Remarks on Propagation of Analytic Singularities and Solvability in the Space of Microfunctions

Seiichiro Wakabayashi (University of Tsukuba)

1. Introduction

Let $P$ be a linear partial differential operator on $\mathbb{R}^n$ with $C^\infty$ coefficients, and let $x^0 \in \mathbb{R}^n$. In Treves [5] and Yoshikawa [8] it was proved that if $P$ is hypoelliptic at $x^0$, then there is a neighborhood $U$ of $x^0$ satisfying the following; for every $f \in C^\infty(U)$ there is $u \in \mathcal{D}'(U)$ such that $tP u = f$ in $U$. Here $tP$ denotes the transposed operator of $P$. Hörmander [3] generalized their results (see Theorem 1.2.4 of [3]). Recently Albanese, Corli and Rodino proved in [1] that the result of Treves and Yoshikawa is still valid in the framework of the Gevrey classes and the spaces of ultradistributions. Moreover, Cordaro and Trépreau proved in [2] that Hörmander’s result can be generalized in the space of hyperfunctions for partial differential operators with analytic coefficients. In particular, they proved that $P$ is locally solvable at $x^0$ in the space of hyperfunctions if the coefficients of $P$ are analytic and $P$ is analytic hypoelliptic at $x^0$. The aim of this article is to microlocalize their results for a pseudodifferential operator $p(x, D)$, i.e., if $U$ is a bounded open subset of the cosphere bundle $S^*\mathbb{R}^n$ ($\simeq \mathbb{R}^n \times S^{n-1}$) over $\mathbb{R}^n$ and if $p(x, D)$ satisfies

$$f \in L^2(\mathbb{R}^n), \quad WF_A(f) \cap \partial U = \emptyset, \quad WF_A(p(x, D)f) \cap U = \emptyset$$

implies

$$WF_A(f) \cap U = \emptyset,$$

then the transposed operator $t p(x, D) : \mathcal{C}(\bar{U}) \to \mathcal{C}(\bar{U})$ is surjective, where $\bar{U} = \{(x, \xi); (x, -\xi) \in U\}$ and $\mathcal{C}(\bar{U})$ denotes the space of microfunctions on $U$.

We shall explain briefly about hyperfunctions, microfunctions and pseudodifferential operators acting on them. For the details we refer to [6]. Let $\varepsilon \in \mathbb{R}$, and
denote \( (\xi) = (1 + |\xi|^2)^{1/2} \), where \( \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n \) and \( |\xi| = (\sum_{j=1}^n |\xi_j|^2)^{1/2} \). We define
\[
\hat{S}_\varepsilon := \{ v(\xi) \in C^\infty(\mathbb{R}^n); \, e^{\varepsilon(\xi)} v(\xi) \in S \},
\]
where \( S \) (or \( S(\mathbb{R}^n) \)) denotes the Schwartz space. We introduce the topology to \( \hat{S}_\varepsilon \) in a natural way. Then the dual space \( \hat{S}_\varepsilon' \) of \( \hat{S}_\varepsilon \) can be identified with \( \{ v(\xi) \in D'; \, e^{-\varepsilon(\xi)} v(\xi) \in S' \} \), since \( D \) (or \( C_0^\infty(\mathbb{R}^n) \)) is dense in \( \hat{S}_\varepsilon \). If \( \varepsilon \geq 0 \), then \( \hat{S}_\varepsilon \) is a dense subset of \( S \) and we can define \( S_\varepsilon := \mathcal{F}^{-1}[\hat{S}_\varepsilon] \,( = \mathcal{F}[\hat{S}_\varepsilon]) \, (\subset S) \), where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier transformation and the inverse Fourier transformation on \( S \) (or \( S' \)), respectively. For example, \( \mathcal{F}[u](\xi) = \int e^{-ix\xi} u(x) \, dx \) for \( u \in S \), where \( x \cdot \xi = \sum_{j=1}^n x_j \xi_j \) for \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \) and \( \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n \). Let \( \varepsilon \geq 0 \).

We introduce the topology in \( S_\varepsilon \) so that \( \mathcal{F} : \hat{S}_\varepsilon \to S_\varepsilon \) is homeomorphic. Denote by \( S'_\varepsilon \) the dual space of \( S_\varepsilon \). Since \( \hat{S}_\varepsilon \) is dense in \( S \), we can regard \( S' \) as a subspace of \( S'_\varepsilon \). We can define the transposed operators \( tF \) and \( tF^{-1} \) of \( F \) and \( F^{-1} \), which map \( S'_\varepsilon \) and \( \hat{S}_\varepsilon' \) onto \( \hat{S}_\varepsilon' \) and \( S'_\varepsilon \), respectively. Since \( \hat{S}_{-\varepsilon} \subset \hat{S}_\varepsilon' \,(\subset D') \), we can define \( S_{-\varepsilon} = tF^{-1}[\hat{S}_{-\varepsilon}] \), and introduce the topology in \( S_{-\varepsilon} \) so that \( tF^{-1} : \hat{S}_{-\varepsilon} \to S_{-\varepsilon} \) is homeomorphic. \( S'_{-\varepsilon} \) denotes the dual space of \( S_{-\varepsilon} \). We note that \( S'_{-\varepsilon} = \mathcal{F}[\hat{S}_{-\varepsilon}'] \subset S' \cap S'_\varepsilon \) and \( F = tF \) on \( S' \). So we also represent \( tF \) by \( F \). Let \( \mathcal{A}(\mathbb{C}^n) \) be the space of entire analytic functions on \( \mathbb{C}^n \), and let \( K \) be a compact subset of \( \mathbb{C}^n \). We denote by \( \mathcal{A}'(K) \) the space of analytic functionals carried by \( K \), i.e., \( u \in \mathcal{A}'(K) \) if and only if (i) \( u : \mathcal{A}(\mathbb{C}^n) \ni \varphi \mapsto u(\varphi) \in \mathbb{C} \) is a linear functional, and (ii) for any neighborhood \( \omega \) of \( K \) in \( \mathbb{C}^n \) there is \( C_\omega \geq 0 \) such that \( |u(\varphi)| \leq C_\omega \sup_{z \in \omega} |\varphi(z)| \) for \( \varphi \in \mathcal{A}(\mathbb{C}^n) \). Define \( \mathcal{A}'(\mathbb{R}^n) := \bigcup_{K \in \mathbb{R}^n} \mathcal{A}'(K) \), \( S_\infty := \bigcap_{\varepsilon > 0} S_{-\varepsilon} \), \( \mathcal{E}_0 := \bigcap_{\varepsilon > 0} S_{-\varepsilon} \) and \( \mathcal{F}_0 := \bigcap_{\varepsilon > 0} S'_{-\varepsilon} \). Here \( A \Subset B \) means that the closure \( \overline{A} \) of \( A \) is compact and included in the interior \( \overset{\circ}{B} \) of \( B \). We note that \( \mathcal{F}^{-1}[C_0^\infty(\mathbb{R}^n)] \subset S_\infty \) and that \( S_\infty \) is dense in \( S_{-\varepsilon} \) and \( S'_{-\varepsilon} \) for \( \varepsilon \in \mathbb{R} \).

