Remarks on the conditions (L) and (L)$_0$ in the paper “On the Cauchy problem for hyperbolic operators with double characteristics whose principal parts have time dependent coefficients”

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Let the conditions (L) and (L)$_0$ be given in [3]. In [3] we asserted the following lemma.

**Lemma 1.** The condition (L)$_0$ is satisfied if the condition (L) is satisfied.

In this note we shall prove Lemma 1. Let $U$ be an open subset of $\mathbb{R}^n$, and let $a(t, \xi)$ be a real analytic function defined in $[0, \delta_0] \times \overline{U}$, where $\delta_0 > 0$. Then there is a compact complex neighborhood $\Omega_a$ of $[0, \delta_0]$ such that $a(t, \xi)$ is regarded as an analytic function defined in $\Omega_a$ for $\xi \in \overline{U}$. We assume that $a(t, \xi) \geq 0$ for $(t, \xi) \in [0, \delta_0] \times \overline{U}$. Let $b(t, \xi)$ be real analytic in $[0, \delta_0] \times \overline{U}$.

Let $R_U(\xi) : U \ni \xi \mapsto R_U(\xi) \in \mathcal{P}(\mathbb{C})$ satisfy $#R_U(\xi) \leq N_U$ for any $\xi \in U$, where $N_U \in \mathbb{N}$ and $#A$ denotes the number of the elements of a set $A$. We choose $\delta \in (0, 1]$ so that $[-\delta, \delta_0 + \delta] \subset \Omega_a$. Let $c \in (0, 1]$, and let $R_{a, \delta, c}(\xi)$ ($\subset \mathbb{C}$) be a set-valued function defined for $\xi \in U$ satisfying the following:

(i) $\sup_{\xi \in U} #R_{a, \delta, c}(\xi) < \infty$.

(ii) If $\xi \in U$, $a(t, \xi) \neq 0$ in $t$, $\lambda \in \Omega_a$, $a(\lambda, \xi) = 0$, $|\text{Im} \lambda| \leq \delta$ and $\text{Re} \lambda \in [-\delta, \delta_0 + \delta]$, then there is $s \in R_{a, \delta, c}(\xi)$ satisfying $|\text{Im} \lambda| \geq c|(\text{Re} \lambda)_+ - s|$.

Lemma 1 easily follows from Lemma 2 below and the compactness argument.

**Lemma 2.** There are positive constants $\delta_1$ and $A \equiv A(a, \delta, c)$ independent of $\xi$ such that

$$(L)_{a, \delta, c} \quad \min_{s \in R_{a, \delta, c}(\xi)} \min \left\{ \frac{|t - s|}{1 + |b(t, \xi)|} |b(t, \xi)| \leq AC \sqrt{a(t, \xi)} \right\}$$

for $(t, \xi) \in [0, \delta_1] \times U$
if, with $C \geq 1$,

$$(L)_U \min \left\{ \min_{s \in R_U(t, \xi)} |t - s|, 1 \right\} |b(t, \xi)| \leq C \sqrt{a(t, \xi)} \quad \text{for } (t, \xi) \in [0, \delta_1] \times U,$$

where $\min_{s \in \emptyset} |t - s| = 1$.

Proof. Assume that $(L)_U$ holds, and put

$$\kappa(\xi) = \int_0^{\delta_0} a(t, \xi) \, dt.$$  

If $\kappa(\xi) \equiv 0$, then $a(t, \xi) \equiv b(t, \xi) \equiv 0$ in $(t, \xi)$ and the lemma becomes trivial. Assume that $\kappa(\xi) \not\equiv 0$. Let $\xi^0 \in \overline{U}$. We apply Hironaka’s resolution theorem to $\kappa(\xi)$ (see [1]). Then there are an open neighborhood $U(\xi^0)$ of $\xi^0$, a real analytic manifold $\widetilde{U}(\xi^0)$, a proper analytic mapping $\varphi \equiv \varphi(\xi^0)$: $\widetilde{U}(\xi^0) \ni \tilde{u} \mapsto \varphi(\tilde{u})(\equiv \varphi(\tilde{u}; \xi^0)) \subset U(\xi^0)$ satisfying the following:

(i) $\varphi$: $\widetilde{U}(\xi^0) \setminus \tilde{A} \to U(\xi^0) \setminus A$ is an isomorphism, where $A = \{ \xi \in U; \kappa(\xi) = 0 \}$ and $\tilde{A} = \varphi^{-1}(A)$.

