ON BRACKET POLYNOMIALS FOR ALEXANDER-TYPE INVARIANTS

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Abstract. We introduce bracket polynomials for Alexander-type invariants, including the (multivariable) Alexander polynomial and quandle twisted Alexander invariants. The bracket polynomials provide an elementary diagrammatic proof of the invariance of the (multivariable) Alexander polynomial and a simple definition for the quandle twisted Alexander invariant satisfying a certain condition.

1. Introduction

The Kauffman bracket [10] provides a simple definition and an elementary proof of the invariance of the Jones polynomial [8]. In this paper, we introduce counterparts of the Kauffman bracket for Alexander-type invariants, including the (multivariable) Alexander polynomial and quandle twisted Alexander invariants with $f_1 + f_2 = 1$. A quandle twisted Alexander invariant [6] is a family of invariants constructed by fixing a quandle and its linear extension.

The multivariable Alexander polynomial is an invariant of an ordered oriented link, while a quandle twisted Alexander invariant is an invariant of a pair of an oriented link and its quandle representation. In Section 2, we begin with the Alexander–Conway polynomial [1, 2] to become familiar with the bracket polynomials in our framework. Before introducing the bracket polynomial for the multivariable Alexander polynomial in Section 5, we present its quandle twisted version in Section 4, since an ordered oriented link can be regarded as an oriented link equipped with a quandle representation. In Section 3, we recall the notions of a quandle [9, 12], an Alexander pair [5], a column relation map [4] and a row relation map [3]. A quandle is an algebraic structure whose axioms correspond to the Reidemeister moves on link diagrams. The other notions are used to define the quandle twisted Alexander invariant, which is a generalization of the Alexander–Conway polynomial, the multivariable Alexander polynomial and the twisted Alexander polynomial [11, 14]. A quandle twisted Alexander matrix is defined using an Alexander pair (f_1, f_2), which is a pair of maps corresponding to a linear extension of a quandle. The row relation map and column relation map also yield matrices which annihilate the quandle twisted Alexander matrix from the left and right, respectively. In Section 6, we recall the definition of the quandle twisted Alexander invariant, and in Section 7, we show that the quandle twisted version of the bracket polynomial coincides with the determinant of a generalized quandle twisted Alexander matrix of a diagram with vertices. In Section 8, we demonstrate that the R_p-twisted Alexander invariant can be recovered from the quandle twisted bracket polynomial, and in Section 9, we present several properties of the invariant.

In the rest of this section, we provide an overview of the results of this paper with minimal introduction of terminology. See Sections 2, 3 and 6 for the precise

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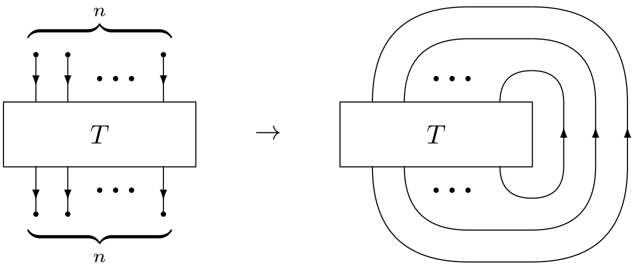
Key words and phrases. Quandle; Alexander polynomial; multivariable Alexander polynomial; twisted Alexander polynomial; knots; links.



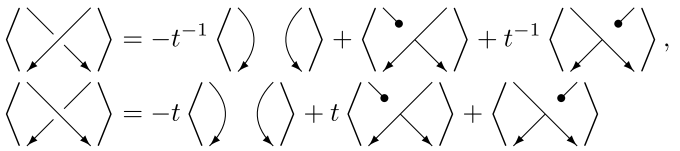
(n,n)-tangle $\widehat{T}$

Figure 1. An (n,n)-tangle and its closure

definitions of unfamiliar terms. Throughout this paper we work in the piecewise linear category.

An *(n,n)-tangle* is a tangle with n top endpoints and n bottom endpoints as depicted in the left picture of Figure 1. In this paper, a tangle may contain vertices other than endpoints. We call such vertices *inner vertices*. Throughout this paper, we assume that an inner vertex is of indegree 1 and of degree $1, 2$ or 3. A tangle is *classical* if it has no inner vertices. In this paper, a graph and a tangle may contain circle components, which have no vertices. In particular, a $(0,0)$-tangle is a link. We denote by $\widehat{T}$ the closure of an (n,n)-tangle T (see Figure 1). For a diagram D of T, we denote by $\widehat{D}$ the diagram of $\widehat{T}$ as depicted in Figure 1. A tangle is *cyclic* if its underlying graph contains a cycle. A tangle is *acyclic* if it is not cyclic, that is, its underlying graph is a disjoint union of trees.

Definition 1.1. Let D be a diagram of an oriented uni-trivalent (n,n)-tangle. We define $\langle D\rangle \in \mathbb{Z}[t^{\pm 1}]$ by the local relations



and, for a diagram D without crossings,

$$\langle D\rangle = \begin{cases} 1 & \text{if } D \text{ is a diagram of an acyclic tangle,} \\ 0 & \text{if } D \text{ is a diagram of a cyclic tangle.} \end{cases}$$

Let T be an oriented classical (n,n)-tangle, and let D be a diagram of T. We define the *rotation number* $\mathrm{rot}(D)$ of D to be the total rotation angle of the tangent vector on D divided by 2π. The *writhe* $\mathrm{wr}(D)$ of D is the total number of positive crossings minus the total number of negative crossings of D. We then have $\mathrm{rot}(D) = \mathrm{rot}(\widehat{D}) - n$ and $\mathrm{wr}(D) = \mathrm{wr}(\widehat{D})$.

Theorem 1.2. *Let T be an oriented classical (n,n)-tangle. Let D be a diagram of T. Then*

$$t^{\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}}\langle D\rangle$$

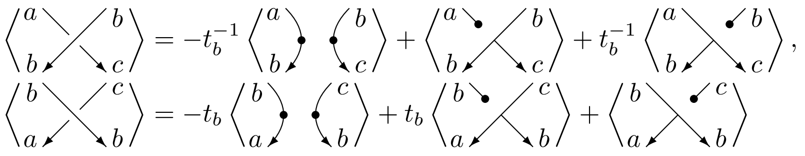
is invariant under the Reidemeister moves. In particular, for an oriented classical $(1,1)$-tangle T, we have

$$\Delta_{\widehat{T}}(t) = t^{\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}}\langle D\rangle,$$

where $\Delta_L(t)$ is the Alexander–Conway polynomial of an oriented link L.

Let T be an oriented (n,n)-tangle, and let $K_1,\dots,K_r$ be the connected components of T. Let D be a diagram of T. We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc of D is a piece of a curve each of whose endpoints is an undercrossing or a vertex. Suppose that T is classical. We denote by $\mathcal{A}(D;K_i)$ the set of arcs of D that originate from K_i, and denote by $C(D;K_i)$ the set of crossings of D whose under arcs originate from K_i. We define $\mathrm{wr}(D;K_i) := \sum_{c\in C(D;K_i)} \mathrm{sgn}(c)$. We then have $\mathrm{wr}(D) = \sum_{i=1}^{r} \mathrm{wr}(D;K_i)$.

Definition 1.3. Let D be a diagram of an oriented 1, 2, 3-valent (n,n)-tangle T, and let $\rho : \mathcal{A}(D) \to \mathbb{Z}_{>0}$ be a map. We define $\langle(D,\rho)\rangle \in \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots]$ by the local relations



and, for a diagram D without crossings,

$$\langle(D,\rho)\rangle = \begin{cases} 1 & \text{if } D \text{ is a diagram of an acyclic tangle,} \\ 0 & \text{if } D \text{ is a diagram of a cyclic tangle.} \end{cases}$$

A *colored classical* (n,n)*-tangle* is a pair (T,ρ) of a classical (n,n)-tangle T and a map $\rho : \{K_1,\dots,K_r\} \to \mathbb{Z}_{>0}$, where $K_1,\dots,K_r$ are the connected components of T. An *ordered classical* (n,n)*-tangle* is a colored classical (n,n)-tangle (T,ρ) such that $\rho(K_i) = i$ for any $i \in \{1,\dots,r\}$. We often write $T = T_1 \cup \cdots \cup T_r$ for an ordered classical (n,n)-tangle (T,ρ) by omitting ρ. The multivariable Alexander polynomial $\Delta_L(t_1,\dots,t_r)$ is an invariant of an ordered oriented link $L = K_1 \cup \cdots \cup K_r$. Let D be a diagram of T. A map $\rho : \{K_1,\dots,K_r\} \to \mathbb{Z}_{>0}$ induces a map from $\mathcal{A}(D)$ to $\mathbb{Z}_{>0}$ which sends $\alpha \in \mathcal{A}(D;K_i)$ into $\rho(K_i)$. We denote the map by the same symbol $\rho : \mathcal{A}(D) \to \mathbb{Z}_{>0}$.

Theorem 1.4. *Let (T,ρ) be a colored oriented classical (n,n)-tangle, and let D be a diagram of T. Let $K_1,\dots,K_r$ be the connected components of T. Then*

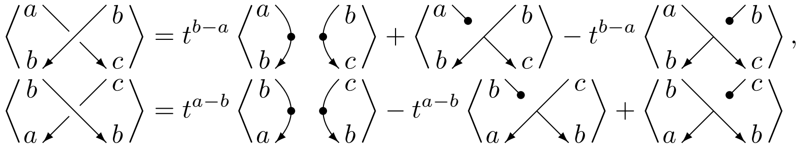
$$\prod_{i=1}^{r} t_{\rho(K_i)}^{\frac{\mathrm{rot}(D(K_i))+\mathrm{wr}(D;K_i)}{2}} \langle(D,\rho)\rangle$$

is invariant under the colored Reidemeister moves. In particular, for an ordered oriented classical $(1,1)$-tangle $T = T_1 \cup \cdots \cup T_r$ such that T_j is a strand connecting the end points of T, we have

$$\Delta_{\widehat{T}}(t_1,\dots,t_r) = \frac{\prod_{i=1}^{r} t_i^{\frac{\mathrm{rot}(D(K_i))+\mathrm{wr}(D;K_i)}{2}} \langle(D,\rho)\rangle}{t_j^{1/2} - t_j^{-1/2}}.$$

Let $R_p = (\mathbb{Z}/p\mathbb{Z}, \triangleleft)$ be the dihedral quandle of order p, where the binary operation is given by $a \triangleleft b = 2b - a$.

Definition 1.5. Let p be an odd prime number, and let $F := \mathbb{Q}(\sqrt{-1})[t]/(t^{p-1} + \cdots + 1)$. Let D be a diagram of an oriented 1, 2, 3-valent (n,n)-tangle T, and let $\rho : \mathcal{A}(D) \to R_p$ be a map. We define $\langle(D,\rho)\rangle \in F$ by the local relations



and, for a diagram D without crossings,

$$\langle (D,\rho)\rangle = \begin{cases} 1 & \text{if } D \text{ is a diagram of an acyclic tangle,} \\ 0 & \text{if } D \text{ is a diagram of a cyclic tangle.} \end{cases}$$

We denote by $Q(L)$ the fundamental quandle of an oriented link L. Let Q be a quandle. A quandle representation $\rho : Q(L) \to Q$ induces an Q-coloring of a diagram D of L, which we denote by the same symbol $\rho : \mathcal{A}(D) \to Q$. We also use the same symbol $\rho : \mathcal{A}(D') \to Q$ for the restriction of the Q-coloring $\rho : \mathcal{A}(D) \to Q$ to a subdiagram D' of D, which is a part of D.

Let L be an oriented link, and let $\rho : Q(L) \to R_p$ be a quandle representation. Suppose ρ is trivial. Let D_1 be a diagram of a $(1,1)$-tangle whose closure is L. Suppose ρ is nontrivial. Let D_2 be a diagram of a $(2,2)$-tangle whose closure is L such that the images of ρ on the top endpoints of D_2 are distinct elements $a, b \in R_p$. We then define

$$\Delta_p(L,\rho) := \begin{cases} (-1)^{-\frac{\mathrm{rot}(D_1)+\mathrm{wr}(D_1)}{2}} \langle (D_1,\rho)\rangle & \text{if } \rho \text{ is trivial,} \\ \dfrac{(-1)^{-\frac{\mathrm{rot}(D_2)+\mathrm{wr}(D_2)}{2}} \langle (D_2,\rho)\rangle}{(t^a - t^b)(t^{-a} - t^{-b})} & \text{if } \rho \text{ is nontrivial.} \end{cases}$$

Theorem 1.6. *Let T be an oriented classical (n,n)-tangle, and let D be a diagram of T. Let $\rho : \mathcal{A}(D) \to R_p$ be a quandle coloring. Then*

$$(-1)^{-\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}} \langle (D,\rho)\rangle$$

is invariant under the colored Reidemeister moves. Furthermore, for an oriented r-component link $L = K_1 \cup \cdots \cup K_r$ and its quandle representation $\rho : Q(L) \to R_p$, we have

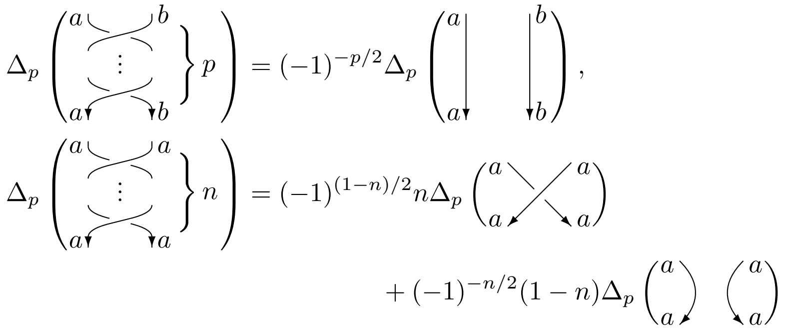
$$\Delta_p(L,\rho) = (-1)^{r-1}\Delta_L(-1)$$

if ρ is trivial, and we have

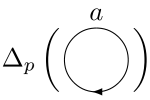
$$\Delta_p(L,\rho) = (-1)^{r/2+\mathrm{lk}(L)}(t - 2 + t^{-1})\Delta(L,\rho; f_1, f_2; 0, 1)$$

if ρ is nontrivial, where $\mathrm{lk}(L) := \sum_{i<j} \mathrm{lk}(K_i, K_j)$ and $\Delta(L,\rho; f_1, f_2; 0, 1)$ is the normalized quandle twisted Alexander invariant with $f_1(a,b) = -t^{b-a}$ and $f_2(a,b) = t^{b-a} + 1$.