For \( u \in \mathcal{A}'(\mathbb{R}^n) \) we can define the Fourier transform \( \hat{u}(\xi) \) of \( u \) by
\[
\hat{u}(\xi) = (\mathcal{F}[u](\xi)) = u_\varepsilon(e^{-ix\xi}),
\]
where \( z \cdot \xi = \sum_{j=1}^n z_j \xi_j \) for \( z = (z_1, \cdots, z_n) \in \mathbb{C}^n \) and \( \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n \). By definition we have \( \hat{u}(\xi) \in \bigcap_{\varepsilon > 0} \hat{S}_{-\varepsilon} \,( = \mathcal{F}[\mathcal{E}_0]) \). Therefore, we can regard \( \mathcal{A}'(\mathbb{R}^n) \) as a subspace of \( \mathcal{E}_0 \), i.e., \( \mathcal{A}'(\mathbb{R}^n) \subset \mathcal{E}_0 \subset \mathcal{F}_0 \), (see Lemma 1.1.2 of [6]). The space \( \mathcal{F}_0 \) plays an important role in our treatment as the space \( S' \) does in the framework of \( C^\infty \) and distributions. For a bounded open subset \( X \) of \( \mathbb{R}^n \) we define the space \( \mathcal{B}(X) \) of hyperfunctions in \( X \) by
\[
\mathcal{B}(X) := \mathcal{A}'(\overline{X})/\mathcal{A}'(\partial X),
\]
where $\partial X$ denotes the boundary of $X$.

Let $u \in \mathcal{F}_0$. We define

$$
\mathcal{H}(u)(x, x_{n+1}) := (\text{sgn } x_{n+1}) \exp[-|x_{n+1}|(D)]u(x)/2
$$

$$
( = (\text{sgn } x_{n+1})\mathcal{F}^{-1}_x[\exp[-|x_{n+1}|\langle \xi \rangle]\hat{u}(\xi)](x)/2 \in S'(\mathbb{R}^n))
$$

for $x_{n+1} \in \mathbb{R} \setminus \{0\}$, and

$$
supp u := \bigcap \{F; \ F \text{ is a closed subset of } \mathbb{R}^n \text{ and there is a real analytic function } U(x, x_{n+1}) \text{ in } \mathbb{R}^{n+1} \setminus F \times \{0\}
$$

$$
such that U(x, x_{n+1}) = \mathcal{H}(u)(x, x_{n+1}) \text{ for } x_{n+1} \neq 0\}.
$$

We note that $\text{supp } u$ coincides with the support of $u$ as a distribution if $u \in \mathcal{S}'$ (see Lemma 1.2.2 of [6]). Let $K$ be a compact subset of $\mathbb{R}^n$. Then $u \in \mathcal{A}'(K)$ if and only if $u$ is an analytic functional and $\text{supp } u \subset K$ (see Proposition 1.2.6 of [6]). It follows from Theorem 1.3.3 of [6] that there is $v \in \mathcal{A}'(K)$ satisfying $\text{supp } (u - v) \cap K \subset \partial K$, and if $v = v_1, v_2$ are such functionals in $\mathcal{A}'(K)$ we have $\text{supp } (v_1 - v_2) \subset \partial K$.

Therefore, we can define the restriction map from $\mathcal{F}_0$ to $\mathcal{A}'(K)/\mathcal{A}'(\partial K) (= \mathcal{B}(K))$ which is surjective. For $x^0 \in \mathbb{R}^n$ we say that $u$ is analytic at $x^0$ if $\mathcal{H}(u)(x, x_{n+1})$ can be continued analytically from $\mathbb{R}^n \times (0, \infty)$ to a neighborhood of $(x^0, 0)$ in $\mathbb{R}^{n+1}$. We define

$$
sing \text{ supp } u := \{x \in \mathbb{R}^n; \ u \text{ is not analytic at } x\}.
$$

Next let $u \in \mathcal{B}(X)$, where $X$ is a bounded open subset of $\mathbb{R}^n$. Then there is $v \in \mathcal{A}'(\bar{X})$ such that the residue class of $v$ is $u$ in $\mathcal{B}(X)$. We define

$$
\text{supp } u := \text{supp } v \cap X, \ \text{sing supp } u := \text{sing supp } v \cap X.
$$