(ii) For each $p \in \widetilde{U}(\xi^0)$ there are local analytic coordinates $X(\equiv \mathcal{X}^p) = (X_1, \cdots, X_n) (= (X_1^p, \cdots, X_n^p))$ centered at $p$, $r(p) \in \mathbb{Z}_+$ with $r(p) \leq n$, $s_k(p) \in \mathbb{N}$ ($1 \leq k \leq r(p)$); a neighborhood $\widetilde{U}(\xi^0; p)$ of $p$ and a real analytic function $e(X)(\equiv e(X^p; p))$ in $\widetilde{V}(\xi^0; p)$ such that $e(X) > 0$ for $X \in \widetilde{V}(\xi^0; p)$ and

$$\kappa(\varphi(\tilde{u})) = e(X(\tilde{u})) \prod_{k=1}^{r(p)} X_k(\tilde{u})^{2s_k(p)} \quad (\tilde{u} \in \widetilde{U}(\xi^0; p)),$$

where $\widetilde{V}(\xi^0; p) = \{ X(\tilde{u}); \tilde{u} \in \widetilde{U}(\xi^0; p) \}$ and $\prod_{k=1}^{r(p)} \cdots = 1$ if $r(p) = 0$. Here $\widetilde{V}(\xi^0; p)$ is a neighborhood of 0 in $\mathbb{R}^n$. Define $\tilde{\varphi} (\equiv \tilde{\varphi}(\xi^0, p)) : \widetilde{V}(\xi^0; p) \to U(\xi^0)$ by $\tilde{\varphi}(X(\tilde{u}))(\equiv \tilde{\varphi}(X^p(\tilde{u}); \xi^0, p)) = \varphi(\tilde{u})(\equiv \varphi(\tilde{u}; \xi^0))$ for $\tilde{u} \in \widetilde{U}(\xi^0; p)$.

Let $U_0(\xi^0)$ be a compact neighborhood of $\xi^0$ in $U(\xi^0)$, and put $\tilde{U}_0(\xi^0) = \varphi^{-1}(U_0(\xi^0))$. Fix $p \in \tilde{U}_0(\xi^0)$, and put

$$\alpha(p) = (s_1(p), \cdots, s_{r(p)}(p), 0, \cdots, 0) \in (\mathbb{Z}_+)^n.$$

From $(L)_U$ it is easy to see that there is a real analytic function $d(t, X; p)$ defined in $[0, \delta_0] \times \widetilde{V}(\xi^0; p)$ satisfying

$$|b(t, \tilde{\varphi}(X; \xi^0, p))|^2 = d(t, X; p)X^{2\alpha(p)} \quad \text{for } (t, X) \in [0, \delta_0] \times \widetilde{V}(\xi^0; p).$$
From (2.6) and (2.7) of [3] we can also write

\[ a(t, \hat{\phi}(X; \xi^0, p)) = c(t, X; p) f(t, X; p) X^{2 \alpha(p)}, \]

\[ f(t, X; p) = t^{m(p)} + a_1(X; p) t^{m(p)-1} + \cdots + a_{m(p)}(X; p) \]