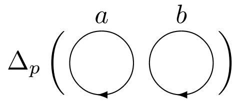
Proposition 1.7. *We have*



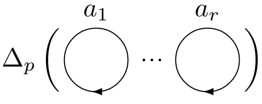
for $n \in \mathbb{Z}$ and any distinct elements $a, b \in R_p$, where the n crossings indicates $-n$ negative crossings if $n < 0$. For $a, b, a_1 \dots, a_r \in R_p$, we have



$$= 1,$$



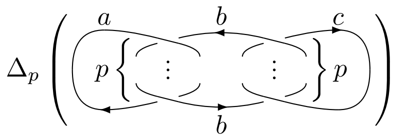
$$= \frac{1}{(t^a - t^b)(t^{-a} - t^{-b})} \quad \textit{if } a \neq b,$$



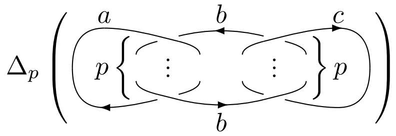
$$= 0 \quad \textit{if } r \geq 3 \textit{ or } a_i = a_j \textit{ for some } i \neq j.$$

Using the properties in Proposition 1.7, we have the following calculation example.

Example 1.8. For $a, b, c \in R_p$, we have



$$= \begin{cases} 0 & (a \neq b \neq c), \\ \frac{(-1)^{1/2} p}{(t^a - t^b)(t^{-a} - t^{-b})} & (a \neq b = c), \\ \frac{(-1)^{1/2} p}{(t^b - t^c)(t^{-b} - t^{-c})} & (a = b \neq c), \\ p^2 & (a = b = c), \end{cases}$$



$$= \begin{cases} 0 & (a \neq b \neq c), \\ \frac{(-1)^{1/2} p}{(t^a - t^b)(t^{-a} - t^{-b})} & (a \neq b = c), \\ \frac{(-1)^{-1/2} p}{(t^b - t^c)(t^{-b} - t^{-c})} & (a = b \neq c), \\ p^2 & (a = b = c). \end{cases}$$

We remark that these two knots are generalization of the granny knot and square knot and can be distinguished by the invariant Δ_p.

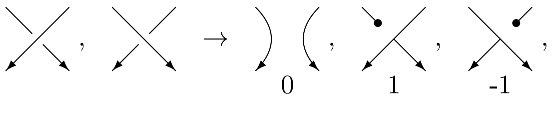
2. A BRACKET POLYNOMIAL FOR THE ALEXANDER–CONWAY POLYNOMIAL

A *tangle* is a graph embedded in I^3 such that the intersection of the graph and the boundary of I^3 is a union of several univalent vertices of the graph. We call a univalent vertex on the boundary an *endpoint* of the tangle and call a vertex in the interior an *inner vertex* of the tangle. A *$d_1, \dots, d_k$-valent tangle* is a tangle, each inner vertex of which has valency $d_1, \dots, d_{n-1}$, or d_n. An *(m, n)-tangle* is a tangle with m top endpoints and n bottom endpoints.

The Alexander–Conway polynomial $\Delta_L(t)$ of an oriented link L is characterized by the following:

- For the trivial knot ◯, we have $\Delta_{◯}(t) = 1$.
- The skein relation Δ  $(t) - \Delta$  $(t) = (t^{1/2} - t^{-1/2})\Delta$  (t) holds.

The bracket $\langle D \rangle$ introduced in Definition 1.1 is also defined as a state sum, which ensures that the bracket is well-defined. We denote by $C(D)$ the set of crossings of D. We denote by $\mathrm{sgn}(c)$ the sign of a crossing c. A *state* σ of an oriented uni-trivalent (n, n)-tangle diagram D is an assignment of an element of $\{0, 1, -1\}$ to each crossings:



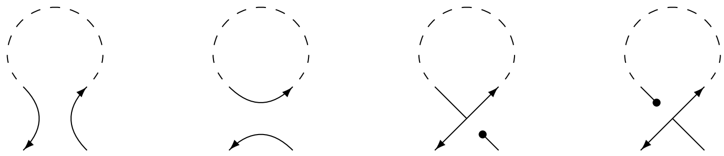
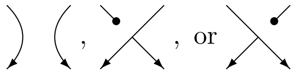


FIGURE 2. Top endpoints are connected

which is a map from $C(D)$ to $\{0,1,-1\}$. We denote by $S(D)$ the set of states of D. For a state σ, we define the *weight* $\mathrm{wt}(c;\sigma)$ of a crossing c by

$$\mathrm{wt}(c;\sigma)=\begin{cases}1 & \text{if } \sigma(c)=\operatorname{sgn}(c),\\ t^{-\operatorname{sgn}(c)} & \text{if } \sigma(c)=-\operatorname{sgn}(c),\\ -t^{-\operatorname{sgn}(c)} & \text{if } \sigma(c)=0.\end{cases}$$

We denote by D_σ the digram obtained from D by replacing each crossing with



according to σ. We then have

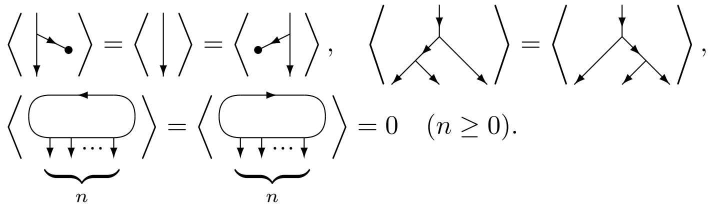
$$\langle D\rangle=\sum_{\sigma\in S(D)}\prod_{c\in C(D)}\mathrm{wt}(c;\sigma)\delta(D_\sigma), \tag{1}$$

where

$$\delta(D_\sigma)=\begin{cases}1 & \text{if } D_\sigma \text{ is a diagram of an acyclic tangle,}\\ 0 & \text{if } D_\sigma \text{ is a diagram of a cyclic tangle.}\end{cases}$$

From the state sum formula (1), we have the following lemma.

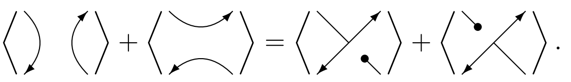
Lemma 2.1. *We have*



$= 0 \quad (n \geq 0).$

Proof. It is easy to see the equalities for diagrams without crossings. From the state sum formula (1), we have the equalities for any oriented uni-trivalent (n,n)-tangle diagrams. ☐

Lemma 2.2. *We have*



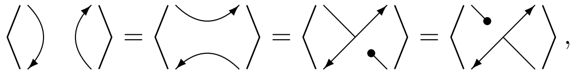
Proof. It is sufficient to show the equality for diagrams without crossings. If the two top endpoints of the tangles are connected by a path outside the tangles as shown in Figure 2, we have

which imply the desired equality. In a similar manner, we have the desired equality in the following cases:

(a) The two bottom endpoints of the tangles are connected by a path outside the tangles.

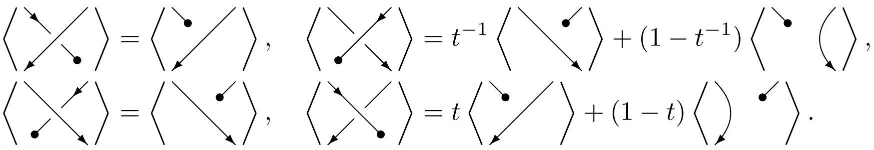
(b) The two left endpoints of the tangles are connected by a path outside the tangles.
(c) The two right endpoints of the tangles are connected by a path outside the tangles.

If no two of the endpoints of the tangles are connected by a path outside the tangles, we have

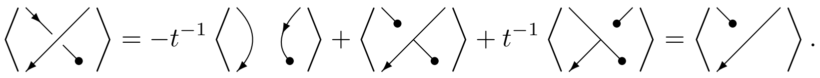


which imply the desired equality. ☐

Lemma 2.3. *We have*

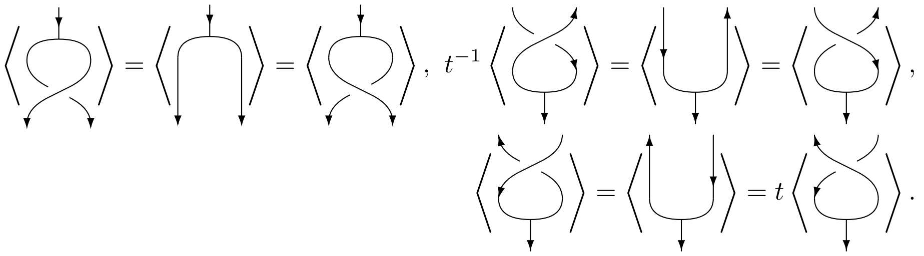


Proof. By Lemma 2.1, we have

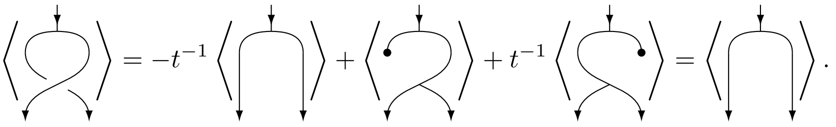


In a similar manner, we have the other equalities. ☐

Lemma 2.4. *We have*

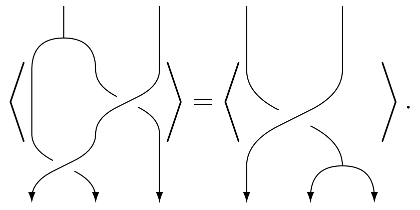


Proof. By Lemma 2.1, we have



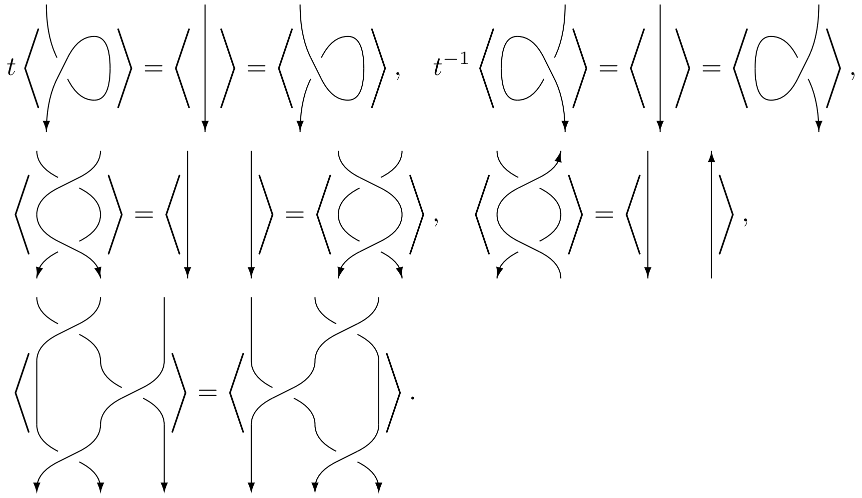
In a similar manner, we have the other equalities. ☐