These definitions do not depend on the choice of $v$. So we say that $u$ is analytic at $x^0$ if $x^0 \notin \text{sing supp } u$. Let $X$ be an open subset of $\mathbb{R}^n$. We also define $\mathcal{B}(X)$ (see Definition 1.4.5 of [6]). For open subsets $U$ and $V$ of $X$ with $V \subset U$ the restriction map $\rho^U_V: \mathcal{B}(U) \ni u \mapsto u|_V \in \mathcal{B}(V)$ can be defined so that $\rho^U_U$ is the identity mapping and $\rho^U_W \circ \rho^V_U = \rho^W_U$ for open subsets $U, V$ and $W$ of $X$ with $W \subset V \subset U$. By definition we can also define the restriction map from $\mathcal{F}_0$ to $\mathcal{B}(X)$, and we denote by $v|_X$ the restriction of $v \in \mathcal{F}_0$ to $\mathcal{B}(X)$ (or on $X$). We define the presheaf $\mathcal{B}_X$ by associating $\mathcal{B}(U)$ to every open subset $U$ of $X$. By definition $\mathcal{B}_X$ is a sheaf on $X$.

Next we shall define analytic wave front sets and microfunctions.
Definition 1.1. (i) Let $u \in \mathcal{F}_0$. The analytic wave front set $WF_A(u) \subset T^*\mathbb{R}^n \setminus 0$ ($\simeq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$) is defined as follows: $(x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0$ does not belong to $WF_A(u)$ if there are a conic neighborhood $\Gamma$ of $\xi^0$, $R_0 > 0$ and $\{g^R(\xi)\}_{R \geq R_0} \subset C^\infty(\mathbb{R}^n)$ such that $g^R(\xi) = 1$ in $\Gamma \cap \{\langle \xi \rangle \geq R\}$,

\[(1.1) \quad |\partial^\alpha \xi g^R(\xi)| \leq C_{|\alpha|}(C/R)^{|\alpha|/|\alpha|} \]

if $\langle \xi \rangle \geq R|\alpha|$, and $g^R(D)u \ (= \mathcal{F}^{-1}[g^R(\xi)\hat{u}(\xi)])$ is analytic at $x^0$ for $R \geq R_0$, where $C$ is a positive constant independent of $R$.

(ii) Let $X$ be an open subset of $\mathbb{R}^n$, and let $u \in \mathcal{B}(X)$ and $(x^0, \xi^0) \in T^*X \setminus 0$ ($\simeq X \times (\mathbb{R}^n \setminus \{0\})$). Then we say that $(x^0, \xi^0) \notin WF_A(u) \ (\subset T^*X \setminus 0)$ if there are a bounded open neighborhood $U$ of $x^0$ and $v \in \mathcal{A}'(\overline{U})$ such that $v|_U = u|_U$ in $\mathcal{B}(U)$ and $(x^0, \xi^0) \notin WF_A(v)$

Remark. (i) $WF_A(u)$ for $u \in \mathcal{B}(X)$ is well-defined. Indeed, it follows from Theorem 2.6.5 in [6] that for any $v \in \mathcal{A}'(\mathbb{R}^n)$ with $x^0 \notin \text{supp } v$ there is $R_1 > 0$ such that $g^R(D)v$ is analytic at $x^0$ if $R \geq R_1$, where $\{g^R(\xi)\}_{R \geq R_0}$ is a family of symbols satisfying (1.1).

(ii) Several remarks on this definition are given in Proposition 3.1.2 of [6].

(iii) From Theorem 3.1.6 in [6] and the results in [4] it follows that our definition of $WF_A(u)$ coincides with the usual definition.

Let $\mathcal{U}$ be an open subset of the cosphere bundle $S^*\mathbb{R}^n$ over $\mathbb{R}^n$, which is identified with $\mathbb{R}^n \times S^{n-1}$. We define

$$\mathcal{C}(\mathcal{U}) := \mathcal{B}(\mathbb{R}^n)/\{u \in \mathcal{B}(\mathbb{R}^n); \ WF_A(u) \cap \mathcal{U} = \emptyset\}.$$ 

Since $\mathcal{B}$ is a flabby sheaf, we have

$$\mathcal{C}(\mathcal{U}) = \mathcal{B}(U)/\{u \in \mathcal{B}(U); \ WF_A(u) \cap \mathcal{U} = \emptyset\}$$

if $U$ is an open subset of $\mathbb{R}^n$ and $\mathcal{U} \subset U \times S^{n-1}$. Elements of $\mathcal{C}(\mathcal{U})$ are called microfunctions on $\mathcal{U}$. We can define the restriction map $\mathcal{C}(\mathcal{U}) \ni u \mapsto u|_V \in \mathcal{C}(V)$ for open subsets $\mathcal{U}$ and $V$ of $\mathbb{R}^n \times S^{n-1}$ with $V \subset U$. Let $\Omega$ be an open subset of $\mathbb{R}^n \times S^{n-1}$. We define the presheaf $\mathcal{C}_\Omega$ on $\Omega$ associating $\mathcal{C}(\mathcal{U})$ to every open subset $\mathcal{U}$ of $\Omega$. Then $\mathcal{C}_\Omega$ is a flabby sheaf (see, e.g., Theorem 3.6.1 of [6]). For each open subset $U$ of $\mathbb{R}^n$ we define the mapping $\text{sp}: \mathcal{B}(U) \rightarrow \mathcal{C}(U \times S^{n-1})$ such that the residue class in $\mathcal{C}(U \times S^{n-1})$ of $u \in \mathcal{B}(U)$ is equal to $\text{sp}(u)$. We also write $u|_U = \text{sp}(u)|_U$ for $u \in \mathcal{B}(U)$ and $v|_U = \text{sp}(v|_U)|_U$ for $v \in \mathcal{F}_0$, where $\mathcal{U}$ is an open subset of $U \times S^{n-1}$. 

