Remarks on semi-algebraic functions II

Seiichiro Wakabayashi

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This note is a supplement to [W]. In this note we slightly modify the definition of semi-algebraic functions as follows.

**Definition 1.** (i) Let $U$ be a semi-algebraic set in $\mathbb{R}^n$, and let $f(X)$ be a real-valued function defined in $U$. We say that $f(X)$ is semi-algebraic in $U$ if the graph of $f \equiv \{(X, y) \in U \times \mathbb{R}; y = f(X)\}$ is a semi-algebraic set.

(ii) Let $X_0 \in \mathbb{R}^n$, and let $f(X)$ be a real-valued function defined in a neighborhood of $X_0$. We say that $f(X)$ is semi-algebraic at $X_0$ if there is $r > 0$ such that $f(X)$ is semi-algebraic in $B_r(X_0) \equiv \{X \in \mathbb{R}^n; |X - X_0| < r\}$.

(iii) When $f(x)$ is a complex-valued function, we say that $f(X)$ is semi-algebraic in $U$ (resp. at $X_0$) if $\text{Re} f(X)$ and $\text{Im} f(X)$ are semi-algebraic in $U$ (resp. at $X_0$).

**Lemma 2.** Let $m, n \in \mathbb{Z}_+$, and let $S$ and $T$ be semi-algebraic sets in $\mathbb{R}^{n+m}$. For $X \in \mathbb{R}^n$ we define

$$T(X) = \{Y \in \mathbb{R}^n; (X, Y) \in T\}.$$  

Then the set

$$A \equiv \{X \in \mathbb{R}^n; (X, Y) \in S \text{ for } \forall Y \in T(X)\}$$

is a semi-algebraic set in $\mathbb{R}^n$.

**Remark.** Let $U$ be a semi-algebraic set in $\mathbb{R}^n$. Then $\{X \in U; (X, Y) \in S \text{ for } \forall Y \in T(X)\}$ is semi-algebraic.

**Proof.** We have

$$A^c(= \mathbb{R}^n \setminus A) = \{X \in \mathbb{R}^n; \exists Y \in T(X) \text{ s.t. } (X, Y) \in S^c\}$$

$$= \{X \in \mathbb{R}^n; \exists Y \in \mathbb{R}^m \text{ s.t. } (X, Y) \in T \cap S^c\}.$$  

From Lemma 2 in [W] $T \cap S^c$ is semi-algebraic. So the Tarski-Seidenberg Theorem implies that $A^c$ is semi-algebraic (see, e.g., Theorem 3 in [W]). Thus $A$ is semi-algebraic. \[\square\]
Theorem 3. Let $U$ be a semi-algebraic set in $\mathbb{R}^n$, and let $t(X)$ be a semi-algebraic function in $U$ satisfying $t(X) > 0$. Put

$$
\Omega = \{(X, t) \in U \times \mathbb{R}; 0 < t < t(X)\},
$$

and let $f(X, t)$ be a real-valued semi-algebraic function in $\Omega$. If $g(X) = \lim_{t \to 0} f(X, t)$ exists for $X \in U$, then $g(X)$ is semi-algebraic in $U$.

Proof. By definition $G \equiv \{(X, t, y) \in \Omega \times \mathbb{R}; y = f(X, t)\}$ is semi-algebraic.

$$
A = \{(X, t, y, \varepsilon, \delta, f) \in \mathbb{R}^{n+5}; X \in U, \varepsilon > 0, 0 < \delta \leq t(X), 0 < t < \delta \text{ and } (X, t, f) \in G\}.
$$

Then $A$ is semi-algebraic. For $X \in U$, $y \in \mathbb{R}$, $\varepsilon > 0$ and $\delta \in (0, t(X)]$ we define

$$
A(X, y, \varepsilon, \delta) = \{(t, f) \in \mathbb{R}^2; (X, t, y, \varepsilon, \delta, f) \in A\}.
$$

Moreover, we put

$$
B = \{(X, y, \varepsilon, \delta) \in \mathbb{R}^{n+3}; X \in U, \varepsilon > 0, \delta \in (0, t(X)] \text{ and } (f - y)^2 \leq \varepsilon^2 \text{ for } \forall (t, f) \in A(X, y, \varepsilon, \delta)\}
$$

$$
C = \{(X, y, \varepsilon) \in \mathbb{R}^{n+2}; \exists \delta \in \mathbb{R} \text{ s.t. } (X, y, \varepsilon, \delta) \in B\},
$$

$$
D = \{(X, y) \in \mathbb{R}^{n+1}; (X, y, \varepsilon) \in C \text{ for } \forall \varepsilon > 0\}.
$$

