SKEIN RELATIONS OF QUANDLE COCYCLE INVARIANTS FOR HANDLEBODY-LINKS AND UNORIENTED LINKS

ATSUSHI ISHII, KENGO KAWAMURA AND KANAKO OSHIRO

Abstract. We establish skein relations for quandle cocycle invariants of $\mathbb{Z}_2$-flowed handlebody-links, a class that includes unoriented links as a special case. For an odd prime p, we study the cocycle invariant $\Phi_p(D, C)$ associated with the dihedral quandle R_p and Mochizuki's 3-cocycle, and derive skein relations for local splicings, crossing changes, and the p-move.

For $p = 3$, we prove that these relations characterize Φ_3 under the assumption that the invariant vanishes on $\mathbb{Z}_2$-flowed trivial handlebody-links. As a consequence, Φ_3 can be computed recursively. We also obtain the corresponding characterization for unoriented links.

As an application, we derive lower bounds for $4kp$-move distances of handlebody-links and for kp-move distances of unoriented links in terms of cocycle invariants, and we show that these bounds are sharp for an infinite family of examples.

1. Introduction

Skein relations are one of the basic tools in knot theory. They provide simple definitions of invariants and often lead to effective methods for calculation. For classical links, fundamental invariants such as the Alexander polynomial and the Jones polynomial admit natural skein-theoretic descriptions. The purpose of this paper is to develop an analogous skein theory for quandle cocycle invariants in the setting of $\mathbb{Z}_2$-flowed handlebody-links and unoriented links.

A handlebody-link is a disjoint union of handlebodies embedded in S^3. A $\mathbb{Z}_2$-flow on a handlebody-link is an element of its first homology group with $\mathbb{Z}_2$-coefficients, and can be represented diagrammatically by a $\mathbb{Z}_2$-labeling of edges satisfying a natural condition at each trivalent vertex. When all components have genus 1, a handlebody-link reduces to an unoriented link equipped with the constant $\mathbb{Z}_2$-flow 1. Thus, $\mathbb{Z}_2$-flowed handlebody-links provide a natural framework extending unoriented links.

For an odd prime p, we consider the cocycle invariant $\Phi_p(D, C)$ associated with the dihedral quandle R_p and Mochizuki's 3-cocycle. Our first result establishes skein relations for $\Phi_p(D, C)$ under local splicings, crossing changes, and the p-move; see Proposition 4.1. Our second result shows that, when $p = 3$, these relations determine Φ_3 under the assumption that the invariant vanishes on $\mathbb{Z}_2$-flowed trivial handlebody-links; see Theorem 5.5. As a consequence, Φ_3 admits a recursive computation. We also obtain the corresponding characterization for unoriented links; see Theorem 5.6.

Our third result applies these skein relations to the n-move distances. For handlebody-links, we obtain the lower bound

$$d_{4kp}(H, H') \geq \widehat{d}_{S_{4k,p}}(\Phi_p(H), \Phi_p(H'));$$

2020 *Mathematics Subject Classification.* 57K12, 57K10.

Key words and phrases. Skein relation, quandle cocycle invariant, n-move, handlebody-link, link.