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Assume that $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ and there are positive constants $C_k$ $(k \geq 0)$ such that

$$\begin{equation}
(1.2) \quad |\partial_x^\alpha D_y^\beta \partial_y^\gamma a(\xi, y, \eta)| \leq C_{|\alpha|+|\beta|+|\gamma|} (A/R)^{|\beta|} \langle \xi \rangle^{m_1} \langle \eta \rangle^{m_2} \exp[\delta_1 \langle \xi \rangle + \delta_2 \langle \eta \rangle] \end{equation}$$

if $\alpha, \beta, \tilde{\beta}, \gamma \in (\mathbb{Z}_+)^n$, $\xi, y, \eta \in \mathbb{R}^n$ and $\langle \xi \rangle \geq R|\beta|$, where $D_y = -i\partial_y$, $R \geq 1$, $A \geq 0$, $m_1, m_2, \delta_1, \delta_2 \in \mathbb{R}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. It should be remarked that some functions satisfying the estimates (1.2) with $m_1 = m_2 = 0$ and $\delta_1 = \delta_2 = 0$ are given in Proposition 2.2.3 of [6]. We define pseudodifferential operators $a(D_x, y, D_y)$ and $r a(D_x, y, D_y)$ by

$$a(D_x, y, D_y)u(x) = (2\pi)^{-n} \mathcal{F}_x^{-1} \left[ \int \left( \int e^{-i\eta \cdot (\xi - x)} a(\xi, y, \eta) \hat{u}(\eta) \, d\eta \right) \, dy \right](x)$$

and $r a(D_x, y, D_y)u = b(D_x, y, D_y)u$ for $u \in \mathcal{S}_\infty$, respectively, where $b(\xi, y, \eta) = a(\eta, y, \xi)$. Applying the same argument as in the proof of Theorem 2.3.3 of [6] we have the following

**Proposition 1.2.** $a(D_x, y, D_y)$ can be extended to a continuous linear operator from $\mathcal{S}_{\infty}$ to $\mathcal{S}_{\infty}$ and from $\mathcal{S}_{\infty}'$ to $\mathcal{S}_{\infty}'$, respectively, if

$$\begin{equation}
(1.3) \quad \begin{cases}
\nu > 1, & \varepsilon_2 - \delta_2 = \nu (\varepsilon_1 + \delta_1), \\
\varepsilon_1 + \delta_1 \leq 1/R, & R \geq e^{\sqrt{n}\nu A/(\nu - 1)},
\end{cases}
\end{equation}$$

where $c_+ = \max\{c, 0\}$. Similarly, $r a(D_x, y, D_y)$ can be extended to a continuous linear operator from $\mathcal{S}_{\infty}$ to $\mathcal{S}_{\infty}$ and from $\mathcal{S}_{\infty}'$ to $\mathcal{S}_{\infty}'$, respectively, if (1.3) is valid.

**Remark.** (i) We had a slight improvement of the remark of Theorem 2.3.3 of [6], i.e., we can take $R_1(S, T, \nu) = e^{\sqrt{n}\nu/(\nu - 1)}$ there instead of $R_1(S, T, \nu) = e^{\sqrt{n}\nu/(\nu - 1)}$ if $n = n' = n''$, $S(y, \xi) = -y \cdot \xi$ and $T(y, \eta) = y \cdot \eta$. This is reflected in the condition (1.3).

(ii) Since for any open sets $X_j$ $(j = 1, 2)$ with $X_1 \subset X_2$ one can construct a symbol $a(\xi, y, \eta)$ satisfying (1.2) with $m_1 = m_2 = 0$ and $\delta_1 = \delta_2 = 0$, supp $a \subset \mathbb{R}^n \times X_2 \times \mathbb{R}^n$ and $a(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times X_1 \times \mathbb{R}^n$, one can use the operator $a(D_x, y, D_y)$ instead of cut-off functions.

**Definition 1.3.** Let $\Gamma$ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and let $X$ be an open subset of $\mathbb{R}^n$. Moreover, let $R_0 \geq 0$. 

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(i) Let $R_0 \geq 1$, $m, \delta \in \mathbb{R}$ and $A, B \geq 0$, and let $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(x, \xi) \in S^{m, \delta}(R_0, A, B)$ if $a(x, \xi)$ satisfies

$$|a^{(\alpha+\bar{\alpha})}_{(\beta+\bar{\beta})}(x, \xi)| \leq C_{|\alpha|+|\beta|}(A/R_0)^{|\alpha|}(B/R_0)^{|\beta|} |\xi|^{m+|\beta|-|\bar{\alpha}|} e^{\delta(\xi)}$$

for any $\alpha, \bar{\alpha}, \beta, \bar{\beta} \in (\mathbb{Z}_+)^n$ and $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|\xi| \geq R_0(|\alpha| + |\beta|)$, where $a^{(\alpha)}_{(\beta)}(x, \xi) = \partial_\xi^{\bar{\beta}} D_\xi^{\beta} a(x, \xi)$ and the $C_k$ are independent of $\alpha$ and $\beta$. We also write $S^m(R_0, A, B) = S^{m, 0}(R_0, A, B)$ and $S^m(R_0, A) = S^m(R_0, A, A)$. We define $S^+(R_0, A, B) := \bigcap_{\delta > 0} S^{0, \delta}(R_0, A, B)$.