Lemma 2.5. *We have*

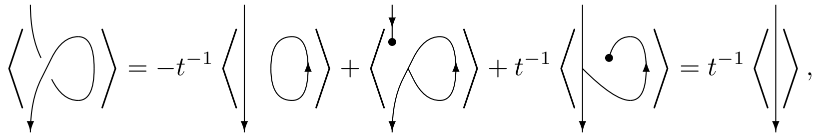


Proof. By Lemmas 2.1 and 2.4, we have the equality. ☐

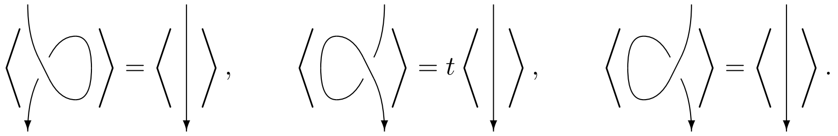
Proposition 2.6. *We have*



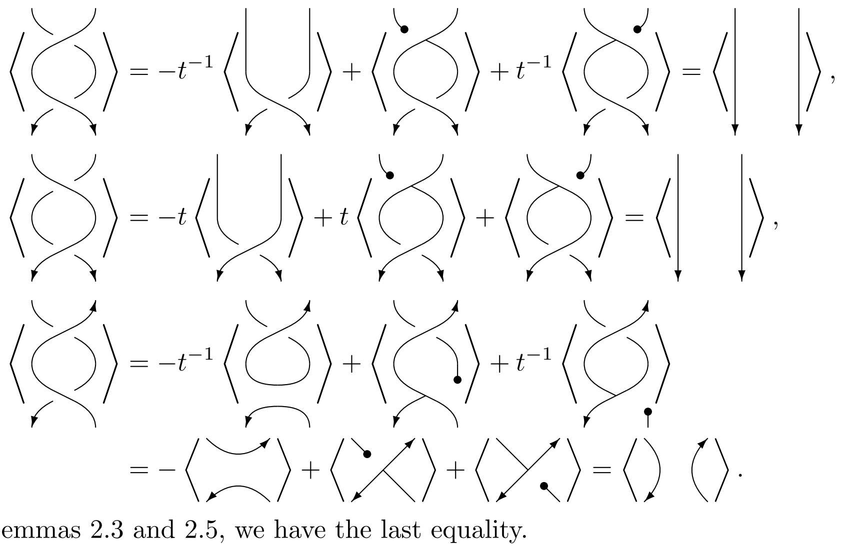
Proof. From the defining relations of $\langle D\rangle$, we have



where the last equality follows from Lemma 2.1. In a similar manner, we have



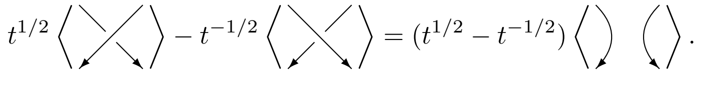
By Lemmas 2.2–2.4, we have



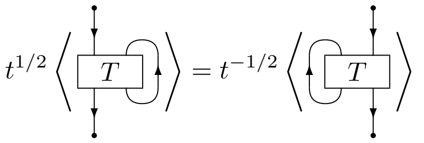
By Lemmas 2.3 and 2.5, we have the last equality. ☐

From the defining relations of $\langle D\rangle$, we have the following proposition.

Proposition 2.7. *We have*

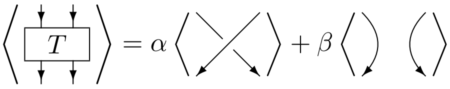


Proposition 2.8. *We have*

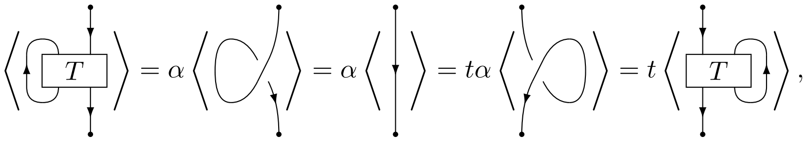


for any oriented classical $(2,2)$*-tangle* T*.*

Proof. By Propositions 2.6 and 2.7, we have



for some $\alpha, \beta \in \mathbb{Z}[t^{\pm 1/2}]$. We then have



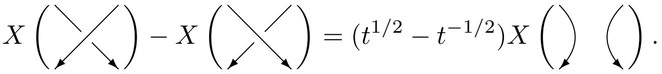
which imply the desired equality. □

Proof of Theorem 1.2. Let T be an oriented classical (n,n)-tangle. Let D be a diagram of T. By Proposition 2.6, $t^{\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}}\langle D\rangle$ is invariant under the Reidemeister moves.

Let L be an oriented link, and let T be an oriented classical $(1,1)$-tangle whose closure is L. Let D be a diagram of T. We define

$$X(L) := t^{\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}}\langle D\rangle.$$

By Proposition 2.8, $X(L)$ does not depend on the choice of the oriented classical $(1,1)$-tangle T whose closure is L. By Proposition 2.7, $X(L)$ satisfies the skein relation



We then have

$$\Delta_L(t) = X(L),$$

since both satisfy the same skein relation and $\Delta_{\bigcirc}(t) = 1 = X(\bigcirc)$. Therefore we have

$$\Delta_{\widehat{T}}(t) = X(\widehat{T}) = t^{\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}}\langle D\rangle.$$

□

3. Quandles and Alexander pairs

In this section, we will briefly recall quandles and Alexander pairs. For details, we refer the reader to [3, 4, 5]. Throughout this paper, for a positive integer n, we denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n as $\mathbb{Z}_n$.

A *quandle* [9, 12] is a non-empty set Q equipped with a binary operation $\lhd : Q \times Q \to Q$ satisfying the following axioms:

- For any $a \in Q$, $a \lhd a = a$.
- For any $a \in Q$, the map $\lhd a : Q \to Q$ defined by $\lhd a(x) = x \lhd a$ is bijective.
- For any $a, b, c \in Q$, $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$.

We denote by $(\lhd a)^n : Q \to Q$ by $\lhd^n a$ for $n \in \mathbb{Z}$. Then $R_n = (\mathbb{Z}_n, \lhd)$ is a quandle, where $a \lhd b = 2b - a$. A *trivial quandle* is a quandle Q with a binary operation $\lhd$ satisfying $a \lhd b = a$ for any $a, b \in Q$. Let $(Q_1, \lhd_1)$ and $(Q_2, \lhd_2)$ be quandles. A *quandle homomorphism* from Q_1 to Q_2 is defined to be a map $f : Q_1 \to Q_2$

Figure 3

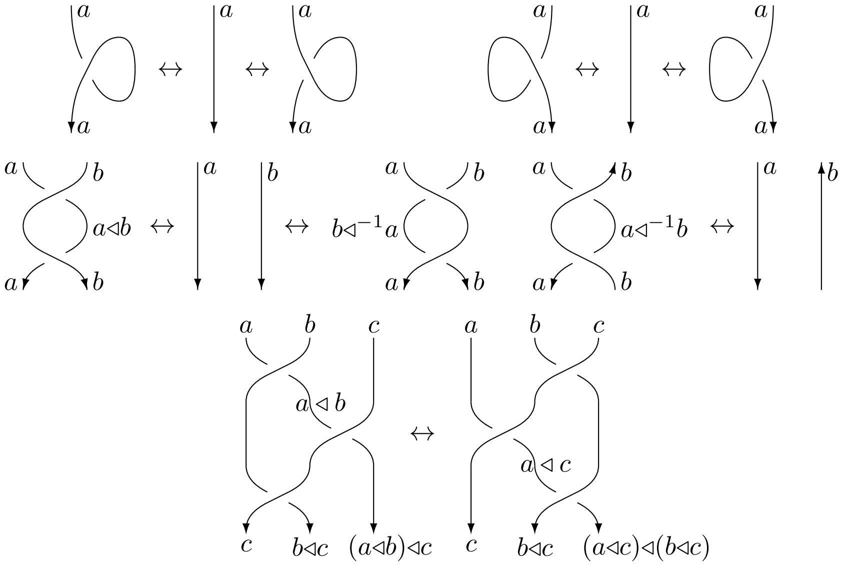


Figure 4. Colored Reidemeister moves

satisfying $f(a \lhd_1 b) = f(a) \lhd_2 f(b)$ for any $a, b \in Q_1$. A *quandle representation* ρ of a quandle X into a quandle Q is a quandle homomorphism $\rho : X \to Q$.

Let Q be a quandle. Let D be a diagram of an oriented classical (n, n)-tangle T. A *Q-coloring* of D is a map $C : \mathcal{A}(D) \to Q$ satisfying the condition

$$C(u_c) \lhd C(v_c) = C(w_c)$$

for each crossing $c \in C(D)$, where u_c, v_c and w_c are the arcs forming the crossing c as shown in the left picture of Figure 3. Here, the normal orientation is obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. A Q-coloring is *trivial* if it is a constant map. A colored classical (n, n)-tangle is an oriented classical (n, n)-tangle T with a $\mathbb{Z}_{>0}$-coloring ρ, where $\mathbb{Z}_{>0}$ is a trivial quandle. We denote by $\mathrm{Col}_Q(D)$ the set of Q-colorings of D. Let D' be a diagram of T obtained by applying a single Reidemeister move to D. Then, each Q-coloring C of D has a unique Q-coloring C' of D' that coincides with C except in the disk in which the move is applied. This gives a one-to-one correspondence between $\mathrm{Col}_Q(D)$ and $\mathrm{Col}_Q(D')$. The colored Reidemeister moves are listed in Figure 4, which are the Reidemeister moves with corresponding Q-colorings.

For a quandle $(Q, \lhd)$, a *Q-set* is a non-empty set Y equipped with a map $\lhd : Y \times Q \to Y$ satisfying the following axioms:

- For any $a \in Q$, the map $\lhd a : Y \to Y$ defined by $\lhd a(y) = y \lhd a$ is bijective.
- For any $y \in Y$ and $a, b \in Q$, we have $(y \lhd a) \lhd b = (y \lhd b) \lhd (a \lhd b)$.

The *associated group* $\mathrm{As}\, Q$ of a quandle Q is a group defined by the presentation:

$$\langle x \ (x \in Q) \mid x \lhd y = y^{-1} x y \ (x, y \in Q) \rangle.$$

Then $\operatorname{As} Q$ is a Q-set with $y \triangleleft a = ya$. Let $(Y_1, \triangleleft_1)$ and $(Y_2, \triangleleft_2)$ be Q-sets. A *Q-set homomorphism* from Y_1 to Y_2 is defined to be a map $f : Y_1 \to Y_2$ satisfying $f(y \triangleleft_1 a) = f(y) \triangleleft_2 a$ for any $y \in Y_1$ and $a \in Q$. Let D be a diagram of an oriented link L. We denote by $\mathcal{SA}(D)$ the set of semi-arcs of D, where a semi-arc is a piece of a curve such that the endpoints of the piece are crossings. We denote by $\mathcal{R}(D)$ the set of complementary regions of D. For a semi-arc α, we denote by $r(\alpha)$ and $r'(\alpha)$ the regions facing the semi-arc α as shown in the right picture of Figure 3. For an arc α, we set $r(\alpha) := r(\alpha_0)$ and $r'(\alpha) := r'(\alpha_0)$, where α_0 is the semi-arc that originates from the arc α and shares its initial point with the arc α. Let Y be a Q-set. A *Q_Y-coloring* ρ_Y of D is an extension of a Q-coloring ρ of D that assigns an element of Y to each region of D satisfying the condition

$$\rho_Y(r(\alpha)) \triangleleft \rho(\alpha) = \rho_Y(r'(\alpha))$$

for each semi-arc $\alpha \in \mathcal{A}(D)$, where the color $\rho(\alpha)$ of a semi-arc α is defined by the color of the arc from which the semi-arc originates. We denote by $\widetilde{\rho}$ the $Q_{\operatorname{As} Q}$-coloring that is the extension of ρ satisfying $\widetilde{\rho}(r_{\text{out}}) = 1$, where r_{out} is the outermost region of D.

Let Q be a quandle, and let R be a unital ring. The pair (f_1, f_2) of maps $f_1, f_2 : Q \times Q \to R$ is an *Alexander pair* [5] if f_1 and f_2 satisfy the following conditions:

- For any $a \in Q$, $f_1(a, a) + f_2(a, a) = 1$.
- For any $a, b \in Q$, $f_1(a, b)$ is invertible.
- For any $a, b, c \in Q$,

$$\begin{aligned} &f_1(a \triangleleft b, c) f_1(a, b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a, c), \\ &f_1(a \triangleleft b, c) f_2(a, b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b, c), \text{ and} \\ &f_2(a \triangleleft b, c) = f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c). \end{aligned}$$

Assuming that $f_1(a, b) + f_2(a, b) = 1$ for any $a, b \in Q$, we have the last equality follows from the other conditions, since we have

$$\begin{aligned} &f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c) - f_2(a \triangleleft b, c) \\ &= -f_1(a \triangleleft c, b \triangleleft c) f_1(a, c) - f_2(a \triangleleft c, b \triangleleft c) f_1(b, c) + f_1(a \triangleleft b, c) \\ &= -f_1(a \triangleleft b, c) f_1(a, b) - f_1(a \triangleleft b, c) f_2(a, b) + f_1(a \triangleleft b, c) = 0. \end{aligned}$$

Let (f_1, f_2) be an Alexander pair. A *column relation map* $f_{\text{col}} : Q \to R$ is a map satisfying

$$f_{\text{col}}(a \triangleleft b) = f_1(a, b) f_{\text{col}}(a) + f_2(a, b) f_{\text{col}}(b)$$

for any $a, b \in Q$. For each $c \in Q$, the map $f_{\text{col}} : Q \to R$ defined by $f_{\text{col}}(a) = f_2(a \triangleleft^{-1} c, c)$ is a column relation map ([4]).