for \((t, X) \in [0, \delta(p)] \times \tilde{V}(p)\), where \(0 < \delta(p) \leq \min\{\delta_0, 1\}\), \(\tilde{V}(p)\) is a compact neighborhood of 0 in \(\tilde{V}(\xi^0; p)\), \(m(p) \in \mathbb{Z}_+\), \(c(t, X; p)\) is a real analytic function defined in \([0, \delta(p)] \times \tilde{V}(p)\) satisfying \(c(t, X; p) > 0\) and the \(a_k(X; p)\) are real analytic functions defined in \(\tilde{V}(p)\). By the Weierstrass division theorem there are a polynomial \(g(t, X; p)\) of \(t\) with real analytic coefficients of \(X\) defined in \(\tilde{V}(p)\), and a real analytic function \(h(t, X; p)\) defined in \([0, \delta(p)] \times \tilde{V}(p)\) satisfying \(\deg_t g(t, X; p) < m(p)\) and

\[ d(t, X; p) = h(t, X; p) f(t, X; p) + g(t, X; p) \quad \text{in} \quad [0, \delta(p)] \times \tilde{V}(p), \]

modifying \(\delta(p)\) and \(\tilde{V}(p)\) if necessary. Fix \(X \in \tilde{V}(p)\) with \(X^{\alpha(p)} \neq 0\), and put \(\xi = \hat{\phi}(X; \xi^0, p)\). Then (L)_{U} implies that

\[ \begin{aligned}
& \min \left\{ \min_{s \in \mathcal{R}_c(\xi)} |t - s|^2, 1 \right\} |g(t, X; p)| \leq C^2 C_1(p) f(t, X; p) \\
& \quad \text{for} \quad t \in [0, \delta(p)], \\
& \quad \text{where} \\
& \quad C_1(p) = \max_{(s, Y) \in [0, \delta(p)] \times \tilde{V}(p)} \left( c(s, Y; p) + |h(s, Y; p)| \right) + 1.
\end{aligned} \]

Let us prove that there is a positive constant \(A(p, \delta, c)\) independent of \(X\) such that

\[ \begin{aligned}
& \min \left\{ \min_{s \in \mathcal{R}_{a, \delta, c}(\xi)} |t - s|^2, 1 \right\} |g(t, X; p)| \leq A(p, \delta, c) C^2 C_1(p) f(t, X; p) \\
& \quad \text{for} \quad t \in [0, \delta(p)]. \\
& \quad \text{If} \quad t \in [0, \delta(p)] \cap \mathcal{R}_{a, \delta, c}(\xi), \quad \text{then} \quad (2) \quad \text{holds.} \\
& \quad \text{So we may assume} \\
& \quad \text{that} \quad t \in [0, \delta(p)] \setminus \mathcal{R}_{a, \delta, c}(\xi) \quad \text{and} \quad g(s, X, p) \neq 0 \quad \text{in} \quad s. \\
& \quad \text{If} \quad \mathcal{R}_{a, \delta, c}(\xi) = \emptyset, \\
& \quad \text{then} \quad \text{we have} \quad a(s, \xi) \neq 0 \quad \text{in} \quad s \quad \text{and} \\
& \quad a(\lambda, \xi) \neq 0 \quad \text{if} \quad \lambda \in \Omega_{a}, \quad \text{Re} \lambda \in [\delta, \delta_0 + \delta] \quad \text{and} \quad |\text{Im} \lambda| \leq \delta,
\end{aligned} \]

since \(\kappa(\xi) \neq 0\). This implies that

\[ f(t, X; p) \geq \delta^{m(p)} \quad \text{if} \quad \mathcal{R}_{a, \delta, c}(\xi) = \emptyset. \]