(ii) Let $R_0 \geq 1$, $m_j, \delta_j \in \mathbb{R}$ (for $j = 1, 2$), $A_j \geq 0$ (for $j = 1, 2$) and $B \geq 0$, and let $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(\xi, y, \eta) \in S^{m_1, m_2, \delta_1, \delta_2}(R_0, A_1, B_1, A_2)$ if $a(\xi, y, \eta)$ satisfies

$$|\partial_\xi^{\alpha+\bar{\alpha}} D_y^{\beta+\bar{\beta}} \partial_\eta^{\gamma+\bar{\gamma}} a(\xi, y, \eta)| \leq C_{|\alpha|+|\beta|+|\gamma|+|\bar{\gamma}|}(A_1/R_0)^{|\alpha|}(B/R_0)^{|\beta|} \xi^{m_1+|\beta|-|\bar{\alpha}|} |\eta|^{m_2+|\beta|-|\bar{\gamma}|} \exp[\delta_1(\xi) + \delta_2(\eta)]$$

for any $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma} \in (\mathbb{Z}_+)^n$, $(\xi, y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with $|\xi| \geq R_0(|\alpha| + |\beta|)$ and $|\eta| \geq R_0(|\gamma| + |\bar{\beta}|)$. We also write $S^{m_1, m_2, \delta_1, \delta_2}(R_0, A) = S^{m_1, m_2, \delta_1, \delta_2}(R_0, A, A, A)$. Similarly, we define $S^+(R_0, A_1, B_1, A_2) = \bigcap_{\delta > 0} S^{0, 0, \delta}(R_0, A_1, B_1, A_2).

(iii) Let $A, B \geq 0$, and let $a(\xi, \in) \in C^\infty(\Gamma)$. We say that $a(\xi, \in) \in PS^+(\Gamma; R_0, A, B)$ if $a(\xi, \in)$ satisfies

$$|a^{(\alpha+\bar{\alpha})}_{(\beta+\bar{\beta})}(x, \xi)| \leq C_{|\alpha|+\delta A^{n, |\beta|, |\bar{\beta}|, |\bar{\alpha}|}(\xi)^{m-|\beta|-|\bar{\alpha}|} e^{\delta(\xi)}$$

for any $\alpha, \bar{\alpha}, \beta \in (\mathbb{Z}_+)^n$, $(x, \xi) \in \Gamma$ with $|\xi| \geq 1$ and $|\in| \geq R_0|\alpha|$ and $\delta > 0$. We also write $PS^+(\Gamma; R_0, A) = PS^+(\Gamma; R_0, A, A)$. Moreover, we say that $a(\xi, \in) \in PS^+(X; R_0, A, B)$ if $a(\xi, \in) \in C^\infty(X \times \mathbb{R}^n)$ and $a(\xi, \in) \in PS^+(X \times (\mathbb{R}^n \setminus \{0\}); R_0, A, B)$.

(iv) Let $m, \delta \in \mathbb{R}$ and $A, C_0 \geq 0$, and let $\{a_j(x, \xi)\}_{j \in \mathbb{Z}_+} \in \prod_{j \in \mathbb{Z}_+} C^\infty(\Gamma)$. We say that $a(x, \in) \equiv \{a_j(x, \xi)\}_{j \in \mathbb{Z}_+} \in FS^m, \delta(\Gamma; C_0, A)$ if $a(x, \in)$ satisfies

$$|a^{(\alpha)}_{(\beta)}(x, \xi)| \leq C_{0, A} A^{n+|\beta|, j} |\in| |\beta| \xi^{m-j} e^{\delta(\xi)}$$

for any $j \in \mathbb{Z}_+$, $\alpha, \beta \in (\mathbb{Z}_+)^n$ and $(x, \xi) \in \Gamma$ with $|\xi| \geq 1$, where $C$ is independent of $\alpha, \beta$ and $j$. We define $FS^+(\Gamma; C_0, A) := \bigcap_{\delta > 0} FS^{0, \delta}(\Gamma; C_0, A)$. We also write $a(x, \xi) = \sum_{j=0}^{\infty} a_j(x, \xi)$ formally. Moreover, we write $FS^+(X; C_0, A) = FS^+(X \times (\mathbb{R}^n \setminus \{0\}); C_0, A)$.

(v) For $a(x, \xi) \equiv \sum_{j=0}^{\infty} a_j(x, \xi) \in FS^+(\Gamma; C_0, A)$ we define the symbol $\langle^t a(x, \xi)$ by

$$\langle^t a(x, \xi) = \sum_{j=0}^{\infty} b_j(x, \xi), \quad b_j(x, \xi) = \sum_{k+|\alpha|=j} (-1)^{|\alpha|} a_{k, j}(x, -\xi)/\alpha!.$$
Remark. It is easy to see that \((a)(x, \xi) \in FS^+(\Gamma; \max\{C_0, 4nA^2\}, 2A)\). Moreover, we have \((a')(x, \xi) = a(x, \xi)\).