From Lemma 2 (or its remark) it follows that $B$ is semi-algebraic and, therefore, $C$ is semi-algebraic by the Tarski-Seidenberg theorem. Moreover, it follows from Corollary of Theorem 3 in [W] that $D$ is semi-algebraic. On the other hand, we have

$$
D = \{(X, y) \in \mathbb{R}^{n+1}; X \in U \text{ and } y = g(X)\}.
$$

Indeed, for each $X \in U$ and any $\varepsilon > 0$ there is $\delta > 0$ such that

$$
|f(X, t) - y| < \varepsilon \text{ for any } t \in (0, \delta)
$$

and, therefore, $y = g(X)$, if $(X, y) \in D$. It is obvious that $(X, g(X)) \in D$ if $X \in U$. So $g(X)$ is semi-algebraic in $U$. \hfill \Box

Corollary. Let $U$ be an open semi-algebraic set in $\mathbb{R}^n$, and let $f(X)$ be real-valued and semi-algebraic in $U$. Assume that $(\partial / \partial X_1) f(X)$ exists for $X \in U$. Then $(\partial / \partial X_1) f(X)$ is semi-algebraic in $U$.

Proof. Put

$$
E = \{(X, \delta) \in U \times (0, 1]; B_\delta(X) \subset U\}.
$$
It is obvious that \( E \cap \{X\} \times \mathbb{R} \neq \emptyset \) for each \( X \in E \). We define
\[
t(X) = \sup \{\delta; (X, \delta) \in E\}.
\]
\( t(X) \) is semi-algebraic in \( U \) (see, e.g., Corollary A.2.4 of [H]). Put
\[
f(X, t) = \frac{1}{t} (f(X + te_1) - f(X)),
\]
where \( e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n \). Applying Theorem 3 we complete the proof, since
\[
\lim_{t \downarrow 0} f(X, t) = (\partial / \partial X_1) f(X).
\]
\( \square \)

**Lemma 4.** Let \( I \) be an interval of \( \mathbb{R} \), and let \( F(t) (\neq 0) \) be real analytic and semi-algebraic in \( I \). Then the set \( A \equiv \{t \in I; F(t) = 0\} \) is finite.

**Proof.** Since \( A \) is semi-algebraic, \( A \) is defined by a finite family \( \{A_j; 1 \leq j \leq M\} \) of semi-algebraic subsets of \( \mathbb{R} \), where \( A_j = \{t \in \mathbb{R}; p_j(t) = 0\} \) or \( A_j = \{t \in \mathbb{R}; p_j(t) > 0\} \) with polynomials \( p_j(t) (\neq 0) \) \( (1 \leq j \leq M) \). Suppose that there is \( t_0 \in A \) satisfying \( p_j(t_0) \neq 0 \) for any \( j \). Then there is \( \delta > 0 \) satisfying \( (t_0 - \delta, t_0 + \delta) \subset A \), which contradicts discreteness of the set \( A \). Therefore, we have
\[
A \subset \bigcup_{j=1}^{M} \{t \in \mathbb{R}; p_j(t) = 0\}
\]
which implies that \( A \) is finite. \( \square \)

**Theorem 5.** Let \( I \) be an interval of \( \mathbb{R} \), and assume that \( a_j(t) \in C^\infty(I) \) \( (1 \leq j \leq m) \) are semi-algebraic in \( I \), where \( m \in \mathbb{N} \). If \( \lambda(t) \in C(I) \) satisfies
\[
\lambda(t)^m + a_1(t) \lambda(t)^{m-1} + \cdots + a_m(t) = 0 \quad \text{in } I,
\]
then \( \lambda(t) \) is semi-algebraic in \( I \).

**Proof.** There are \( m' \in \mathbb{N} \) and semi-algebraic functions \( \tilde{a}_j(t) \) in \( I \) \( (1 \leq j \leq m') \) such that the \( \tilde{a}_j(t) \in C^\infty(I) \) and
\[
(\text{Re} \lambda(t))^{m'} + \tilde{a}_1(t) (\text{Re} \lambda(t))^{m'-1} + \cdots + \tilde{a}_{m'}(t) = 0 \quad \text{in } I.
\]
Here the \( \tilde{a}_j(t) \) are given as polynomials of \( a_1(t), \tilde{a}_1(t), \cdots, a_m(t), \tilde{a}_m(t) \). For \( \text{Im} \lambda(t) \) we have the same. So we may assume that \( \lambda(t) \) is real-valued. Moreover, we may assume that the \( a_j(t) \) are real-valued. From the proof of Theorem 10 in [W] we see that the \( a_j(t) \) are real analytic in \( I \). We define
\[
\mathcal{B} = \{a(t); a(t) \text{ is a complex-valued semi-algebraic function}\}
\]
defined in $I$ and real analytic in $I$.\)