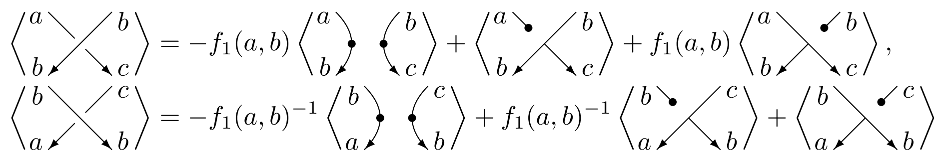
Let (f_1, f_2) be an Alexander pair. Let Y be a Q-set. A *row relation map* $f_{\text{row}} : Y \times Q \to R$ is a map satisfying

$$\begin{aligned} &f_{\text{row}}(y, a) = f_{\text{row}}(y \triangleleft b, a \triangleleft b) f_1(a, b), \text{ and} \\ &f_{\text{row}}(y \triangleleft a, b) = f_{\text{row}}(y, b) + f_{\text{row}}(y \triangleleft b, a \triangleleft b) f_2(a, b) \end{aligned}$$

for any $a, b \in Q$ and $y \in Y$. Let Y be the Q-set $Q \times R^\times$ with $(y, z) \triangleleft a := (y \triangleleft a, f_1(y, a) z)$. The map $f_{\text{row}} : Y \times Q \to R$ defined by $f_{\text{row}}((y, z), a) = z^{-1} f_1(y, a)^{-1} f_2(y, a)$ is a row relation map ([3]). For each $c \in Q$, the map $f_{\text{row}} : \operatorname{As} Q \times Q \to R$ defined by $f_{\text{row}}(y, a) = f_{\text{row}}(\varphi_c(y), a)$ is a row relation map, where $\varphi_c : \operatorname{As} Q \to Y$ is the Q-set homomorphism satisfying $\varphi_c(1) = (c, 1)$.

4. A BRACKET POLYNOMIAL FOR AN (f_1, f_2)-TWISTED ALEXANDER INVARIANT WITH $f_1 + f_2 = 1$

Definition 4.1. Let Q be a quandle, and let R be a commutative ring. Let (f_1, f_2) be an Alexander pair of $f_1, f_2 : Q \times Q \to R$ satisfying $f_1(a,b) + f_2(a,b) = 1$ for any $a, b \in Q$. Let D be a diagram of an oriented $1,2,3$-valent (n,n)-tangle T, and let $\rho : \mathcal{A}(D) \to Q$ be a map. We define $\langle(D,\rho)\rangle \in R$ by the local relations



and, for a diagram D without crossings,

$$\langle(D,\rho)\rangle = \begin{cases} 1 & \text{if } D \text{ is a diagram of an acyclic tangle,} \\ 0 & \text{if } D \text{ is a diagram of a cyclic tangle.} \end{cases}$$

Let D be a diagram of an oriented classical (n,n)-tangle T, and let $K_1, \dots, K_r$ be the connected components of T. Let $D(K_i)$ be the diagram of K_i that is obtained by removing the other connected components from D. We then have $\operatorname{rot}(D) = \sum_{i=1}^{r} \operatorname{rot}(D(K_i))$.

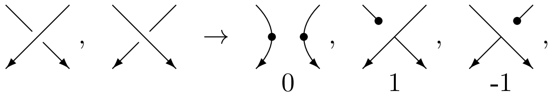
Proposition 4.2. *Let T be an oriented classical (n,n)-tangle, and let D be a diagram of T. Let $\rho : \mathcal{A}(D) \to Q$ be a quandle coloring. We fix $\omega_1, \dots, \omega_r \in R^\times$ so that $\omega_i = f_1(\rho(\alpha), \rho(\alpha))$ for some $\alpha \in \mathcal{A}(D; K_i)$. Then*

$$\left(\prod_{i=1}^{r} \omega_i^{\frac{\operatorname{rot}(D(K_i)) + \operatorname{wr}(D(K_i)) + 1}{2}} \right)^{-1} \langle(D,\rho)\rangle$$

is invariant under the colored Reidemeister moves.

We remark that $f_1(\rho(\alpha), \rho(\alpha))$ does not depend on the choice of the arc α, since $f_1(\rho(\alpha), \rho(\alpha)) = f_1(\rho(\alpha) \triangleleft c, \rho(\alpha) \triangleleft c)$ follows from $f_1(a \triangleleft a, c) f_1(a, a) = f_1(a \triangleleft c, a \triangleleft c) f_1(a, c)$.

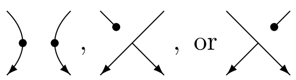
The bracket $\langle(D,\rho)\rangle$ introduced in Definition 4.1 is also defined as a state sum, which ensures that the bracket is well-defined. A *state* σ of D is an assignment of an element of $\{0, 1, -1\}$ to each crossings:



which is a map from $C(D)$ to $\{0, 1, -1\}$. For a state σ, we define the *weight* $\operatorname{wt}(c;\sigma)$ of a crossing c by

$$\operatorname{wt}(c;\sigma) = \begin{cases} 1 & \text{if } \sigma(c) = \operatorname{sgn}(c), \\ f_1(\rho(u_c), \rho(v_c))^{\operatorname{sgn}(c)} & \text{if } \sigma(c) = -\operatorname{sgn}(c), \\ -f_1(\rho(u_c), \rho(v_c))^{\operatorname{sgn}(c)} & \text{if } \sigma(c) = 0. \end{cases}$$

We denote by D_σ the digram obtained from D by replacing each crossing with



according to σ. We then have

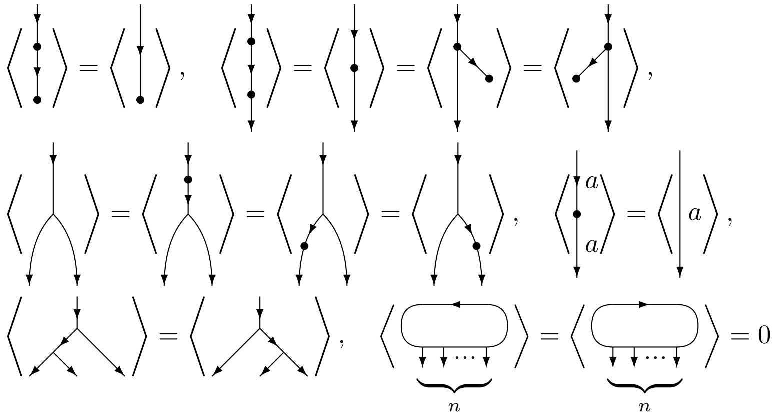
$$\langle(D,\rho)\rangle = \sum_{\sigma \in S(D)} \prod_{c \in C(D)} \operatorname{wt}(c;\sigma)\delta(D_\sigma),$$

where

$$\delta(D_\sigma) = \begin{cases} 1 & \text{if } D_\sigma \text{ is a diagram of an acyclic tangle,} \\ 0 & \text{if } D_\sigma \text{ is a diagram of a cyclic tangle.} \end{cases}$$

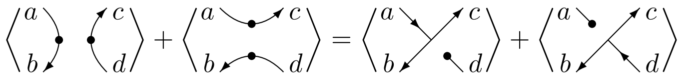
In a similar way as in Section 2, we have the following lemmas and proposition.

Lemma 4.3. *We have*



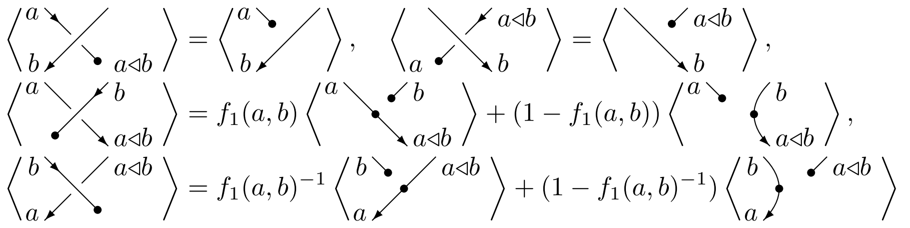
for $a \in Q$, $m \in \{0, 1, 2\}$ and $n \geq 0$, where we omit colors of the arcs. There are no restrictions between the omitted colors other than the requirement that the colors of the corresponding endpoints are the same.

Lemma 4.4. *We have*



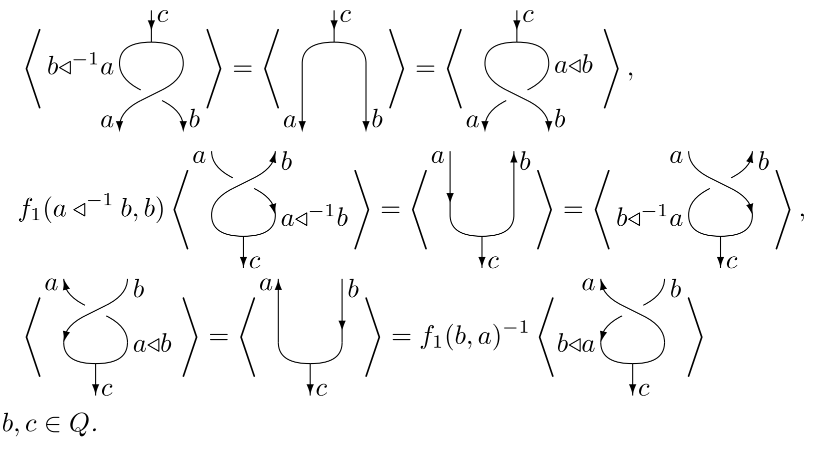
for $a, b, c, d \in Q$.

Lemma 4.5. *We have*



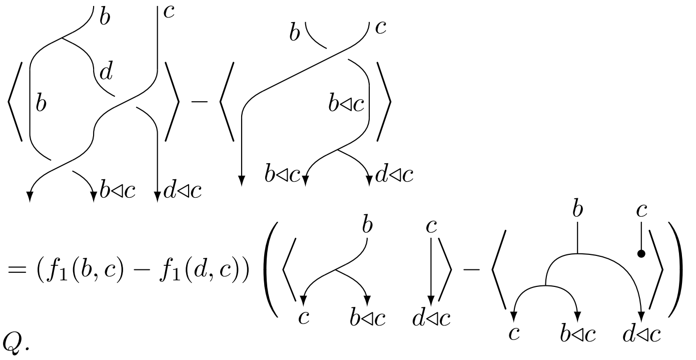
for $a, b \in Q$.

Lemma 4.6. *We have*



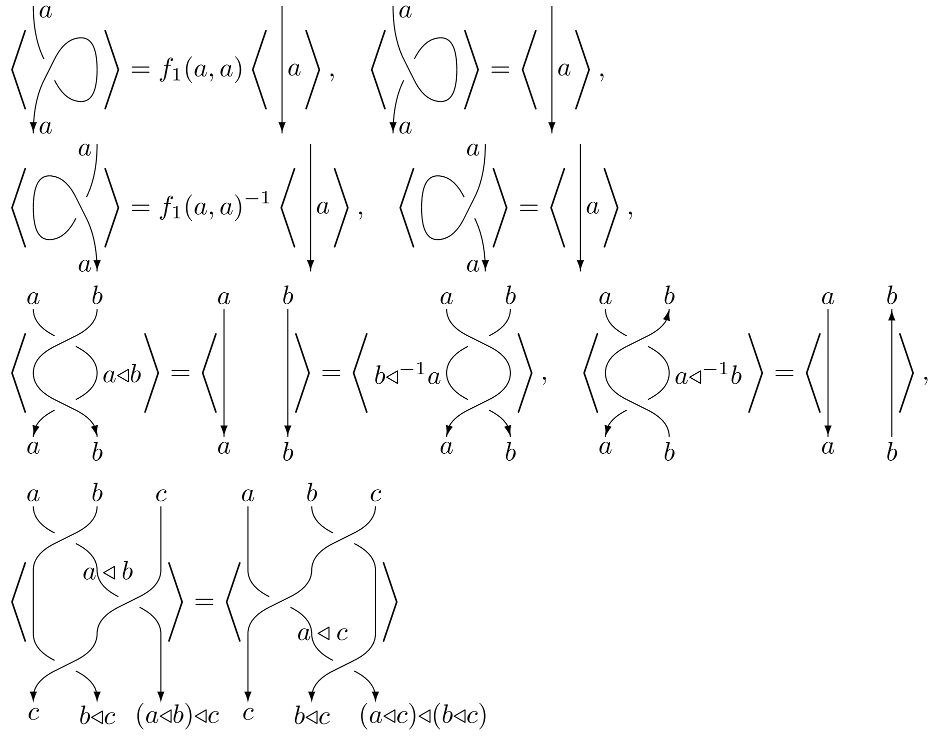
for $a, b, c \in Q$.

Lemma 4.7. *We have*



for $b, c, d \in Q$.

Proposition 4.8. *We have*



for $a, b, c \in Q$.

Proposition 4.2 follows from this proposition.

Proposition 4.9. *Let* T *be an oriented classical* (n, n)*-tangle containing a split link component* L*, which is a link component of* T *with a* 2*-sphere separating* L *from* $T - L$*. Let* D *be a diagram of* T*. Let* $\rho : \mathcal{A}(D) \to Q$ *be a quandle coloring. Then, we have* $\langle (D, \rho) \rangle = 0$.