Let \(\Gamma\) be an open conic subset of \(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\), and assume that \(a(x, \xi) \in PS^+(\Gamma; R_0, A)\), where \(A \geq 0\) and \(R_0 \geq 1\). Let \(\Gamma_j\) (\(0 \leq j \leq 2\)) be open conic subsets of \(\Gamma\) such that \(\Gamma_j \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma\), and write \(\Gamma_0 = \Gamma \cap (\mathbb{R}^n \times S^{n-1})\), where \(\Gamma_2 \subset \Gamma\) implies that \(\Gamma_0 \subset \Gamma\). It follows from Proposition 2.2.3 of [6] that there are symbols \(\Phi^R(\xi, y, \eta) \in S^{0,0,0}(R, C_s, C(\Gamma_1, \Gamma_2), C(\Gamma_1, \Gamma_2))\) \((R \geq 4)\) satisfying
\[
0 \leq \Phi^R(\xi, y, \eta) \leq 1, \text{ supp } \Phi^R \subset \mathbb{R}^n \times \Gamma_2 \text{ and } \Phi^R(\xi, y, \eta) = 1 \text{ for } (\xi, y, \eta) \in \mathbb{R}^n \times \Gamma_1 \text{ with } \langle \eta \rangle \geq R.
\]
Put \(a^R(\xi, y, \eta) = \Phi^R(\xi, y, \eta)a(y, \eta)\). Then we have \(a^R(\xi, y, \eta) \in S^+(R, C_s, 2A + C(\Gamma_1, \Gamma_2), A + C(\Gamma_1, \Gamma_2))\) for \(R \geq \max\{4, R_0\}\). Let \(u \in C(\Gamma_0)\), and choose \(v \in \mathcal{F}_0\) so that \(v|_{\Gamma_0} = u\). Applying Proposition 1.2 with \(a(\xi, y, \eta) = a^R(\xi, y, \eta)\) and noting that \(a^R(D_x, y, D_y)v = a(D_x, y, D_y)v\) is well-defined and belongs to \(\mathcal{F}_0\) if \(R \geq \max\{4, R_0, 2e \sqrt{n}(2A + C(\Gamma_1, \Gamma_2))\}\). Moreover, \(a^R(D_x, y, D_y)v\) determines an element \((a^R(D_x, y, D_y)v)|_{\mathcal{F}_0} \in C(\Gamma_0)\), where \(U\) is a bounded open subset of \(\mathbb{R}^n\) satisfying \(\Gamma_0 \subset U \times S^{n-1}\). It follows from Lemma 2.1 of [7] that \((a^R(D_x, y, D_y)v)|_{\mathcal{F}_0}\) does not depend on the choice of \(\Phi^R(\xi, y, \eta)\) if \(\Phi^R(\xi, y, \eta) \in S^{0,0,0}(R, B)\) and \(R \geq R(A, B, \Gamma_0, \Gamma_1)\),\(\Gamma_2 > 0\) such that \(WF_A(a^R(D_x, y, D_y)v) \cap \Omega = \emptyset\) if \(R \geq R(A, \Omega, \Gamma_0, \Gamma_1, \Gamma_2)\), \(w \in \mathcal{F}_0\) and \(WF_A(w) \cap \Gamma_0 = \emptyset\). Therefore, we can define the operator \(a(x, D) : C(\Gamma_0) \to C(\Gamma_0)\) by \(u \mapsto a^R(D_x, y, D_y)v|_{\mathcal{F}_0}\) for \(R \gg 1\), and the operator \(a(x, D) : C(\Gamma_0) \to C(\Gamma_0)\). Moreover, it follows from Lemma 2.2 of [7] that
\[
a(x, D)(w|\mathcal{U}) = (a(x, D)w)|\mathcal{U} \quad \text{for } w \in C(\mathcal{V}),
\]
where \(\mathcal{U}\) and \(\mathcal{V}\) are open subsets of \(\mathbb{R}^n \times S^{n-1}\) satisfying \(\mathcal{U} \subset \mathcal{V} \subset \Gamma_0\). So we can define \(a(x, D) : C_{\mathcal{V}} \to C_{\mathcal{U}},\) which is a sheaf homomorphism. Let \(X\) be an open subset of \(\mathbb{R}^n\), and assume that \(a(x, \xi) \in PS^+(X; R_0, A)\). Similarly, taking \(\Gamma = X \times (\mathbb{R}^n \setminus \{0\})\), we can define the operator \(a(x, D) : \mathcal{B}(U) \to \mathcal{B}(U)/\mathcal{A}(U)\) and the operator \(a(x, D) : \mathcal{B}(U)/\mathcal{A}(U) \to \mathcal{B}(U)/\mathcal{A}(U)\), where \(U\) is a bounded open subset of \(X\) and \(\mathcal{A}(U)\) denotes the space of all real analytic functions defined in \(U\) (see, also, §2.7 of [6]). In doing so, we may choose \(\Phi^R(\xi, y, \eta) \in S^{0,0,0}(R, C_s, C(\Gamma_1, \Gamma_2), C_s)\) so that \(\text{supp } \Phi^R \subset \mathbb{R}^n \times X \times \mathbb{R}^n\) and \(\Phi^R(\xi, y, \eta) = 1\) for \((\xi, y, \eta) \in \mathbb{R}^n \times X \times \mathbb{R}^n\), where \(\Gamma_j = X_j \times (\mathbb{R}^n \setminus \{0\})\). Moreover, we can define the operator \(a(x, D) : \mathcal{B}_X \to \mathcal{B}_X/\mathcal{A}_X\) and the operator \(a(x, D) : \mathcal{B}_X/\mathcal{A}_X \to \mathcal{B}_X/\mathcal{A}_X\), which are sheaf homomorphisms.
Here $A_X$ denotes the sheaf (of germs) of real analytic functions on $X$.

Assume that $a(x, \xi) \equiv \sum_{j=0}^{\infty} a_j(x, \xi) \in FS^+(\Gamma; C_0, A)$. Choose $\{\phi^R_j(\xi)\}_{j \in \mathbb{Z}^+} \subset C^\infty(\mathbb{R}^n)$ so that $0 \leq \phi^R_j(\xi) \leq 1$,

$$\phi^R_j(\xi) = \begin{cases} 
0 & \text{if } \langle \xi \rangle \leq 2Rj, \\
1 & \text{if } \langle \xi \rangle \geq 3Rj,
\end{cases}$$

$$|\partial^{\alpha+\beta} \phi^R_j(\xi)| \leq \hat{C}_{|\beta|}(\hat{C}/R)^{|\alpha|} \langle \xi \rangle^{-|\beta|} \quad \text{if } |\alpha| \leq 2j,$$