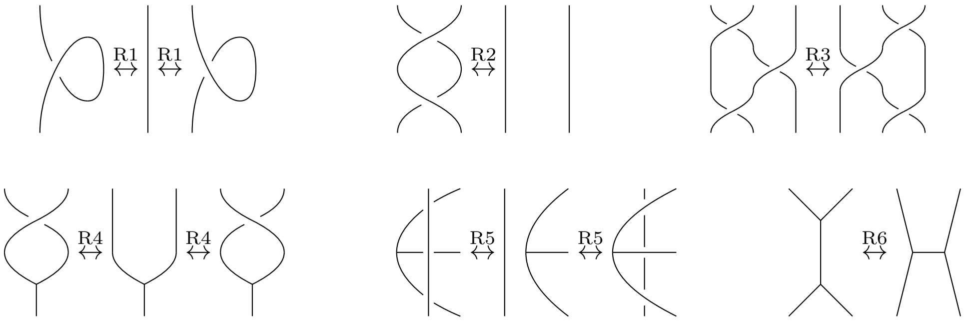


FIGURE 1. The Reidemeister moves for handlebody-links

see Theorem 6.2. For unoriented links, we similarly prove

$$d_{kp}(L, L') \geq \widehat{d}_{S_{k,p}}(\Phi_p(L), \Phi_p(L'));$$

see Theorem 6.5. Propositions 6.3 and 6.6 show that these bounds are sharp for an infinite family of examples.

The paper is organized as follows. In Section 2, we review handlebody-links and $\mathbb{Z}_2$-flows. In Section 3, we introduce R_p-colorings and define the cocycle invariant $\Phi_p(D, C)$ by using Mochizuki's 3-cocycle. In Section 4, we establish skein relations for $\Phi_p(D, C)$. In Section 5, we prove that, for $p = 3$, these relations characterize Φ_3 and yield a recursive method of computation. In Section 6, we study n-moves and derive lower bounds for n-move distances in terms of Φ_p.

2. HANDLEBODY-LINKS AND $\mathbb{Z}_2$-FLOWS

A *handlebody-knot* is an embedding of a handlebody of genus $g \geq 1$ into the 3-sphere S^3. When $g = 1$, this notion coincides with that of a classical knot. A *handlebody-link* is a disjoint union of finitely many handlebodies embedded in S^3. In this paper, we assume that the genera of the handlebodies are positive integers. A handlebody-knot is a handlebody-link with one component. Two handlebody-links are said to be *equivalent* if there exists an orientation-preserving homeomorphism of S^3 that sends one to the other. Any handlebody-link can be obtained as a regular neighborhood of a spatial trivalent graph (possibly with circular components). A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the given handlebody-link. Two diagrams represent equivalent handlebody-links if and only if they are related by a finite sequence of the moves shown in Figure 1.

Let D be a diagram of a handlebody-link H. We denote by $C(D)$ and $V(D)$ the sets of crossings and trivalent vertices of D, respectively. We denote by $\mathcal{A}(D)$ the set of arcs of D, whose endpoints are under-crossings or trivalent vertices.

For an ordinary link L, an orientation of L corresponds to a choice of generator of the first homology group $H_1(L; \mathbb{Z})$. In contrast, for a handlebody-link H, the group $H_1(H; \mathbb{Z})$ is finitely generated but there is, in general, no canonical way to select a finite system of generators of $H_1(H; \mathbb{Z})$. As a natural extension of the notion of an orientation of L, we consider elements of $H_1(H; \mathbb{Z})$, which we call $\mathbb{Z}$-flows. In this paper, we work with elements of $H_1(H; \mathbb{Z}_2)$, which we call $\mathbb{Z}_2$-flows. A $\mathbb{Z}_2$-flow φ of a handlebody-link H can be represented on a diagram of H as an assignment of an element of $\mathbb{Z}_2$ to each edge such that, at every trivalent vertex, the sum of the values on the incident edges is 0 in $\mathbb{Z}_2$. This assignment induces a map $\varphi \colon \mathcal{A}(D) \to \mathbb{Z}_2$ by sending each arc to the value of the edge containing it, and we denote this induced map again by φ. When $s \in \mathbb{Z}_2$ is assigned to an arc by the $\mathbb{Z}_2$-flow φ, we indicate it on the diagram by $\underline{s}$. We denote by $\mathrm{Flow}(H; \mathbb{Z}_2)$ the set of $\mathbb{Z}_2$-flows of H.