Therefore, (2) is valid if \(\mathcal{R}_{a, \delta, c}(\xi) = \emptyset\) and

\[ A(p, \delta, c) \geq \delta^{-m(p)} \max_{(s, Y) \in [0, \delta(p)] \times \tilde{V}(p)} |g(s, Y; p)|. \]
Thus we may assume that $\mathcal{R}_{a,\delta,c}(\xi) \neq \emptyset$. Then there is $\lambda_0 \in \mathcal{R}_{a,\delta,c}(\xi)$ satisfying $\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |t - \lambda| = |t - \lambda_0|$. Now let us apply the argument in the proof of Lemma 2.1 of [2]. First consider the case where $t \geq \delta(p)/2$. We put

$$s_0 = \max\{0, t - |t - \lambda_0|\},$$

and divide the interval $(s_0, t]$ into $(m(p) + N_U)$ subintervals with equal length. Write

$$g(s, X; p) = d(p) \prod_{j=1}^{\bar{\mu}} (s - \mu_j),$$

where $\bar{\mu} < m(p)$ and $\mu_j \in \mathbb{C}$. Then there is $\hat{k} \in \mathbb{N}$ such that $\hat{k} \leq m(p) + N_U$ and

$$(s_0 + (\hat{k} - 1)\rho, s_0 + \hat{k}\rho] \cap (\{\Re \mu_j; 1 \leq j \leq \bar{\mu}\} \cup \{\Re \lambda; \lambda \in \mathcal{R}_U(\xi)\}) = \emptyset,$$

where $\rho = (t - s_0)/(m(p) + N_U)$. Put

$$I = [s_0 + (\hat{k} - 2/3)\rho, s_0 + (\hat{k} - 1/3)\rho].$$

(i) Let $\tilde{t} \in I$. Then we have

\[
(3) \quad \min\{\min_{\lambda \in \mathcal{R}_U(\xi)} |\tilde{t} - \lambda|, 1\} \geq \rho/3,
\]

\[
(4) \quad \min\{\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |\tilde{t} - \lambda|, 1\} \leq \min\{|\tilde{t} - \lambda_0|, 1\} \leq 2 \min\{|t - \lambda_0|, 1\},
\]

since $\rho \leq 1$ and $0 < t - s_0 \leq |t - \lambda_0|$. Since $\rho = |t - \lambda_0|/(m(p) + N_U)$ if $t \geq |t - \lambda_0|$, and $\rho = t/(m(p) + N_U)$ otherwise, we have

\[
(5) \quad \rho \geq \delta(p) \min\{|t - \lambda_0|, 1\}/(2(m(p) + N_U)).
\]

This, together with (3) and (4), gives

\[
(6) \quad \min\{\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |\tilde{t} - \lambda|, 1\} \leq 4(m(p) + N_U)\rho/\delta(p)
\]

\[
\leq 12(m(p) + N_U) \min\{\min_{\lambda \in \mathcal{R}_U} |\tilde{t} - \lambda|, 1\}/\delta(p).
\]

From (1) and (6) we have

\[
(7) \quad \min\{\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |\tilde{t} - \lambda|^2, 1\} |g(\tilde{t}, X; p)|
\]

\[
\leq 144(m(p) + N_U)^2 C^2 C_1(p) f(\tilde{t}, X; p)/\delta(p)^2.
\]
(ii) (5) implies that
\[
\min\left\{ \min_{\lambda \in \mathcal{R}_{a,b,c}(\xi)} |t - \lambda|, 1 \right\} = \min\{ |t - \lambda_0|, 1 \} \leq 2(m(p) + N_U)\rho/\delta(p).
\]