where the $\hat{C}_{|\beta|}$ and $\hat{C}$ do not depend on $j$ and $R$ (see §2.2 of [6]). Then it follows from Lemma 2.2.4 of [6] that

$$\hat{a}(x, \xi) := \sum_{j=0}^{\infty} \phi^{R/2}_j(\xi) a_j(x, \xi) \in PS^+(\Gamma; R, 2A + 3\hat{C}, A)$$

if $R > C_0$. So we can define $a(x, D)u \in C(\Gamma^0)$ by $a(x, D)u = \hat{a}(x, D)u$. Indeed, applying the same argument as in §3.7 of [6] we can see that $a(x, D)u \in C(\Gamma^0)$ does not depend on the choice of $\{\phi^R_j(\xi)\}$. Similarly, $a(x, D)$ defines a sheaf homomorphism $a(x, D): C_{\Gamma^0} \rightarrow C_{\Gamma^0}$. If $\Gamma = X \times (\mathbb{R}^n \setminus \{0\})$, then we can also define the operator $a(x, D): B(U) / A(U) \rightarrow B(U) / A(U)$ and the operator $a(x, D): B_{\gamma} / A_{\gamma} \rightarrow B_{\gamma} / A_{\gamma}$, where $U$ is an open subset satisfying $U \subset X$.

Let $\Gamma$ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and let $p(x, \xi) \in FS^+(\Gamma; C_0, A)$, where $A, C_0 \geq 0$.

**Theorem 1.4.** Let $U$ and $V$ be bounded open subsets of $\Gamma^0$ in $\mathbb{R}^n \times S^{n-1}$ such that $V \subset U \subset \Gamma^0$. Assume that $WF_A(f) \cap U = \emptyset$ if $f \in L^2(\mathbb{R}^n), WF_A(f) \cap \partial U = \emptyset$ and $p(x, D)(f|_U) = 0$ in $C(U)$. Then $(\ell p)(x, D)$ maps $C(V)$ onto $C(V)$, i.e., for any $f \in C(V)$ there is $u \in C(V)$ satisfying $(\ell p)(x, D)u = f$ in $C(V)$.

**Corollary 1.5** ([7]). Let $z^0 = (x^0, \xi^0) \in \Gamma$, and assume that $p(x, D)$ is analytic microhypoelliptic at $z^0$, i.e., there is an open neighborhood $U$ of $(x^0, \xi^0/|\xi^0|)$ in $\Gamma^0$ such that the sheaf homomorphism $p(x, D): C_U \rightarrow C_U$ is injective. Then $(\ell p)(x, D)$ is microlocally solvable at $(x^0, -\xi^0)$, i.e., there is an open neighborhood $U$ of $(x^0, \xi^0/|\xi^0|)$ in $\Gamma^0$ such that $(\ell p)(x, D): C(U) \rightarrow C(U)$ is surjective.

**Corollary 1.6.** Assume that $p(x, \xi) \equiv \sum_{j=0}^{\infty} p_j(x, \xi) \in FS^m(\Gamma; C_0, A)$, and that $p_0(x, \xi)$ is positively homogeneous of degree $m$ in $\xi$. Let $U$ and $V$ be bounded open subsets of $\Gamma^0$ satisfying $V \subset U$, and assume that there is a continuous vector field $\vartheta: U \ni z \mapsto \vartheta(z) \in \mathbb{R}^{2n}$ such that $p_0(x, \xi)$ is microhyperbolic with respect to $\vartheta(z)$
at each \( z \in \mathcal{U} \). Moreover, we assume that for any \( z^0 \in \mathcal{U} \) there is no generalized semi-bicharacteristics \( \{z(s)\}_{s \in (-\infty, 0]} \) of \( p_0 \) starting from \( z^0 \) in the negative direction such that \( (x(s), \xi(s)/|\xi(s)|) \in \mathcal{U} \) for \( s \in (-\infty, 0] \), where the parameter \( s \) of the curve is chosen so that \(-s\) coincides with the arc length from \( z^0 \) to \( z(s) = (x(s), \xi(s)) \). For terminology we refer to §4.3 of [6]. Then \((\ell p)(x, D) : C(\mathcal{V}) \to C(\mathcal{V'})\) is surjective.

**Corollary 1.7.** Let \( z^0 = (x^0, \xi^0) \in \Gamma \), and assume that \( p(x, \xi) = \sum_{j=0}^\infty p_j(x, \xi) \in FS^{m,0}(\Gamma; C_0, A) \), and that \( p_0(x, \xi) \) is positively homogeneous of degree \( m \) in \( \xi \) and microhyperbolic with respect to \((0, e_1) \in \mathbb{R}^{2n} \) at \( z^0 \), where \( e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n \). Then \((\ell p)(x, D)\) is microlocally solvable at \((x^0, -\xi^0)\).

**Remark.** The above corollary was proved in Theorem 5.4.1 of [6] in a different way. Theorem 1.4 can be proved in the same way as in [7]. We shall give the outline of the proof in the next section. Then Corollary 1.5 easily follows from Theorem 1.4. Combining Theorem 4.3.8 of [6] and Theorem 1.4 one can easily prove Corollary 1.6. Corollary 1.7 is an immediate consequence of Corollary 1.6.