It follows from Lemma 9 in [W] (or its proof) that $\mathcal{B}$ is a subring of $\mathcal{A}(I)$, where $\mathcal{A}(I)$ denotes the space of real analytic functions defined in $I$. We denote by $\mathcal{B}$ the quotient field of $\mathcal{B}$. Write

$$P(\lambda, t) = \lambda^m + a_1(t)\lambda^{m-1} + \cdots + a_m(t) \in \mathcal{B}[\lambda] \subset \mathcal{B}[\lambda].$$

Then there are $s \in \mathbb{N}$, $m_j \in \mathbb{N}$ and irreducible polynomials $P_j(\lambda, t) \in \mathcal{B}[\lambda]$ ($1 \leq j \leq s$) such that $P_1(\lambda, t), \ldots, P_s(\lambda, t)$ are mutually prime and

$$P(\lambda, t) = P_1(\lambda, t)^{m_1} \cdots P_s(\lambda, t)^{m_s}.$$  

We note that the $P_j(\lambda, t)$ can be chosen in $\mathcal{B}[\lambda]$ (see, e.g., IV§6 of [L]). Put

$$Q(\lambda, t) = P_1(\lambda, t) \cdots P_s(\lambda, t),$$

and denote by $D(t)$ the discriminant of $Q(\lambda, t) = 0$ in $\lambda$. Then we have $D(t) \neq 0$ in $I$, since $Q(\lambda, t)$ and $(\partial/\partial \lambda)Q(\lambda, t)$ are mutually prime. By Lemma 4 we can write

$$\{ t \in I; D(t) = 0 \} = \{ \tau_1, \tau_2, \ldots, \tau_N \}, \quad \tau_1 < \tau_2 < \cdots < \tau_N.$$  

Put

$$I_0 = (-\infty, \tau_1) \cap I, \quad I_1 = (\tau_1, \tau_2), \quad \ldots, \quad I_{N-1} = (\tau_{N-1}, \tau_N), \quad I_N = (\tau_N, \infty) \cap I.$$  

Then $Q(\lambda, t) = 0$ in $\lambda$ has only simple roots for $0 \leq j \leq N$ and $t \in I_j$. We fix $j \in \{0, 1, \ldots, N\}$. For $t \in I_j$ we can write

$$Q(\lambda, t) = \prod_{k=1}^{\hat{m}} (\lambda - \lambda_{j,k}(t)), \quad \lambda_{j,1}(t) < \lambda_{j,2}(t) < \cdots < \lambda_{j,r(j)}(t), \quad \text{Im}\lambda_{j,k}(t) \neq 0 \ (r(j) + 1 \leq k \leq \hat{m}),$$

where $\hat{m} = \deg_{\lambda} Q(\lambda, t)$ and $1 \leq r(j) \leq \hat{m}$. By assumption there is $k(j) \in \mathbb{N}$ such that $1 \leq k(j) \leq r(j)$ and $\lambda(t) = \lambda_{j,k(j)}(t)$ for $t \in I_j$. Put

$$E = \{(z, t, Q(z, t)) \in \mathbb{R}^3; \ t \in I \},$$

$$F_j = \{(t, y) \in I_j \times \mathbb{R}; \ \exists \lambda_1, \ldots, \lambda_{\hat{m}} \in \mathbb{C} \text{ s.t.} \}$$

$$\left(z, t, \prod_{k=1}^{\hat{m}} (z - \lambda_k) \right) \in E \text{ for } \forall z \in \mathbb{R}, \ \lambda_1 < \lambda_2 < \cdots < \lambda_{r(j)},$$

$$\text{Im}\lambda_k \neq 0 \ (r(j) + 1 \leq k \leq \hat{m}) \text{ and } y = \lambda_{j,k(j)} \}.\]
It is obvious that $E$ and $F_j$ are semi-algebraic and 

$$F_j = \{(t,y) \in I_j \times \mathbb{R}; \ y = \lambda(t)\},$$

which implies that $\lambda(t)$ is semi-algebraic in $I_j$. Since $\bigcup_{j=1}^{N} \{(\tau_j, \lambda(\tau_j))\} \cup \bigcup_{j=0}^{N} F_j$ is semi-algebraic, $\lambda(t)$ is semi-algebraic in $I$. \hfill \Box

I could not prove Theorem 5 when $I$ is an open connected semi-algebraic subset of $\mathbb{R}^n$. Under stronger assumptions we have the following

**Theorem 6.** Let $U$ be an open semi-algebraic set in $\mathbb{R}^n$, and assume that $U$ is connected, and that $a_j(X)$ ($1 \leq j \leq m$) are real analytic and semi-algebraic in $U$, where $m \in \mathbb{N}$. Put 

$$P(\lambda, X) = \lambda^m + a_1(X)\lambda^{m-1} + \cdots + a_m(X).$$

Then $\lambda(X)$ is semi-algebraic in $U$ if $\lambda(X)$ is real analytic in $U$ and $P(\lambda(X), X) \equiv 0$ in $U$.