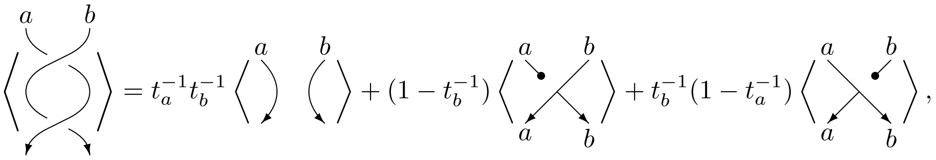
Proof. By Proposition 4.8, we may assume that $D = D_{T-L} \sqcup D_L$, that is, there is a circle separating L from $T - L$ on the diagram D, where D_{T-L} and D_L are diagrams of $T - L$ and L, respectively. Since any state of D_L has a cycle, we have $\langle (D, \rho) \rangle = 0$. □

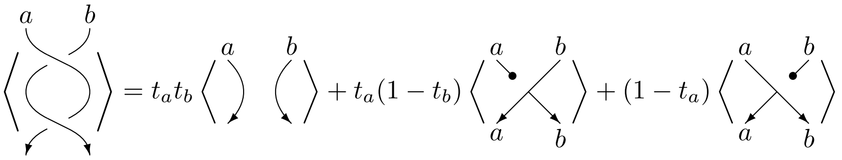
5. A BRACKET POLYNOMIAL FOR THE MULTIVARIABLE ALEXANDER POLYNOMIAL

Let $Q := \mathbb{Z}_{>0}$ be the trivial quandle, and let $R := \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots]$. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$ defined by $f_1(a, b) = t_b^{-1}$ and

$f_2(a,b) = 1 - t_b^{-1}$. Then, the bracket polynomial introduced in Definition 4.1 coincides with the bracket polynomial introduced in Definition 1.3. In this section, we discuss this bracket polynomial $\langle (D,\rho)\rangle$.

Lemma 5.1. *We have*

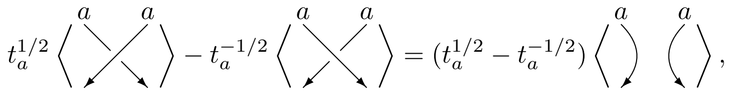




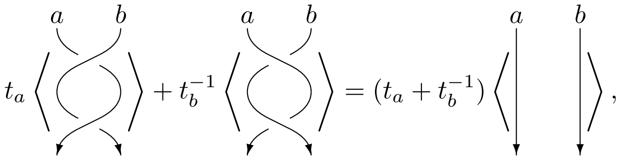
for $a, b \in \mathbb{Z}_{>0}$.

Proof. Using Lemmas 4.3 and 4.6, we have the equalities. ☐

Proposition 5.2. *We have*



(2)



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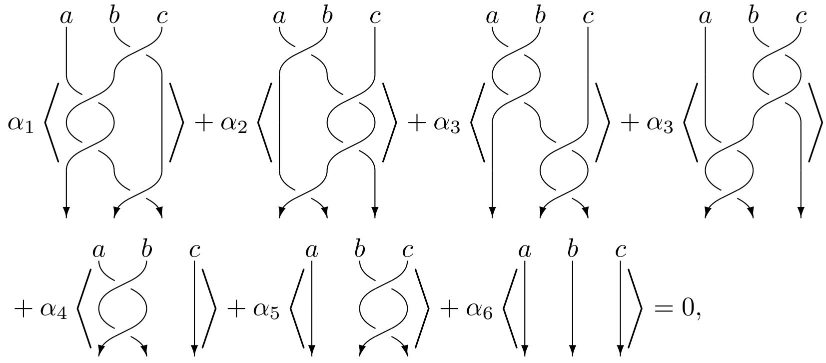
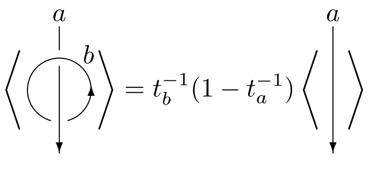


(4)



(5)

for $a, b, c \in \mathbb{Z}_{>0}$, *where*

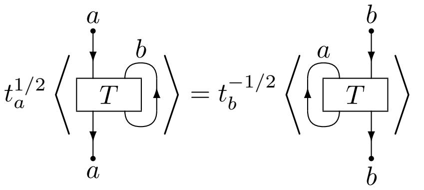
$$\alpha_1 = t_a t_b t_c - t_a t_c + t_b t_c - t_c, \qquad \alpha_2 = -t_a t_b t_c - t_a t_b + t_a t_c + t_a,$$

$$\alpha_3 = t_a t_b - t_b t_c, \qquad \alpha_4 = -t_a + t_b - t_a t_c^{-1} + t_b t_c,$$

$$\alpha_5 = -t_a t_b + t_a^{-1} t_c - t_b + t_c, \qquad \alpha_6 = t_a t_c^{-1} - t_a^{-1} t_c.$$

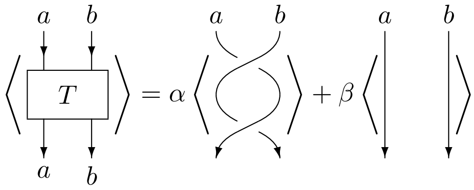
Proof. Using Lemma 5.1, we have the equalities. ☐

Proposition 5.3. *We have*



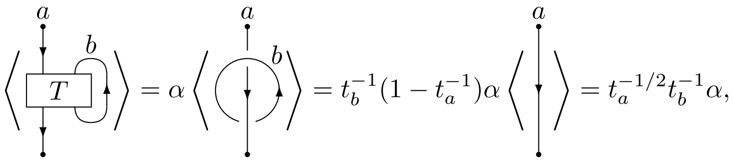
for any oriented classical $(2,2)$*-tangle* T.

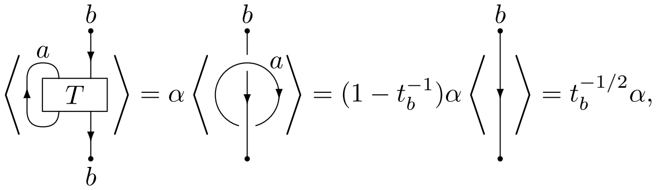
Proof. By Propositions 4.8, 4.9 and 5.2, we have



(6)

for some $\alpha, \beta \in \mathbb{Z}[t_1^{\pm 1/2}, t_2^{\pm 1/2}, \ldots]$. See [13]. We then have





which imply the desired equality. ☐

Proof of Theorem 1.4. Let (T, ρ) be a colored oriented classical (n,n)-tangle, and let D be a diagram of T. Let $K_1, \ldots, K_r$ be the connected components of T. By Proposition 4.2,

$$\prod_{i=1}^{r} t_{\rho(K_i)}^{\frac{\mathrm{rot}(D(K_i)) + \mathrm{wr}(D;K_i)}{2}} \langle\langle D, \rho \rangle\rangle = \prod_{i=1}^{r} t_{\rho(K_i)}^{\frac{\mathrm{lk}(K_i, L - K_i) - 1}{2}} \prod_{i=1}^{r} t_{\rho(K_i)}^{\frac{\mathrm{rot}(D(K_i)) + \mathrm{wr}(D(K_i)) + 1}{2}} \langle\langle D, \rho \rangle\rangle$$

is invariant under the colored Reidemeister moves.

Let L be an oriented link, and let T be an oriented classical $(1,1)$-tangle whose closure is L. Let D be a diagram of T. Let $K_1, \ldots, K_r$ be the connected components of T such that T_j is a strand connecting the end points of T. We then define

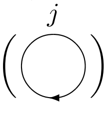
$$X(L, \rho) := \frac{\prod_{i=1}^{r} t_i^{\frac{\mathrm{rot}(D(K_i)) + \mathrm{wr}(D;K_i)}{2}} \langle\langle D, \rho \rangle\rangle}{t_j^{1/2} - t_j^{-1/2}}.$$

By Proposition 5.3, $X(L, \rho)$ does not depend on the choice of the oriented classical $(1,1)$-tangle T whose closure is L. By Proposition 5.2, $X(L, \rho)$ satisfies the skein relations obtained from (2)–(5) by replacing $\langle D \rangle$ with $X(D)$ and the coefficients $t_a^{1/2}$, $-t_a^{-1/2}$, $(t_a^{1/2} - t_a^{-1/2})$, t_a, t_b^{-1}, $(t_a + t_b^{-1})$, $\alpha_1, \ldots, \alpha_6$, $t_b^{-1}(1 - t_a^{-1})$ with 1, −1,

$(t_a^{1/2}-t_a^{-1/2})$, 1, 1, $(t_a^{1/2}t_b^{1/2}+t_a^{-1/2}t_b^{-1/2})$, $\beta_1,\dots,\beta_6$, $(t_a^{1/2}-t_a^{-1/2})$, where

$$\begin{aligned}
\beta_1 &= t_a^{1/2}t_b^{1/2}-t_a^{1/2}t_b^{-1/2}+t_a^{-1/2}t_b^{1/2}-t_a^{-1/2}t_b^{-1/2},\\
\beta_2 &= -t_b^{1/2}t_c^{1/2}-t_b^{1/2}t_c^{-1/2}+t_b^{-1/2}t_c^{1/2}+t_b^{-1/2}t_c^{-1/2},\\
\beta_3 &= t_a^{1/2}t_c^{-1/2}-t_a^{-1/2}t_c^{1/2},\\
\beta_4 &= -t_a^{1/2}t_b^{-1/2}+t_a^{-1/2}t_b^{1/2}-t_a^{1/2}t_b^{-1/2}t_c^{-1}+t_a^{-1/2}t_b^{1/2}t_c,\\
\beta_5 &= -t_at_b^{1/2}t_c^{-1/2}+t_a^{-1}t_b^{-1/2}t_c^{1/2}-t_b^{1/2}t_c^{-1/2}+t_b^{-1/2}t_c^{1/2},\\
\beta_6 &= t_at_c^{-1}-t_a^{-1}t_c.
\end{aligned}$$

By the definition of $X(L,\rho)$, we have

$X\Bigl($  $\Bigr) = \Bigl\langle$  $\Bigr\rangle = \dfrac{1}{t_j^{1/2}-t_j^{-1/2}}.$

By Proposition 4.9, we have $X(L,\rho)=0$ for any split link L. Then, by [13, Theorem 7.2], we have $\Delta_L(t_1,\dots,t_r)=X(L)$. Therefore, we have

$$\Delta_{\widehat{T}}(t_1,\dots,t_r)=X(\widehat{T})=\frac{\prod_{i=1}^r t_i^{\frac{\mathrm{rot}(D(K_i))+\mathrm{wr}(D;K_i)}{2}}\langle(D,\rho)\rangle}{t_j^{1/2}-t_j^{-1/2}}.$$

□

6. The quandle twisted Alexander invariant

In this section, we recall an equivalence relation on triples of matrices and their row and column relation matrices, and see how to obtain the triples from an oriented link and its quandle representation. For details, we refer the reader to [6].

Let R be a unital ring. We denote by $R^\times$ the group of units of R. We denote by $M(m,n;R)$ the set of $m\times n$ matrices over R and denote by $GL(n;R)$ the set of $n\times n$ invertible matrices over R. For $A=(a_{ij})\in M(m,n;R)$, $\boldsymbol{i}=(i_1,\dots,i_s)$ and $\boldsymbol{j}=(j_1,\dots,j_t)$, we define

$$A_{\boldsymbol{i},\boldsymbol{j}} := \begin{pmatrix} a_{i_1j_1} & \cdots & a_{i_1j_t}\\ \vdots & \ddots & \vdots\\ a_{i_sj_1} & \cdots & a_{i_sj_t}\end{pmatrix}.$$

Let S_n be the symmetric group on $\{1,\dots,n\}$. We denote by $\operatorname{sgn}\sigma$ the sign of $\sigma\in S_n$. Put $\overline{n}:=(1,\dots,n)$. For $\sigma\in S_n$, we set

$$\begin{aligned}
\sigma(i_1,\dots,i_s) &:= (\sigma(i_1),\dots,\sigma(i_s)),\\
(i_1,\dots,i_s)+k &:= (i_1+k,\dots,i_s+k).
\end{aligned}$$

Let $A\in M(d+m,d+n;R)$, where $d,m,n>0$. We call $B\in M(m,d+m;R)$ a *row relation matrix* of A if $BA=O$. A row relation matrix $B\in M(m,d+m;R)$ is *regular* if $B_{\overline{m},\sigma(\overline{m})}$ is invertible for some $\sigma\in S_{d+m}$. We call $C\in M(d+n,n;R)$ a *column relation matrix* of A if $AC=O$. A column relation matrix $C\in M(d+n,n;R)$ is *regular* if $C_{\tau(\overline{n}),\overline{n}}$ is invertible for some $\tau\in S_{d+n}$. We define $P_{ij},E_{ij}(r),E_i(u)\in GL(n;R)$ by

$$\begin{aligned}
P_{ij} &= (\boldsymbol{e}_1,\dots,\boldsymbol{e}_{i-1},\boldsymbol{e}_j,\boldsymbol{e}_{i+1},\dots,\boldsymbol{e}_{j-1},\boldsymbol{e}_i,\boldsymbol{e}_{j+1},\dots,\boldsymbol{e}_n),\\
E_{ij}(r) &= (\boldsymbol{e}_1,\dots,\boldsymbol{e}_{j-1},\boldsymbol{e}_j+r\boldsymbol{e}_i,\boldsymbol{e}_{j+1},\dots,\boldsymbol{e}_n)\ (i\neq j),\\
E_i(u) &= (\boldsymbol{e}_1,\dots,\boldsymbol{e}_{i-1},u\boldsymbol{e}_i,\boldsymbol{e}_{i+1},\dots,\boldsymbol{e}_n)
\end{aligned}$$

Figure 5

for $r \in R$ and $u \in R^\times$, where $\boldsymbol{e}_i$ is the unit column vector whose components are all 0, except the ith component that equals 1. We write $(B, A, C) \sim (B', A', C')$ if they are related by a finite sequence of the following transformations:

- $(B, A, C) \leftrightarrow (BE_{ij}(r)^{-1}, E_{ij}(r)A, C)$ $(r \in R)$,
- $(B, A, C) \leftrightarrow (B, AE_{ij}(r), E_{ij}(r)^{-1}C)$ $(r \in R)$,
- $(B, A, C) \leftrightarrow (BE_i(u), E_i(u)^{-1}AE_j(u), E_j(u)^{-1}C)$ $(u \in R^\times)$,
- $(B, A, C) \leftrightarrow \left(\begin{pmatrix} B & \mathbf{0} \end{pmatrix}, \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \begin{pmatrix} C \\ \mathbf{0} \end{pmatrix}\right)$.

