersection $K^-_z \cap \{ x^0 = t_2 \}$ must belong to $WF(u)$ by Theorem 2. Moreover, at least one point of 4 points of $\{(x'', \xi''); (0, x'', \xi_0, \xi'') \in K^-_z \}$ for some $\xi_0 \in \mathbb{R}$ must belong to $\bigcup_{j=0}^{j-1} WF((D^j_0 u)(0, x''))$. Now we assume that $z''^0 \in \bigcup_{j=0}^{j-1} WF((D^j_0 u)(0, x''))$ and, for simplicity, $P_\xi u \in C^\infty(\mathbb{R}^{n+1})$.

In the right figure the broken curves are equal to $\bigcup_{\xi \neq 0} K^+_{(0, x'', \pm \sqrt{a(0, x'')}, \xi''')}$. The intersection of the broken curves and $\{ x^0 = t \}$ consists of 4 points in this figure. Theorem 2 insists that at least one of these 4 points must belong to $WF(u)$.

### 3. Examples

**Example 1.** Let $n = n' = 2$, $a(x_0, \xi'') = (-\xi_1 \sin x_0 + \xi_2 \cos x_0)^2$. Then $\bigcup_{\xi \neq 0} K^+_{(0, \xi)} \cap \{ x_0 = t \}$ is the following:
If $E(x)$ satisfies

$$\begin{cases} P(x,D)E(x) = 0 & \text{in } \mathbb{R}^3_+, \\ E(0,x'') = 0, \quad (D_0 E)(0,x'') = i\delta(x'') & \text{in } \mathbb{R}^2, \end{cases}$$

then $\text{sing supp } E \subset \bigcup_{\xi \neq 0} K^+_{(0,\xi)}$. We could not prove the equality.

**Example 2.** Let $n = n' = 2$, $a(x_0, \xi'') = ((x_0^2 - 2x_0)\xi_1 + \xi_2)^2$ and $P(x, \xi) = p(x_0, \xi)$. Then $\text{sing supp } E = \bigcup_{\xi \neq 0} K^+_{(0,\xi)}$ and $\bigcup_{\xi \neq 0} K^+_{(0,\xi)} \cap \{x_0 = t\}$ is as follows:
Here, in order to prove the equality we have used the fact that \( E(x, 0, -x''') = E(x) \) and results on branching of singularities for operators with non involutive characteristics given by Hanges and Ivrii.

4. Outline of Proof of Theorem 2

In order to prove Theorem 2 (I) (i) or (ii) we use results given in [KW]. To prove Theorem 2 (I) (iii) and (II) we apply the same arguments as used in [KW]. Let 

\[
z^0 \in T^* \mathbb{R}^{n+1}_+ \text{ satisfy } |\xi^0| = 1, \text{ and choose } \vartheta^0 \in \Gamma(p_{x,a}, \partial) \text{ so that } \sigma(r(z^0), \vartheta^0) = 0,
\]

where \( r(x, \xi) = \sum_{j=0}^n \xi_j \frac{\partial}{\partial \xi_j} \). Put

\[
\varphi(z; \kappa) = \tilde{\varphi}(z; \kappa)(1 + \tilde{\varphi}(z; \kappa)^2)^{-1/2},
\]

\[
\tilde{\varphi}(x, \xi; \kappa) = \sigma(\vartheta^0, (x - x^0, \xi/|\xi| - \xi^0)) + \kappa(|x - x^0|^2 + |\xi/|\xi| - \xi^0|^2),
\]

\[
\Lambda(x, \xi) = B\Psi(\xi/h)(\varphi(x, \xi; \kappa) - \nu) \log(\xi)_h + l \log(1 + \delta(\xi)_h),
\]

\[
P_\Lambda(x, D) = (e^{-A})(x, D)P(x, D)(e^A)(x, D),
\]

where \( h \geq 1, \kappa, B, l, \nu > 0, \delta \in [0, 1], (\xi)_h = (h^2 + |\xi|^2)^{1/2} \) and \( \Psi(\xi) \in S^0_{1,0} \) satisfies \( \Psi(\xi) = 1 \) for \( |\xi| \geq 1 \) and \( \Psi(\xi) = 0 \) if \( |\xi| \leq 1/2 \). We note that \(-H_\varphi(z^0) \equiv (-\nabla_\xi \varphi(z^0)), (\nabla_\xi \varphi(z^0)) = \vartheta^0\). In order to prove Theorem 2 (i) it suffices to show the following microlocal Carleman type estimates, choosing \( c_0, c_1, h \) so that \( 0 < c_0 < x_0'' < c_1 \) and \( h \gg 1 \): For any \( \kappa > 0 \) there are \( \nu_0 > 0, \chi_k(x, \xi) \in S^0_{1,0} \) ( \( k = 1, 2 \)) and \( l_k \in \mathbb{R} \) ( \( k = 1, 2, 3 \)) such that the \( \chi_k(z) \) are positively homogeneous of degree 0 for \( |\xi| \geq 1 \), \( \chi_k(z) = 1 \) near \( z^0 \), and “for any \( \nu \in (0, \nu_0] \) there is \( B_0 > 0 \) such that \( r \) for any \( B \geq B_0 \) there is \( l_0 \) such that for any \( l \geq l_0 \) there are \( \delta_0 \in (0, 1] \) and \( C > 0 \) satisfying

\[
||\chi_1(x, D/h) v||_{l_1} \leq C\{||P_\Lambda(x, D) v||_{l_2} + ||v||_{l_1-1} + ||(1 - \chi_2(x, D/h)) v||_{l_3}\}
\]

if \( v \in C_0^\infty((c_0, c_1) \times \mathbb{R}^n) \) and \( 0 < \delta \leq \delta_0 \).” Here \( || \cdot ||_l \) denotes the Sobolev norm of order \( l \). So an essential part is to show the above estimates. We omit it as it is
long. The proof of Theorem 2 will be given in a forthcoming paper.

References


[W] S. Wakabayashi, On the Cauchy problem for second-order hyperbolic operators with the coefficients of their principal parts depending only on the time variable. (located in http://www.math.tsukuba.ac.jp/~wkbysh/)