On the other hand, we have
\[
\min\left\{ \min_{\lambda \in \mathcal{R}_{a,b,c}(\xi)} |\tilde{t} - \lambda|, 1 \right\} \geq \rho/3 \quad \text{for } \tilde{t} \in I,
\]
since \(|\tilde{t} - \lambda| \geq |t - \lambda| - |t - \tilde{t}| \geq |t - \lambda_0| - |t - \tilde{t}|\) for \(\lambda \in \mathcal{R}_{a,b,c}(\xi)\), and
\(|t - \tilde{t}| = t - \tilde{t} \leq t - s_0 - \rho/3 \leq |t - \lambda_0| - \rho/3\) for \(\tilde{t} \in I\). This, together with (5), gives
\[
\min\left\{ \min_{\lambda \in \mathcal{R}_{a,b,c}(\xi)} |t - \lambda|, 1 \right\} = \min\{ |t - \lambda_0|, 1 \}
\leq 2(m(p) + N_U)\rho/\delta(p) \leq 6(m(p) + N_U)\min\left\{ \min_{\lambda \in \mathcal{R}_{a,b,c}(\xi)} |\tilde{t} - \lambda|, 1 \right\}/\delta(p)
\]
for \(\tilde{t} \in I\). It is obvious that \(|\tilde{t} - \text{Re} \mu_j| \geq \rho/3\) and
\[
|t - \text{Re} \mu_j| \leq |\tilde{t} - \text{Re} \mu_j| \quad \text{if } (t + \tilde{t})/2 \leq \text{Re} \mu_j,
\]
\[
|t - \text{Re} \mu_j| \leq 2|\tilde{t} - \text{Re} \mu_j| \quad \text{if } \text{Re} \mu_j \leq 2\tilde{t} - t,
\]
\[
0 < t - \text{Re} \mu_j \leq 2(t - \tilde{t}) \leq 2(m(p) + N_U - 1/3)\rho
\]\nif \(2\tilde{t} - t \leq \text{Re} \mu_j \leq (t + \tilde{t})/2\),

for \(\tilde{t} \in I\) and \(1 \leq j \leq \bar{\mu}\). Noting that \(|\tilde{t} - \text{Re} \mu_j| \geq \rho/3\), we have
\[
|t - \text{Re} \mu_j| \leq 6(m(p) + N_U)|\tilde{t} - \text{Re} \mu_j| \quad \text{for } \tilde{t} \in I \text{ and } 1 \leq j \leq \bar{\mu},
\]
which gives
\[
|g(t, X; p)| \leq \{6(m(p) + N_U)\}_{j=1}^{m(p)-1}|g(\tilde{t}, X; p)| \quad \text{for } \tilde{t} \in I.
\]

We write
\[
f(s, X; p) = \prod_{j=1}^{m(p)} (s - \lambda_j).
\]

We may assume that \(f(s, X; p)\) is defined in \(\mathbb{R} \times \tilde{V}(p), \text{Re} \lambda_j \in [-\delta(p), \delta(p)]\) for \(1 \leq j \leq m(p)\), modifying \(\tilde{V}(p)\) if necessary. Let \(1 \leq j \leq m(p)\). If \(|\text{Im} \lambda_j| > \delta\), then we have
\[
|\tilde{t} - \lambda_j|^2 \leq 4\delta(p)^2 + |\text{Im} \lambda_j|^2 \leq (1 + (2/\delta)^2)|\text{Im} \lambda_j|^2 \leq (3/\delta)^2|\text{Im} \lambda_j|^2
\]
for $\tilde{t} \in I$. If $|\text{Im} \lambda_j| \leq \delta$ and $\text{Re} \lambda_j < -\delta$, then we have

$$|\tilde{t} - \lambda_j|^2 \leq 4\delta(p)^2 + \delta^2 \leq (1 + (2/\delta)^2)|t - \lambda_j|^2 \leq (3/\delta)^2|t - \lambda_j|^2 \quad \text{for } \tilde{t} \in I.$$ (11)