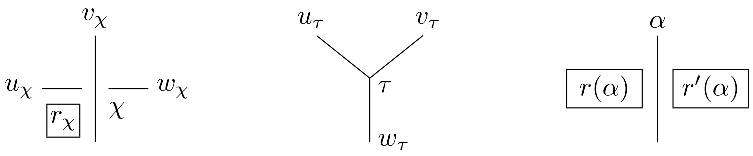


FIGURE 2. Labels at crossings, vertices and regions

We call a pair (H, φ) consisting of a handlebody-link H and a $\mathbb{Z}_2$-flow φ of H a *$\mathbb{Z}_2$-flowed handlebody-link*, and we often omit φ and simply write H. An unoriented link L corresponds to the pair consisting of L and the $\mathbb{Z}_2$-flow that assigns 1 to every circular component. Throughout this paper, we equip every unoriented link L with this $\mathbb{Z}_2$-flow and regard L as a $\mathbb{Z}_2$-flowed handlebody-link.

3. COLORINGS FOR $\mathbb{Z}_2$-FLOWED HANDLEBODY-LINKS

In this section, and throughout the rest of the paper, p denotes an odd prime. We introduce R_p-colorings and shadow R_p-colorings for $\mathbb{Z}_2$-flowed handlebody-links. These colorings generalize the usual dihedral quandle colorings of unoriented links and are compatible with the Reidemeister moves for handlebody-link diagrams. Using Mochizuki's 3-cocycle, we then associate a weight to each shadow R_p-colored diagram and define a quandle cocycle invariant. We finally show that this invariant is well-defined and does not depend on the choice of region coloring. For further details, see [7].

A *quandle* is a set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

- For any $a \in Q$, $a \triangleleft a = a$.
- For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
- For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. An *involutory quandle* is a quandle Q satisfying $a \triangleleft b = a \triangleleft^{-1} b$ for all $a, b \in Q$, and is also referred to as a *kei*. Put $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$. The *dihedral quandle* R_p is the kei on $\mathbb{Z}_p$ with $a \triangleleft b = 2b - a$.

Let H be a $\mathbb{Z}_2$-flowed handlebody-link with flow φ, and let D be a diagram of H. An *R_p-coloring* of D is a map $C : \mathcal{A}(D) \to R_p$ satisfying

$$C(u_\chi) \triangleleft^{\varphi(v_\chi)} C(v_\chi) = C(w_\chi) \qquad \text{and} \qquad C(u_\tau) = C(v_\tau) = C(w_\tau)$$

for every crossing $\chi \in C(D)$ and every trivalent vertex $\tau \in V(D)$, where v_χ is the over-arc of χ and u_χ, w_χ are the under-arcs of χ as shown on the left of Figure 2, and u_τ, v_τ and w_τ are the arcs incident to τ as shown in the middle of Figure 2. Note that this definition does not depend on the choice of u_χ and w_χ, because $a \triangleleft b = a \triangleleft^{-1} b$ for all $a, b \in R_p$, and also does not depend on the choice of labeling the arcs incident to a trivalent vertex. We denote by $\mathrm{Col}_p(D)$ the set of R_p-colorings of D.

Recall that, in this paper, we regard every unoriented link as a $\mathbb{Z}_2$-flowed handlebody-link by equipping it with the $\mathbb{Z}_2$-flow that assigns 1 to each circular component. Under this identification, the above definition agrees with the usual definition of an R_p-coloring for unoriented link diagrams.

Let H be a $\mathbb{Z}_2$-flowed handlebody-link with flow φ, and let D be a diagram of H. We denote by $\mathcal{R}(D)$ the set of complementary regions of D. We denote by $\mathcal{SA}(D)$ the set of semi-arcs of D, where a semi-arc is a piece of an arc obtained by cutting the diagram at all crossings. When a color is assigned to an arc, each semi-arc inherits the color originating from that arc. A *shadow R_p-coloring* of D is an extension $\widetilde{C} : \mathcal{A}(D) \cup \mathcal{R}(D) \to R_p$ of an R_p-coloring $C : \mathcal{A}(D) \to R_p$ satisfying