2. **Proof of Theorem 1.4**

Let \( \Gamma_j \ (j = 1, 2) \) be open conic subsets of \( \Gamma \) such that \( \mathcal{V} \Subset \mathcal{U} \Subset \Gamma_1^0 \Subset \Gamma_2^0 \Subset \Gamma^0 \), where \( \Gamma_j^0 = \Gamma_j \cap (\mathbb{R}^n \times S^{n-1}) \). Choose \( \Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1, \Gamma_2), C(\Gamma_1, \Gamma_2), (R \geq 4) \) so that \( 0 \leq \Phi^R(\xi, y, \eta) \leq 1 \), supp \( \Phi^R \subset \mathbb{R}^n \times \Gamma_2 \) and \( \Phi^R(\xi, y, \eta) = 1 \) for \((\xi, y, \eta) \in \mathbb{R}^n \times \Gamma_1 \) with \( \langle \eta \rangle \geq R \). We put

\[
p^R(\xi, y, \eta) = \Phi^R(\xi, y, \eta) \sum_{j=0}^\infty \phi_j^{R/2}(\eta) p_j(y, \eta),
\]

where \( R > \max\{4, C_0\} \). Then we have

\[
p^R(\xi, y, \eta) \in S^+(R, C_*, 2A + C(\Gamma_1, \Gamma_2), 2A + 3\hat{C} + C(\Gamma_1, \Gamma_2)).
\]

By definition there is \( R(\mathcal{A}, \mathcal{U}, \Gamma_1, \Gamma_2) > \max\{4, C_0\} \) such that

\[
(2.1) \quad (p^R(D_x, y, D_y)v)|_{\mathcal{U}} = p(x, D)(v)|_{\mathcal{U}} \quad \text{in } C(\mathcal{U})
\]

if \( R \geq R(\mathcal{A}, \mathcal{U}, \Gamma_1, \Gamma_2) \) and \( v \in \mathcal{F}_0 \). Let \( \Omega_j \ (j = 1, 2) \) be open conic subsets satisfying \( \mathcal{V} \Subset \Omega_2^0 \Subset \Omega_1^0 \Subset \mathcal{U} \), and let \( \Psi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Omega_2, \Omega_1), C(\Omega_2, \Omega_1)) \ (R \geq 0). \)
4) satisfy \(\text{supp } \Psi^R \subset \mathbb{R}^n \times \Omega_1 \) and \(\Psi^R(\xi, y, \eta) = 1\) for \((\xi, y, \eta) \in \mathbb{R}^n \times \Omega_2\) with \(\langle \eta \rangle \geq R\).

We assume that \(R \geq \max\{R(A, U, \Gamma_1, \Gamma_2), 25\varepsilon \sqrt{n} \max\{2A + C(\Gamma_1, \Gamma_2), C(\Omega_2, \Omega_1)\}\}\).

For \(\varepsilon, \nu \in \mathbb{R}\) we define

\[
L^2_{\varepsilon, \nu} := \{f \in S_\varepsilon': \langle x \rangle^\nu e^{\varepsilon(D)} f(x) \in L^2(\mathbb{R}^n)\}.
\]

\(L^2_{\varepsilon, \nu}\) is a Hilbert space in which the scalar product is given by

\[
(f, g)_{L^2_{\varepsilon, \nu}} := \langle \langle x \rangle^\nu e^{\varepsilon(D)} f, \langle x \rangle^\nu e^{\varepsilon(D)} g \rangle_{L^2},
\]

where \((\cdot, \cdot)_{L^2}\) denotes the scalar product of \(L^2(\mathbb{R}^n)\). We denote by \(\mathcal{X}\) the inductive limit \(\lim_{j \to \infty} L^2_{\varepsilon, \nu}(\mathbb{R}^n)\) of the sequence \(\{L^2_{\varepsilon, \nu}(\mathbb{R}^n)\}\) as a locally convex space. Define an operator \(T : L^2(\mathbb{R}^n) \to \mathcal{X} \times \mathcal{X}\) as follows:

(i) the domain \(D(T)\) of \(T\) is given by

\[
D(T) = \{f \in L^2(\mathbb{R}^n); (1 - \Psi^R(D_x, y, D_y)) f \in \mathcal{X} \text{ and } p^R(D_x, y, D_y) f \in \mathcal{X}\}.
\]

(ii) \(T f = ((1 - \Psi^R(D_x, y, D_y)) f, p^R(D_x, y, D_y) f)\) for \(f \in D(T)\).

Let \(f \in D(T)\). Then (2.1) gives \(p(x, D)(f|_U) = 0\) in \(C(\mathcal{U})\). Moreover, it follows from Lemma 2.1 of [7] that there is \(R(\Omega_1, \Omega_2, \mathcal{U}) > 0\) such that \(WF_A(f) \cap \partial \mathcal{U} = \emptyset\) if \(R \geq R(\Omega_1, \Omega_2, \mathcal{U})\). Therefore, by the assumption of Theorem 1.4 we have \(WF_A(f) \cap \partial \mathcal{U} = \emptyset\). From Lemma 2.9 of [7] there are \(R_1(\Omega_1, \Omega_2, \mathcal{U}) > 0\) and \(\delta(f, \Omega_1, \mathcal{U}) > 0\) such that \(\Psi^R(D_x, y, D_y)f \in L^2_{\varepsilon, \nu}\) if \(R \geq R_1(\Omega_1, \Omega_2, \mathcal{U}), \nu \in \mathbb{R}\) and \(\delta < \min\{1/(2R), \delta(f, \Omega_1, \mathcal{U})\}\). This implies that \(f \in \mathcal{X}, \text{ i.e., } D(T) = \mathcal{X}\). We can easily prove that \(T\) is a closed operator (see §3 of [7]).

Repeating the same argument as in §3 of [7], we can show that for any \(f \in \mathcal{A}'(\mathbb{R}^n)\) there is \(u \in \mathcal{F}_0\) satisfying

\[
(\mathring{\mathcal{L}}p)(x, D)(u|_\tilde{\mathcal{V}}) = f|_{\tilde{\mathcal{V}}} \quad \text{in } \mathcal{C}(\tilde{\mathcal{V}}),
\]

which proves Theorem 1.4.

References