Let R be a field. Let $A \in M(d+m, d+n; R)$. Let $B \in M(m, d+m; R)$ be a regular row relation matrix of A, and let $C \in M(d+n, n; R)$ be a regular column relation matrix of A. We choose $\sigma \in S_{d+m}$ and $\tau \in S_{d+n}$ so that $B_{\overline{m}, \sigma(\overline{m})}$ and $C_{\tau(\overline{n}), \overline{n}}$ are invertible. We then define

$$\Delta(B, A, C) := \frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det A_{\sigma(\bar{d}+m), \tau(\bar{d}+n)}}{\det B_{\overline{m}, \sigma(\overline{m})} \det C_{\tau(\overline{n}), \overline{n}}},$$

which is an invariant of the equivalence class of (B, A, C).

Let Q be a quandle and R a unital ring. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link, and let $\rho : Q(L) \to Q$ be a quandle representation. Let D be a diagram of L such that every component has an undercrossing. Let $c_1, \ldots, c_n$ be the crossings of D. We denote by x_i the arc starting from a crossing c_i for each i (see the left picture of Figure 5). We denote by u_i, w_i and v_i the under-arcs and over-arc, respectively, of a crossing c_i such that the normal orientation of v_i points from u_i to w_i (see the right picture of Figure 5).

We define $A(D, \rho; f_1, f_2)$ as the $n \times n$ matrix whose (i, j)-entry a_{ij} is given by

$$a_{ij} = \delta(u_i, x_j) f_1(a_i, b_i) + \delta(v_i, x_j) f_2(a_i, b_i) - \delta(w_i, x_j),$$

where $a_i = \rho(u_i)$, $b_i = \rho(v_i)$, and

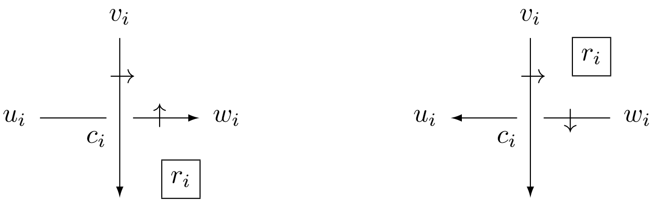
$$\delta(x, y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $C_+(D)$ and $C_-(D)$ the sets of positive and negative crossings of D, respectively. We denote by $\#S$ the number of elements of a set S. We fix $\omega_1, \ldots, \omega_r \in R^\times$ so that $\omega_i = f_1(\rho(\alpha), \rho(\alpha))$ for some $\alpha \in \mathcal{A}(D; K_i)$. We define

$$\widetilde{A}(D, \rho; f_1, f_2) := \begin{pmatrix} A(D, \rho; f_1, f_2) & \mathbf{0} \\ \mathbf{0} & \operatorname{cor}(D, \rho; f_1, f_2)^{-1} \end{pmatrix},$$

where

$$\operatorname{cor}(D, \rho; f_1, f_2) = (-1)^{\#C_+(D)} \prod_{i=1}^{r} \omega_i^{\frac{\operatorname{rot}(D(K_i)) + \operatorname{wr}(D(K_i)) + 1}{2}} \prod_{c \in C_-(D)} f_1(\rho(u_c), \rho(v_c)).$$



$\operatorname{sgn}(c_i) = 1$ $\operatorname{sgn}(c_i) = -1$

Figure 6

For column relation maps $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m} : Q \to R$, we define

$$R_{\mathrm{col}}(D, \rho; f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m}) := \begin{pmatrix} f_{\mathrm{col},1}(\rho(x_1)) & \cdots & f_{\mathrm{col},m}(\rho(x_1)) \\ \vdots & \ddots & \vdots \\ f_{\mathrm{col},1}(\rho(x_n)) & \cdots & f_{\mathrm{col},m}(\rho(x_n)) \end{pmatrix}.$$

We denote $R_{\mathrm{col}}(D, \rho; f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m})$ by $R_{\mathrm{col}}(D, \rho; \boldsymbol{f}_{\mathbf{col}})$ for short. We define

$$\widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f}_{\mathbf{col}}) := \begin{pmatrix} R_{\mathrm{col}}(D, \rho; \boldsymbol{f}_{\mathbf{col}}) \\ \mathbf{0} \end{pmatrix}.$$

We define $r_i := r(\alpha(w_i; c_i))$, where $\alpha(w_i; c_i)$ is the semi-arc that originates from the arc w_i and is incident to the crossing c_i (see Figure 6). For row relation maps $f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m} : \operatorname{As} Q \times Q \to R$, we define

$$\begin{aligned} &R_{\mathrm{row}}(D, \rho; f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m}) \\ &:= \begin{pmatrix} \operatorname{sgn}(c_1) f_{\mathrm{row},1}(\widetilde{\rho}(r_1), \rho(w_1)) & \cdots & \operatorname{sgn}(c_n) f_{\mathrm{row},1}(\widetilde{\rho}(r_n), \rho(w_n)) \\ \vdots & \ddots & \vdots \\ \operatorname{sgn}(c_1) f_{\mathrm{row},m}(\widetilde{\rho}(r_1), \rho(w_1)) & \cdots & \operatorname{sgn}(c_n) f_{\mathrm{row},m}(\widetilde{\rho}(r_n), \rho(w_n)) \end{pmatrix}. \end{aligned}$$

We denote $R_{\mathrm{row}}(D, \rho; f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m})$ by $R_{\mathrm{row}}(D, \rho; \boldsymbol{f}_{\mathbf{row}})$ for short. We define

$$\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f}_{\mathbf{row}}) := \begin{pmatrix} R_{\mathrm{row}}(D, \rho; \boldsymbol{f}_{\mathbf{row}}) & \mathbf{0} \end{pmatrix}.$$

Definition 6.1. Let $a_1, \ldots, a_m \in R$. Let $f_{\mathrm{col},i} : Q \to R$ be the column relation map defined by $f_{\mathrm{col},i}(x) = f_2(x \triangleleft^{-1} a_i, a_i)$. Let Y be the Q-set $Q \times R^{\times}$ with $(y, z) \triangleleft a := (y \triangleleft a, f_1(y, a)z)$. Let $f_{\mathrm{row}} : Y \times Q \to R$ be the row relation map defined by $f_{\mathrm{row}}((y, z), a) = z^{-1} f_1(y, a)^{-1} f_2(y, a)$. Let $f_{\mathrm{row},i} : \operatorname{As} Q \times Q \to R$ be the row relation map defined by $f_{\mathrm{row},i}(y, x) = f_{\mathrm{row}}(\varphi_{a_i}(y), x)$ is a row relation map, where $\varphi_c : \operatorname{As} Q \to Y$ is the Q-set homomorphism satisfying $\varphi_c(1) = (c, 1)$. We then define

$$\begin{aligned} &\Delta(L, \rho; f_1, f_2; a_1, \ldots, a_m) \\ &:= \Delta(\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f}_{\mathbf{row}}), \widetilde{A}(D, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f}_{\mathbf{col}})). \end{aligned}$$

7. Matrices of a diagram with vertices

In this section, we extend the definition of $A(D, \rho; f_1, f_2)$ to a diagram D with vertices and show that its determinant gives the bracket polynomial.

Let Q be a quandle, and let R be a commutative ring. Let (f_1, f_2) be an Alexander pair of $f_1, f_2 : Q \times Q \to R$. Let D be a diagram of an oriented $1, 2, 3$-valent (n, n)-tangle T, and let $\rho : \mathcal{A}(D) \to Q$ be a map. Suppose that every component of D has an undercrossing or a vertex. Let $x_1, \ldots, x_n$ be the arcs of D. We denote

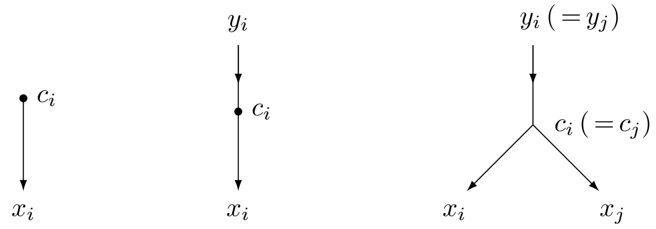


Figure 7

by c_i the initial point of x_i, which is a crossing or a vertex. We denote by y_i the arc whose terminal point is c_i. See the left picture of Figure 5 and Figure 7. We define $A^{\bullet}(D,\rho;f_1,f_2)$ as the $n\times n$ matrix whose (i,j)-entry $a^{\bullet}_{ij}$ is given by

$$a^{\bullet}_{ij}=\begin{cases} -\delta(u_i,x_j)f_1(a_i,b_i)-\delta(v_i,x_j)f_2(a_i,b_i)+\delta(w_i,x_j) & \\ & \text{if } c_i \text{ is a positive crossings,}\\ \delta(u_i,x_j)+\delta(v_i,x_j)f_1(a_i,b_i)^{-1}f_2(a_i,b_i)-\delta(w_i,x_j)f_1(a_i,b_i)^{-1} & \\ & \text{if } c_i \text{ is a negative crossings,}\\ \delta(x_i,x_j)-\delta(y_i,x_j) & \text{if } c_i \text{ is a bivalent or trivalent vertex,}\\ \delta(x_i,x_j) & \text{if } c_i \text{ is a monovalent vertex,}\end{cases}$$

where $a_i=\rho(u_i)$, $b_i=\rho(v_i)$.

Lemma 7.1. *Let R be a commutative ring. Let $a_1,\dots,a_l\in R$, $A^{(1)},\dots,A^{(l)}\in M(n,n;R)$, $B,C\in M(m,n;R)$, $D\in M(m,m-n;R)$. If*

$$\sum_{k=1}^{l}\alpha_k=0 \qquad \textit{and} \qquad \sum_{k=1}^{l}\alpha_k\left|A^{(k)}_{(i_1,\dots,i_d),(j_1,\dots,j_d)}\right|=0$$

for any $d\in\{1,\dots,n\}$ and any $i_1,\dots,i_d,j_1,\dots,j_d\in\{1,\dots,n\}$ such that $1\le i_1<\dots<i_d\le n$ and $1\le j_1<\dots<j_d\le n$, we have

$$\sum_{k=1}^{l}\alpha_k\begin{vmatrix}E_n & -A^{(k)} & O\\ B & C & D\end{vmatrix}=0,$$

where E_n is an identity matrix of size n and O is a zero matrix.

We note that $\sum_{k=1}^{l}\alpha_kA^{(k)}=O$ if and only if $\sum_{k=1}^{l}\alpha_k\left|A^{(k)}_{(i_1),(j_1)}\right|=0$ $(1\le i_1,j_1\le n)$, since $\left|A^{(k)}_{(i_1),(j_1)}\right|$ coincides with the (i_1,j_1)-entry $a^{(k)}_{i_1,j_1}$ of the matrix $A^{(k)}$.