$$\widetilde{C}(r(\alpha)) \lhd^{\varphi(\alpha)} C(\alpha) = \widetilde{C}(r'(\alpha))$$

for every semi-arc α, where $r(\alpha)$ and $r'(\alpha)$ are the regions facing a semi-arc α as shown on the right of Figure 2. Again, this definition does not depend on the choice of $r(\alpha)$ and $r'(\alpha)$, because $a \lhd b = a \lhd^{-1} b$ for all $a, b \in R_p$. We denote by $\widetilde{\mathrm{Col}}_p(D)$ the set of shadow R_p-colorings of D.

Let D and D' be diagrams of $\mathbb{Z}_2$-flowed handlebody-links that are related by a single Reidemeister move. For each $C \in \mathrm{Col}_p(D)$, there exists a unique R_p-coloring $C' \in \mathrm{Col}_p(D')$ that agrees with C outside the disk where the move is applied. We call such a move, together with this induced correspondence of colorings, a *colored Reidemeister move.* If (D_1, C_1) and (D_n, C_n) are related by a finite sequence of colored Reidemeister moves, then we write $(D_1, C_1) \cong (D_n, C_n)$. Similarly, for each $\widetilde{C} \in \widetilde{\mathrm{Col}}_p(D)$, there exists a unique shadow R_p-coloring $\widetilde{C}' \in \widetilde{\mathrm{Col}}_p(D')$ that agrees with $\widetilde{C}$ outside the disk. The notation $(D_1, \widetilde{C}_1) \cong (D_n, \widetilde{C}_n)$ for a finite sequence of shadow colored Reidemeister moves is defined analogously.

Let $\theta_p \colon R_p \times R_p \times R_p \to \mathbb{Z}_p$ denote the Mochizuki 3-cocycle, defined by

$$\theta_p(x, y, z) = (x - y)\frac{(2z - y)^p + y^p - 2z^p}{p}.$$

Let D be a diagram of a $\mathbb{Z}_2$-flowed handlebody-link, and let $\widetilde{C}$ be a shadow R_p-coloring of D. We define the weight $w_p(\chi; \widetilde{C}) \in \mathbb{Z}_p$ on a crossing χ by

$$w_p(\chi; \widetilde{C}) = \varphi(u_\chi)\varphi(v_\chi)\theta_p(\widetilde{C}(r_\chi), \widetilde{C}(u_\chi), \widetilde{C}(v_\chi)),$$

where r_χ is the region around the crossing χ as shown on the left of Figure 2. Note that this definition does not depend on the choice of u_χ, w_χ and r_χ, because $\theta_p(a, b, c) = \theta_p((a \lhd b) \lhd c, b \lhd c, c)$ for any $a, b, c \in R_p$. We then define

$$\Phi_p(D, \widetilde{C}) := \sum_{\chi \in C(D)} w_p(\chi; \widetilde{C}).$$

By [7], if $(D, \widetilde{C}) \cong (D', \widetilde{C}')$, then $\Phi_p(D, \widetilde{C}) = \Phi_p(D', \widetilde{C}')$. For an R_p-coloring C of D, we define

$$\Phi_p(D, C) := \Phi_p(D, \widetilde{C}),$$

where $\widetilde{C}$ is a shadow R_p-coloring such that $\widetilde{C}|_{\mathcal{A}(D)} = C$. From the following lemma, $\Phi_p(D, C)$ is well-defined, which we call the *quandle cocycle invariant* of (D, C). That is, if $(D, C) \cong (D', C')$, then $\Phi_p(D, C) = \Phi_p(D', C')$.