**Proof.** We may assume that $\lambda(X)$ is real-valued and that the $a_j(X)$ are real-valued (see the proof of Theorem 5). Let $X^0 \in U$, and denote by $\mathcal{A}$ the set of germs of real analytic functions at $X^0$. Then there are $s \in \mathbb{N}$, $m_j \in \mathbb{N}$ and irreducible polynomials $P_j(\lambda, X) \in \mathcal{A}[\lambda]$ ($1 \leq j \leq s$) such that $P_1(\lambda, X)$, $\ldots$, $P_s(\lambda, X)$ are mutually prime and

$$P(\lambda, X) = P_1(\lambda, X)^{m_1} \cdots P_s(\lambda, X)^{m_s}.$$ 

Put

$$Q(\lambda, X) = P_1(\lambda, X) \cdots P_s(\lambda, X),$$

and denote by $D(X)$ the discriminant of $Q(\lambda, X) = 0$ in $\lambda$. We choose a neighborhood $V$ of $X^0$ in $U$ so that $D(X)$ is defined in $V$. Since $D(X) \neq 0$ in $V$, there are $X^1 \in V$ and $\delta > 0$ such that $B_\delta(X^1) \subset V$ and $D(X) \neq 0$ for $X \in B_\delta(X^1)$. Then $Q(\lambda, X) = 0$ in $\lambda$ has only simple roots for $X \in B_\delta(X^1)$. For $X \in B_\delta(X^1)$ we can represent

$$Q(\lambda, X) = \prod_{k=1}^{\hat{m}} (\lambda - \lambda_k(X)),$$

$$\lambda_1(X) < \lambda_2(X) < \cdots < \lambda_r(X), \quad \text{Im} \lambda_k(X) \neq 0 \ (r + 1 \leq k \leq \hat{m}),$$

where $\hat{m} = \deg_\lambda Q(\lambda, X)$ and $1 \leq r \leq \hat{m}$. By assumption there is $k_0 \in \mathbb{N}$ such that $1 \leq k_0 \leq r$ and $\lambda(X) = \lambda_{k_0}(X)$ in $B_\delta(X^1)$. There are $l_k \in \mathbb{N}$ ($1 \leq k \leq \hat{m}$) such that

$$P(\lambda, X) = \prod_{k=1}^{\hat{m}} (\lambda - \lambda_k(X))^{l_k} \quad \text{for} \ X \in B_\delta(X^1).$$

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By Lemma 9 in [W] (or its proof) \( E \equiv \{(z,X,P(z,X)) \in \mathbb{R}^{n+2}; X \in B_\delta(X^1)\} \) is semi-algebraic. Define

\[
F = \{(X,y) \in B_\delta(X^1) \times \mathbb{R}; \exists \lambda_1, \cdots, \lambda_{\hat{m}} \in \mathbb{C} \text{ s.t.} \\
(z,X, \prod_{k=1}^{\hat{m}} (z - \lambda_k)^k) \in E \text{ for } \forall z \in \mathbb{R}, \lambda_1 < \lambda_2 < \cdots < \lambda_r, \\
\text{Im}\lambda_k \neq 0 \text{ ( } r + 1 \leq k \leq \hat{m} \text{) and } y = \lambda_{k_0}\}.
\]

Then \( F \) is semi-algebraic and

\[
F = \{(X,y) \in B_\delta(X^1) \times \mathbb{R}; \ y = \lambda(X)\},
\]

which implies \( \lambda(X) \) is semi-algebraic at \( X^1 \). It follows from Theorem 10 in [W] (or its proof) that there is an irreducible polynomial \( P(z,X)(\neq 0) \) of \( (z,X) \) satisfying \( P(\lambda(X),X) \equiv 0 \) near \( X^1 \). Since \( \lambda(X) \) is real analytic in \( U \), by analytic continuation we have \( \tilde{P}(\lambda(X),X) \equiv 0 \) in \( U \). Theorem 11 in [W] (or its proof) implies that \( \lambda(X) \) is semi-algebraic in \( U \).

\[ \square \]

**References**