Let $|\text{Im} \lambda_j| \leq \delta$ and $\text{Re} \lambda_j \geq -\delta$. Then there is $s_j \in \mathcal{R}_{a,\delta,c}(\xi)$ satisfying $|\text{Im} \lambda_j| \geq c|\text{Re} \lambda_j| + s_j$. Therefore, we have

$$|t - \lambda_j| \geq (c|t - (\text{Re} \lambda_j)| + |\text{Im} \lambda_j|)/2$$

$$\geq (c|t - s_j| - c|\text{Re} \lambda_j| + s_j| + |\text{Im} \lambda_j|)/2$$

$$\geq c|t - s_j|/2 \geq c|t - \lambda_0|/2 \geq c|\tilde{t} - \lambda_j|/4 \quad \text{if } |\tilde{t} - \lambda_j| \leq 2|t - \lambda_0|.$$ (12)

$$|t - \lambda_j| \geq |\tilde{t} - \lambda_j| - |t - \tilde{t}| \geq |\tilde{t} - \lambda_j| - |t - s_0| \geq |\tilde{t} - \lambda_j|/2$$

$$\quad \text{if } |\tilde{t} - \lambda_j| \geq 2|t - \lambda_0|.$$ (13)

It follows from (10) – (13) that

$$f(t, X; p) \geq (\min\{\delta/3, c/4\})^{|m(p)|} f(\tilde{t}, X; p) \quad \text{for } \tilde{t} \in I.$$ (14)

(iii) From (7) – (9) and (14) we have

$$\min\{}_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} \min\{|t - \lambda|^2, 1\} |g(t, X; p)|$$

$$\leq 4 \cdot 6^{3+|m(p)|} (m(p) + N_U)^{3+|m(p)|} (\min\{\delta/3, c/4\})^{-m(p)} \delta(p)^{-4}$$

$$\times C^2 C_1(p) f(t, X; p).$$

So (2) holds if $\mathcal{R}_{a,\delta,c}(\xi) \neq \emptyset$, $t \in [\delta(p)/2, \delta(p)]$ and

$$A(p, \delta, c) \geq 4 \cdot 6^{3+|m(p)|} (m(p) + N_U)^{3+|m(p)|} (\min\{\delta/3, c/4\})^{-m(p)} \delta(p)^{-4}.$$ Assume that $t < \delta(p)/2$. Then, putting

$$s_0 = \min\{\delta(p), t + |t - \lambda_0|\}$$

and dividing the interval $[t, \delta_0]$ into $(m(p) + N_U)$ subintervals with equal length we repeat the arguments above to prove (2). (2) yields

$$\min\{}_{s \in \mathcal{R}_{a,\delta,c}(\xi)} \min\{|t - s|^2, 1\} |b(t, \xi)|^2 \leq A'(p, \delta, c) C^2 C_1(p) a(t, \xi)$$

for $t \in [0, \delta(p)]$ and $\xi = \tilde{\varphi}(X, \xi^0, p)$ with $X \in \tilde{V}(p)$,

where

$$A'(p, \delta, c) = A(p, \delta, c) \max_{(s, Y) \in [0, \delta(p)] \times \tilde{V}(p)} |c(s, Y; p)|^{-1} (1 + |h(s, Y; p)|).$$
Put $\tilde{U}(p) = (X^p)^{-1}(\tilde{V}(p)) (\subset \tilde{U}(\xi^0; p))$. Since $\overline{U}$ is compact, there are $N \in \mathbb{N}$ and $\xi^j \in \overline{U}$, $1 \leq j \leq N$ such that $\overline{U} \subset \bigcup_{j=1}^N \tilde{U}_0(\xi^j)$. Here $A$ denotes the interior of $A$ (⊂ $\mathbb{R}^n$). Since $\tilde{U}_0(\xi^j)$ is compact, there are $P_j \in \mathbb{N}$ and $p^{j,k} \in \tilde{U}_0(\xi^j)$, $1 \leq k \leq P_j$ such that $\tilde{U}_0(\xi^j) \subset \bigcup_{k=1}^{P_j} \tilde{U}(p^{j,k})$. Therefore, putting

$$A(a, \delta, c) = \max\{A'(p^{j,k}, \delta, c)^{1/2}C_1(p^{j,k})^{1/2}; 1 \leq j \leq N \text{ and } 1 \leq k \leq P_j\},$$

we complete the proof of Lemma 2. □

References