Lemma 3.1. *Let D be a diagram of a $\mathbb{Z}_2$-flowed handlebody-link H. For shadow R_p-colorings $\widetilde{C}_1, \widetilde{C}_2 \in \widetilde{\mathrm{Col}}_p(D)$ satisfying $\widetilde{C}_1|_{\mathcal{A}(D)} = \widetilde{C}_2|_{\mathcal{A}(D)}$, we have $\Phi_p(D, \widetilde{C}_1) = \Phi_p(D, \widetilde{C}_2)$.*

Proof. Since p is an odd prime, for any $x, y \in R_p$, there exists $a \in R_p$ such that $y = x \lhd a$. Hence we may assume that $y_2 = y_1 \lhd a$ and $y_i = \widetilde{C}_i(r)$ for $i = 1, 2$, where r is a region of D disjoint from a disk containing the entire handlebody-link diagram. Let $H \cup \bigcirc$ be the split union of H and the trivial handlebody-knot of genus 1 with flow 1. Let D_1 be a diagram of $H \cup \bigcirc$ in which the trivial component lies in the outermost region of H, and let $\widetilde{C}'_1$ be the shadow R_p-coloring of D_1 that coincides with $\widetilde{C}_1$ outside the trivial component and assigns the color a to the trivial component. Similarly, let D_2 be a diagram of $H \cup \bigcirc$ in which the trivial component surrounds H, and let $\widetilde{C}'_2$ be the shadow R_p-coloring of D_2 that coincides with $\widetilde{C}_2$ inside the trivial component and assigns the color a to the trivial component. Then $(D_2, \widetilde{C}'_2)$ can be obtained from $(D_1, \widetilde{C}'_1)$ by a finite sequence of

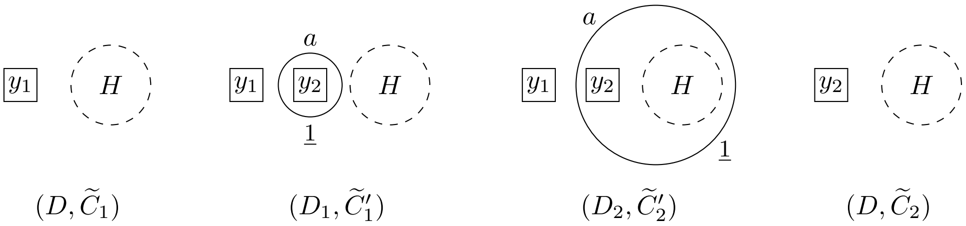


FIGURE 3. Independence from region colorings

colored Reidemeister moves, since the trivial component in D_1 can be moved under H to become that in D_2 (see Figure 3). Hence we have $\Phi_p(D_1, \widetilde{C}'_1) = \Phi_p(D_2, \widetilde{C}'_2)$. Since we have $\Phi_p(D_i, \widetilde{C}'_i) = \Phi_p(D, \widetilde{C}_i)$ for $i = 1, 2$ by the definition of Φ_p, we obtain $\Phi_p(D, \widetilde{C}_1) = \Phi_p(D, \widetilde{C}_2)$. □

Remark 3.2. For a classical link L represented by a diagram D, the usual quandle cocycle invariant with respect to the Mochizuki 3-cocycle is defined as the multiset

$$\{\Phi_p(D, \widetilde{C}) \mid \widetilde{C} \in \widetilde{\mathrm{Col}}_p(D)\},$$

where $\Phi_p(D, \widetilde{C})$ is given as in this section. In this paper, in analogy with the twisted Alexander polynomial, we study a skein relation for the quantity $\Phi_p(D, C)$, regarded as an invariant of a $\mathbb{Z}_2$-flowed handlebody-link together with a coloring, before taking the multiset over all colorings.

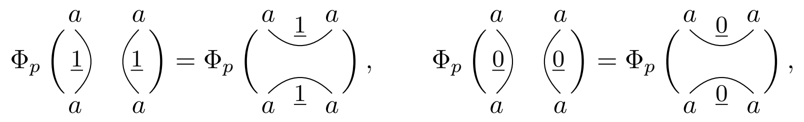
Remark 3.3. We have shown that $\Phi_p(D, \widetilde{C})$ does not depend on the choice of region colors for $\mathbb{Z}_2$-flowed handlebody-links. For classical links, the corresponding fact was proved by Satoh [17].