Proof. We have

$$\begin{aligned}\sum_{k=1}^{l}\alpha_k\begin{vmatrix}E_n & -A^{(k)} & O\\ B & C & D\end{vmatrix}&=\sum_{k=1}^{l}\alpha_k\begin{vmatrix}E_n & -A^{(k)} & O\\ O & C+BA^{(k)} & D\end{vmatrix}\\ &=\sum_{k=1}^{l}\alpha_k\left|C+BA^{(k)}\quad D\right|.\end{aligned}$$

Setting $\begin{pmatrix}\boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n\end{pmatrix} := C$ and $\begin{pmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n\end{pmatrix} := B$, we have

$$\begin{aligned}
&\sum_{k=1}^{l}\alpha_k \left| \begin{pmatrix}\boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n\end{pmatrix} + \begin{pmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n\end{pmatrix} \begin{pmatrix} a_{11}^{(k)} & \cdots & a_{1n}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{(k)} & \cdots & a_{nn}^{(k)} \end{pmatrix} \quad D \right| \\
&= \sum_{k=1}^{l}\alpha_k \left| \boldsymbol{c}_1 + \textstyle\sum_{i=1}^{n} \boldsymbol{b}_i a_{i1}^{(k)} \quad \cdots \quad \boldsymbol{c}_n + \textstyle\sum_{i=1}^{n} \boldsymbol{b}_i a_{in}^{(k)} \quad D \right| \\
&= \sum_{k=1}^{l}\alpha_k \sum_{d=0}^{n} \sum_{1\le j_1<\cdots<j_d\le n} (-1)^{j_1+\cdots+j_d-\frac{d(d+1)}{2}} \\
&\qquad\qquad \cdot \left| \textstyle\sum_{i=1}^{n} \boldsymbol{b}_i a_{ij_1}^{(k)} \quad \cdots \quad \textstyle\sum_{i=1}^{n} \boldsymbol{b}_i a_{ij_d}^{(k)} \quad C_{\widehat{j_1,\ldots,j_d}} \quad D \right|,
\end{aligned}$$

where $C_{\widehat{j_1,\ldots,j_d}}$ is the submatrix of C obtained by removing $j_1,\ldots,j_d$-th column vectors of C. We then have

$$\sum_{k=1}^{l}\alpha_k \begin{vmatrix} E_n & -A^{(k)} & O \\ B & C & D \end{vmatrix} = 0$$

from the equalities

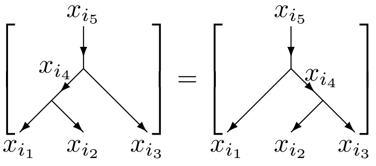
$$\begin{aligned}
&\sum_{k=1}^{l}\alpha_k \left| \textstyle\sum_{i=1}^{n} \boldsymbol{b}_i a_{ij_1}^{(k)} \quad \cdots \quad \textstyle\sum_{i=1}^{n} \boldsymbol{b}_i a_{ij_d}^{(k)} \quad C_{\widehat{j_1,\ldots,j_d}} \quad D \right| \\
&= \sum_{1\le i_1<\cdots<i_d\le n} \sum_{k=1}^{l}\alpha_k \left| A^{(k)}_{(i_1,\ldots,i_d),(j_1,\ldots,j_d)} \right| \left| \boldsymbol{b}_{i_1} \quad \cdots \quad \boldsymbol{b}_{i_d} \quad C_{\widehat{j_1,\ldots,j_d}} \quad D \right| = 0,
\end{aligned}$$

where we note that $\left| A^{(k)}_{(i_1,\ldots,i_d),(j_1,\ldots,j_d)} \right| = 1$ if $d = 0$. □

Proposition 7.2. *Let Q be a quandle, and let R be a commutative ring. Let (f_1, f_2) be an Alexander pair of $f_1, f_2 : Q \times Q \to R$ satisfying $f_1(a,b) + f_2(a,b) = 1$ for any $a, b \in Q$. Let D be a diagram of an oriented $1,2,3$-valent (n,n)-tangle T, and let $\rho : \mathcal{A}(D) \to Q$ be a map. Suppose that every component of D has an undercrossing or a vertex. Let $\langle\langle(D,\rho)\rangle\rangle$ be the bracket polynomial defined in Definition 4.1. Then we have $\langle\langle(D,\rho)\rangle\rangle = |A^\bullet(D,\rho;f_1,f_2)|$.*

Proof. We set $[(D,\rho)] := |A^\bullet(D,\rho;f_1,f_2)|$.

We have the local relation

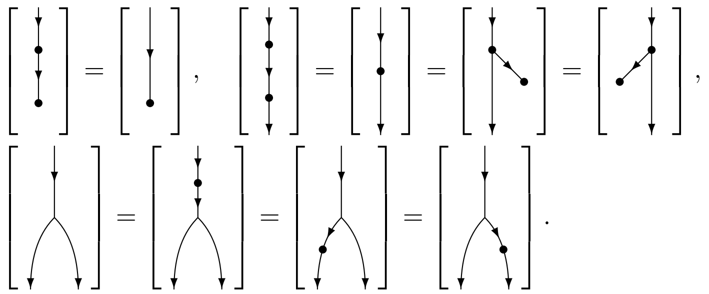


from

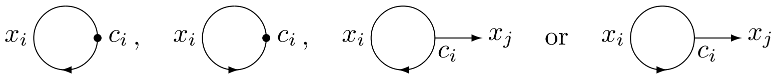
$$\begin{array}{c|ccccc|} & i_1 & i_2 & i_3 & i_4 & i_5 \\ i_1 & 1 & 0 & 0 & -1 & 0 \\ i_2 & 0 & 1 & 0 & -1 & 0 \\ i_3 & 0 & 0 & 1 & 0 & -1 \\ i_4 & 0 & 0 & 0 & 1 & -1 \end{array} = \begin{array}{c|ccccc|} & i_1 & i_2 & i_3 & i_4 & i_5 \\ i_1 & 1 & 0 & 0 & 0 & -1 \\ i_2 & 0 & 1 & 0 & -1 & 0 \\ i_3 & 0 & 0 & 1 & 0 & -1 \\ i_4 & 0 & 0 & 0 & 1 & -1 \end{array} = \begin{array}{c|ccccc|} & i_1 & i_2 & i_3 & i_4 & i_5 \\ i_1 & 1 & 0 & 0 & 0 & -1 \\ i_2 & 0 & 1 & 0 & -1 & 0 \\ i_3 & 0 & 0 & 1 & -1 & 0 \\ i_4 & 0 & 0 & 0 & 1 & -1 \end{array},$$

where we omit common rows and columns. The index i_j $(j = 1,\ldots,5)$ above the determinant indicates the i_j-th column of the whole matrix and the index i_j

$(j = 1, \dots, 4)$ to the left of the determinant indicates the i_j-th row of the whole matrix. In a similar manner, we have



Let D be an oriented (n, n)-tangle diagram without crossings. If D contains



locally, we have $[(D, \rho)] = 0$, since the i-th row of $A^\bullet(D, \rho; f_1, f_2)$ is the zero vector.

(i) Suppose that D is a diagram of a cyclic tangle. Since D can be transformed into a diagram containing a loop by using the above local relations, we have $[(D, \rho)] = 0$.

(ii) Suppose that D is a diagram of an acyclic tangle. By using the above local relations, D can be transformed into a collection of simple edges, each of which consists of one edge and two monovalent vertices. We then have

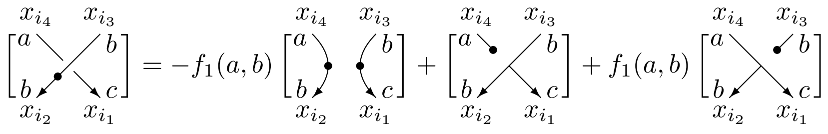
$$[(D, \rho)] = |A^\bullet(D, \rho; f_1, f_2)| = |E| = 1,$$

where E is an identity matrix.

Therefore, for an oriented (n, n)-tangle diagram D without crossings, we have

$$[(D, \rho)] = \begin{cases} 1 & \text{if } D \text{ is a diagram of an acyclic tangle,} \\ 0 & \text{if } D \text{ is a diagram of a cyclic tangle.} \end{cases}$$

The equality



follows from

$$\begin{array}{c|cccc|} & i_1 & i_2 & i_3 & i_4 \\ i_1 & 1 & 0 & -1 + f_1(a, b) & -f_1(a, b) \\ i_2 & 0 & 1 & -1 & 0 \end{array}$$

$$= -f_1(a, b) \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{vmatrix} + f_1(a, b) \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{vmatrix},$$

where we omit common rows and columns. By Lemma 7.1, we have this equality from

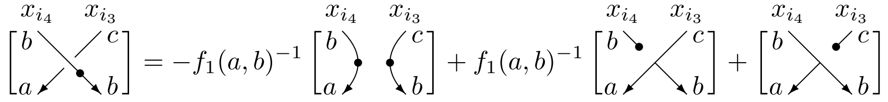
$$\begin{pmatrix} 1 - f_1(a, b) & f_1(a, b) \\ 1 & 0 \end{pmatrix} = -f_1(a, b) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + f_1(a, b) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$1 = -f_1(a, b) + 1 + f_1(a, b),$$
$$-f_1(a, b) = -f_1(a, b) \cdot 1 + 1 \cdot 0 + f_1(a, b) \cdot 0.$$

FIGURE 8. $D \to D_\bullet$

In a similar manner, we have the equality



from

$$\begin{array}{c|cccc|} & i_1 & i_2 & i_3 & i_4 \\ i_1 & 1 & 0 & 0 & -1 \\ i_2 & 0 & 1 & -f_1(a,b)^{-1} & -1+f_1(a,b)^{-1} \end{array}$$

$$= -f_1(a,b)^{-1} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} + f_1(a,b)^{-1} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{vmatrix}.$$

Since $[(D,\rho)]$ satisfies the defining relations of $\langle (D,\rho)\rangle$, we have $\langle (D,\rho)\rangle = |A^\bullet(D,\rho;f_1,f_2)|$. □

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link, and let $\rho : Q(L) \to Q$ be a quandle representation. Let D be a diagram of L such that every component has an undercrossing. We set

$$\widetilde{A^\bullet}(D,\rho;f_1,f_2) := \begin{pmatrix} A^\bullet(D,\rho;f_1,f_2) & \mathbf{0} \\ \mathbf{0} & \left(\prod_{i=1}^r \omega_i^{\frac{\mathrm{rot}(D(K_i))+\mathrm{wr}(D(K_i))+1}{2}}\right)^{-1} \end{pmatrix},$$

$$\widetilde{R^\bullet_{\mathrm{row}}}(D,\rho;\boldsymbol{f_{\mathrm{row}}}) := \begin{pmatrix} R^\bullet_{\mathrm{row}}(D,\rho;\boldsymbol{f_{\mathrm{row}}}) & \mathbf{0} \end{pmatrix},$$

where

$$\begin{aligned} &R^\bullet_{\mathrm{row}}(D,\rho;\boldsymbol{f_{\mathrm{row}}}) \\ &:= \begin{pmatrix} -f_{\mathrm{row},1}(\widetilde{\rho}(r(x_1)),\rho(x_1)) & \cdots & -f_{\mathrm{row},1}(\widetilde{\rho}(r(x_n)),\rho(x_n)) \\ \vdots & \ddots & \vdots \\ -f_{\mathrm{row},m}(\widetilde{\rho}(r(x_1)),\rho(x_1)) & \cdots & -f_{\mathrm{row},m}(\widetilde{\rho}(r(x_n)),\rho(x_n)) \end{pmatrix}. \end{aligned}$$

Let $D_\bullet$ the diagram obtained from D by adding a bivalent vertex at every crossing as shown in Figure 8. As we see in [7], we have

$$\begin{aligned} &(\widetilde{R_{\mathrm{row}}}(D,\rho;\boldsymbol{f_{\mathrm{row}}}), \widetilde{A}(D,\rho;f_1,f_2), \widetilde{R_{\mathrm{col}}}(D,\rho;\boldsymbol{f_{\mathrm{col}}})) \\ &\sim (\widetilde{R^\bullet_{\mathrm{row}}}(D,\rho;\boldsymbol{f_{\mathrm{row}}}), \widetilde{A^\bullet}(D,\rho;f_1,f_2), \widetilde{R_{\mathrm{col}}}(D,\rho;\boldsymbol{f_{\mathrm{col}}})) \\ &\sim (\widetilde{R^\bullet_{\mathrm{row}}}(D_\bullet,\rho;\boldsymbol{f_{\mathrm{row}}}), \widetilde{A^\bullet}(D_\bullet,\rho;f_1,f_2), \widetilde{R_{\mathrm{col}}}(D_\bullet,\rho;\boldsymbol{f_{\mathrm{col}}})). \end{aligned} \tag{7}$$

Here, we remark that, by using derivatives introduced in [5], we obtain the matrices $\widetilde{A}(D,\rho;f_1,f_2)$, $\widetilde{A^\bullet}(D,\rho;f_1,f_2)$, $\widetilde{A^\bullet}(D_\bullet,\rho;f_1,f_2)$ from the presentations

$$\begin{aligned} &\langle x_1,\dots,x_n \,|\, u_1 \lhd v_1 = w_1,\dots,u_n \lhd v_n = w_n\rangle, \\ &\langle x_1,\dots,x_n \,|\, x_1 = y_1 \lhd^{\mathrm{sgn}(c_1)} v_1,\dots,x_n = y_n \lhd^{\mathrm{sgn}(c_n)} v_n\rangle, \\ &\langle x_1,\dots,x_{2n} \,|\, x_1 = y_1 \lhd^{\mathrm{sgn}(c_1)} v_1,\dots,x_{2n} = y_{2n} \lhd^{\mathrm{sgn}(c_{2n})} v_{2n}\rangle \end{aligned}$$

of the fundamental quandle $Q(L)$, where $\mathrm{sgn}(c) := 0$ for a bivalent vertex c and $a \lhd^0 b := a$ even if b does not exist.