4. SKEIN RELATIONS FOR $\Phi_p(D, C)$

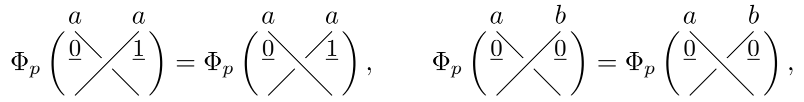
In this section, we establish skein relations for the quandle cocycle invariant $\Phi_p(D, C)$, including a relation corresponding to the p-move.

Proposition 4.1. *Let D be a diagram of a $\mathbb{Z}_2$-flowed handlebody-link H, and let C be an R_p-coloring of D. The quandle cocycle invariant $\Phi_p(D, C)$ satisfies the* *following skein relations:*

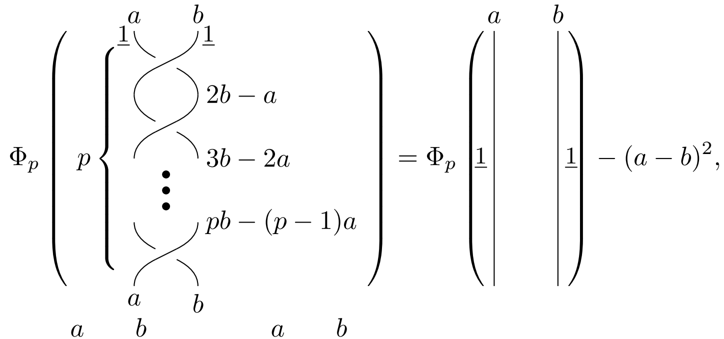
(1)



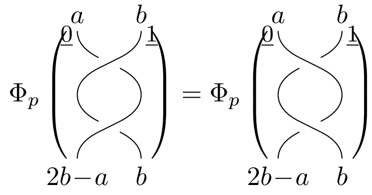
(2)



(3)

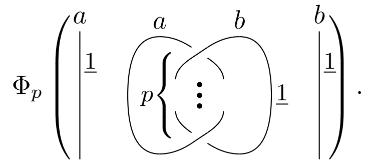


(4)



for any $a, b \in R_p$*.*

Proof. The skein relations (1), (2) and (4) follow directly from the definition of Φ_p. We now prove the skein relation (3). By using the first skein relation in (1) twice, the left-hand side of (3) is equal to

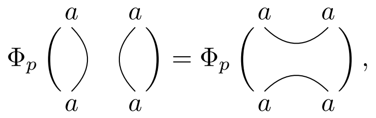


By Lemma 3.1, the value of $\Phi_p(D, C)$ is unaffected by changing the region coloring. Hence, after fixing a suitable region color, the computation reduces to the same calculation as in the proof of Theorem 6.3 of [2]. Therefore the contribution of the p-twist is $-(a-b)^2$, and (3) follows. □

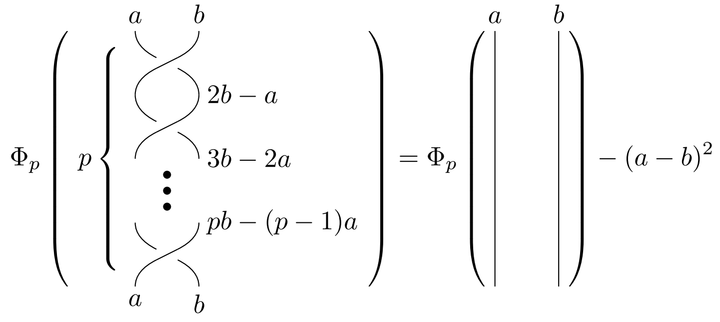
Since a handlebody-link whose components all have genus 1 may be regarded as an unoriented link equipped with the $\mathbb{Z}_2$-flow identically 1, Proposition 4.1 yields the following.