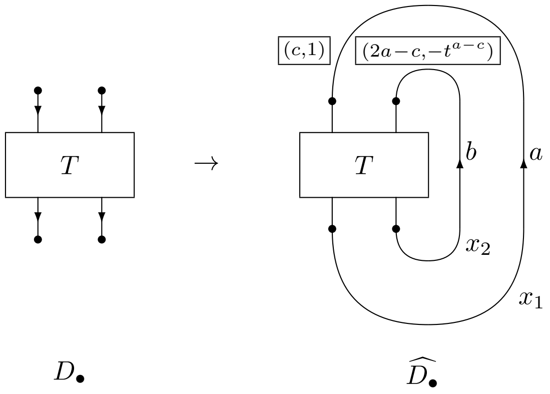


FIGURE 9. $\widehat{D_\bullet}$

8. THE PROOF OF THEOREM 1.6

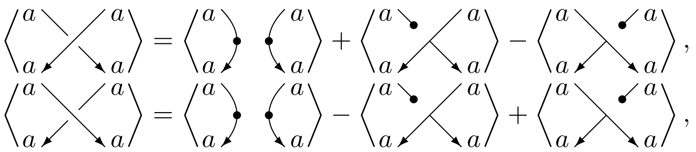
Let p be an odd prime number, and let $F := \mathbb{Q}(\sqrt{-1})[t]/(t^{p-1}+\cdots+1)$, which is isomorphic to a cyclotomic field obtained by adjoining a primitive $4p$th root of unity to $\mathbb{Q}$. We set $a_1 := 0$, $a_2 := 1 \in F$. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : R_p \times R_p \to F$ defined by $f_1(a,b) = -t^{b-a}$ and $f_2(a,b) = t^{b-a}+1$. Let $Y = R_p \times F^\times$ be the R_p-set defined with $(y,z) \triangleleft a = (2a-y, -t^{a-y}z)$. We then have the column relation map $f_{\mathrm{col},1}, f_{\mathrm{col},2} : R_p \to F$ defined by $f_{\mathrm{col},i}(x) = t^{x-a_i}+1$ and the row relation map $f_{\mathrm{row},1}, f_{\mathrm{row},2} : \mathrm{As}\,Q \times Q \to F$ defined by $f_{\mathrm{row},i}(y,x) = f_{\mathrm{row}}(\varphi_{a_i}(y), x)$, where $f_{\mathrm{row}} : Y \times Q \to F$ is the row relation map defined by $f_{\mathrm{row}}((y,z),x) = -z^{-1}(t^{y-x}+1)$ and $\varphi_c : \mathrm{As}\,Q \to Y$ is the Q-set homomorphism satisfying $\varphi_c(1) = (c,1)$. Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link, and let $\rho : Q(L) \to R_p$ be a quandle representation.

Let D be a diagram of an oriented classical (n,n)-tangle T whose closure is L. By Proposition 4.2,

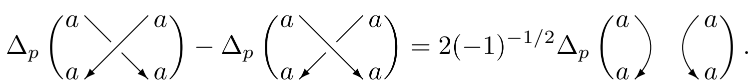
$$(-1)^{-\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}} \langle (D,\rho) \rangle$$

is invariant under the colored Reidemeister moves.

From the local relations

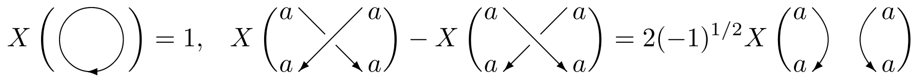


we have



(8)

Suppose ρ is trivial. Setting $X(L,\rho) := (-1)^{r-1}\Delta_p(L,\rho)$, we have



from (8). Hence we have $X(L,\rho) = \Delta_L(-1)$, which implies $\Delta_p(L,\rho) = (-1)^{r-1}\Delta_L(-1)$.

Suppose ρ is nontrivial. Let D be a diagram of an oriented classical $(2,2)$-tangle T whose closure is L such that the images of ρ on the top endpoints of D are distinct elements $a, b \in R_p$. Suppose that every component of D has an undercrossing. Let $D_\bullet$ be the diagram obtained from D by adding a bivalent vertex at every crossing as shown in Figure 8. We denote by $\widehat{D_\bullet}$ the diagram depicted in Figure 9. We note that the diagram obtained from $\widehat{D_\bullet}$ by removing all bivalent vertices represents L. By (7), we have

$$\Delta(L,\rho;f_1,f_2;0,1)=\frac{\det\widetilde{A^\bullet}(\widehat{D_\bullet},\rho;f_1,f_2)_{\hat{2},\hat{2}}}{\det\widetilde{R^\bullet_{\mathrm{row}}}(\widehat{D_\bullet},\rho;\boldsymbol{f}_{\mathbf{row}})_{\bar{2},\bar{2}}\det\widetilde{R^\bullet_{\mathrm{col}}}(\widehat{D_\bullet},\rho;\boldsymbol{f}_{\mathbf{col}})_{\bar{2},\bar{2}}},$$

where

$$\begin{aligned}\widetilde{A^\bullet}(\widehat{D_\bullet},\rho;f_1,f_2)_{\hat{2},\hat{2}}&=\begin{pmatrix}A^\bullet(D_\bullet,\rho;f_1,f_2)&\mathbf{0}\\\mathbf{0}&\prod_{i=1}^{r}(-1)^{-\frac{\mathrm{rot}(\widehat{D_\bullet}(K_i))+\mathrm{wr}(\widehat{D_\bullet}(K_i))+1}{2}}\end{pmatrix},\\\widetilde{R^\bullet_{\mathrm{row}}}(\widehat{D_\bullet},\rho;\boldsymbol{f}_{\mathbf{row}})_{\bar{2},\bar{2}}&=\begin{pmatrix}t^{-a}+1&-t^{a-b}-t^{-a}\\t^{1-a}+1&-t^{a-b}-t^{1-a}\end{pmatrix},\\\widetilde{R^\bullet_{\mathrm{col}}}(\widehat{D_\bullet},\rho;\boldsymbol{f}_{\mathbf{col}})_{\bar{2},\bar{2}}&=\begin{pmatrix}t^{a}+1&t^{a-1}+1\\t^{b}+1&t^{b-1}+1\end{pmatrix}.\end{aligned}$$

By Proposition 7.2, we have

$$\det A^\bullet(D_\bullet,\rho;f_1,f_2)=\langle(D_\bullet,\rho)\rangle=\langle(D,\rho)\rangle.$$

Since

$$\begin{aligned}\sum_{i=1}^{r}\mathrm{rot}(\widehat{D_\bullet}(K_i))&=\mathrm{rot}(\widehat{D_\bullet})=\mathrm{rot}(D)+2,\\\sum_{i=1}^{r}\mathrm{wr}(\widehat{D_\bullet}(K_i))&=\mathrm{wr}(\widehat{D_\bullet})-2\,\mathrm{lk}(L)=\mathrm{wr}(D)-2\,\mathrm{lk}(L),\end{aligned}$$

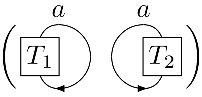
we have

$$\begin{aligned}\Delta(L,\rho;f_1,f_2;0,1)&=\frac{(-1)^{-\frac{\mathrm{rot}(D)+2+\mathrm{wr}(D)-2\,\mathrm{lk}(L)+r}{2}}\langle(D,\rho)\rangle}{(1-t)(t^{-a}-t^{-b})(1-t^{-1})(t^{a}-t^{b})}\\&=\frac{\Delta_p(L,\rho)}{(-1)^{r/2+\mathrm{lk}(L)}(t-2+t^{-1})}.\end{aligned}$$

9. The R_p-twisted Alexander invariant

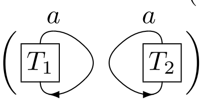
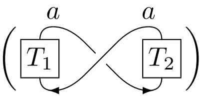
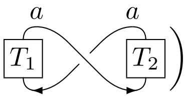
In this section, we focus on the R_p-twisted Alexander invariant and show its properties.

Lemma 9.1. *For any oriented classical* $(1,1)$*-tangles* T_1,T_2*, we have*

$\Delta_p\Bigg($  $\Bigg)=0$

for $a\in R_p$.

Proof. From the skein relation (8), we have

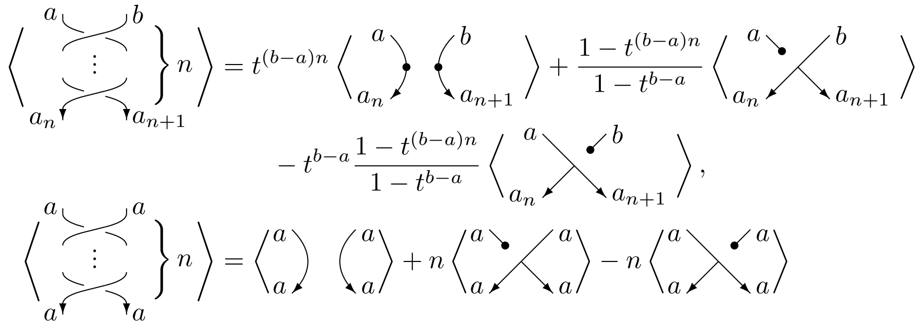
$2(-1)^{-1/2}\Delta_p\Bigg($  $\Bigg)=\Delta_p\Bigg($  $\Bigg)-\Delta_p\Bigg($  $\Bigg)$

$=0.$

□

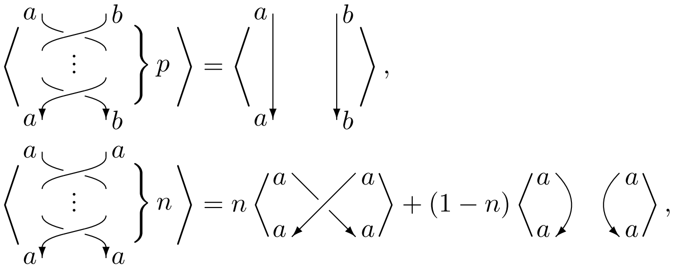
By induction, we have the following lemma.

Lemma 9.2. *We have*



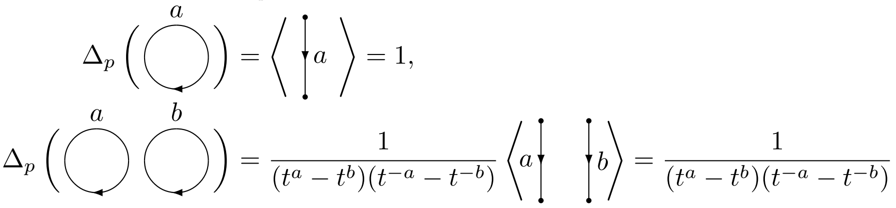
for $n \in \mathbb{Z}$ *and any distinct elements* $a, b \in R_p$*, where* $a_n = nb - (n-1)a$.

Proof of Proposition 1.7. By Lemma 9.2, we have



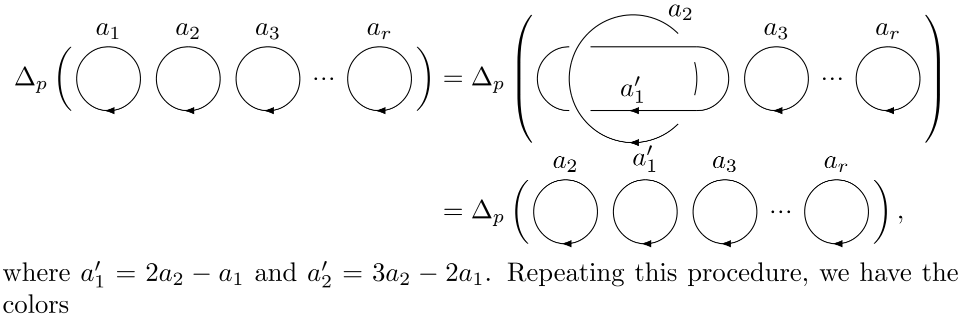
which imply the skein relations.

By the definition of $\Delta_p(L, \rho)$, we have



for any distinct elements $a, b \in R_p$. By Lemma 9.1, we have

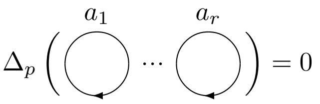
for any $r \geq 3$ and $a, a_1 \ldots, a_r \in R_p$ such that $a_1 = a_2$. Suppose $a_1 \neq a_2$. We have



where $a_1' = 2a_2 - a_1$ and $a_2' = 3a_2 - 2a_1$. Repeating this procedure, we have the colors

$$a_1, a_2, 2a_2 - a_1, 3a_2 - 2a_1, \ldots, (p-1)a_2 - (p-2)a_1.$$

Since these elements are mutually distinct, one of them coincides with a_3. Hence, we have



for any $r \geq 3$ and $a_1, \ldots, a_r \in R_p$. $\square$

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