Proposition 4.2. *Let* D *be a diagram of an unoriented link* L*, and let* C *be an* R_p*-coloring of* D*. The quandle cocycle invariant* $\Phi_p(D, C)$ *satisfies the following* *skein relations:*

(5)



(6)



for any $a, b \in R_p$.

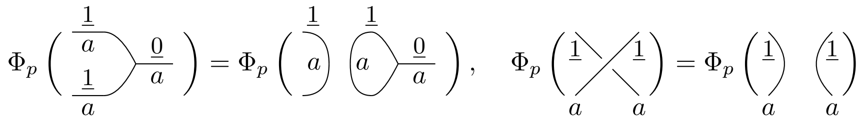
5. Recursive calculation for Φ_3 through the skein relations

In this section, we show that the skein relations obtained in the previous section completely determine the quandle cocycle invariant Φ_3. By repeatedly applying the skein relations, any diagram can be reduced to trivial ones, and this gives a recursive method to compute Φ_3. Thus Φ_3 is characterized as the unique invariant satisfying the skein relations and vanishing on $\mathbb{Z}_2$-flowed trivial handlebody-links.

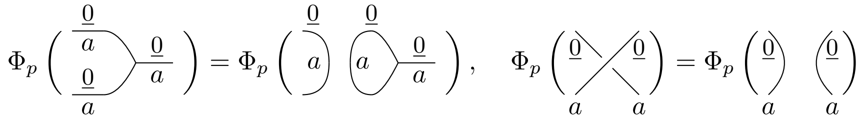
A relation is said to be a *consequence* of skein relations if it can be obtained by a finite sequence of applications of the skein relations and Reidemeister moves.

Lemma 5.1. *For any* $a \in R_p$*, the relations*

(7)



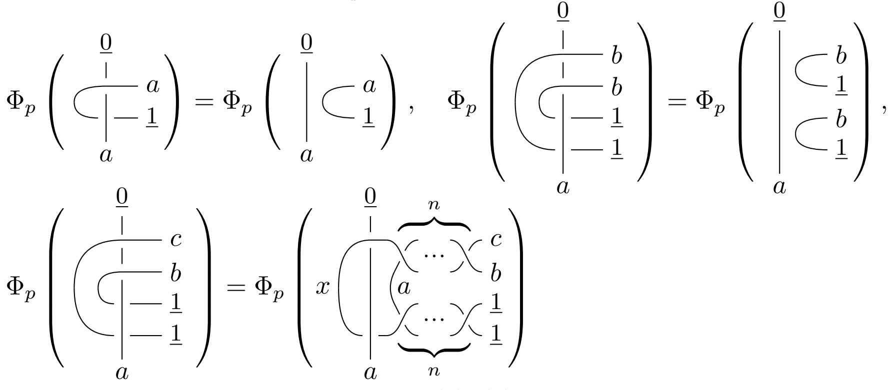
(8)



are consequences of the skein relations (1)–(3).

Proof. By using the skein relations in (1), we obtain these relations. □

Lemma 5.2. *For any* $a, b, c \in R_p$ *such that* $b \neq c$*, the relations*



are consequences of the skein relations (1)–(3)*, where* $x = a - b + c$ *and* n *is the element of* $\{0, \dots, p-1\}$ *such that* $n(b - c) = a - b$ *in* $\mathbb{Z}_p$.

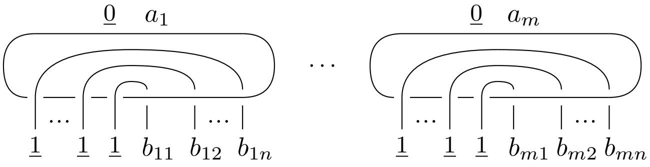
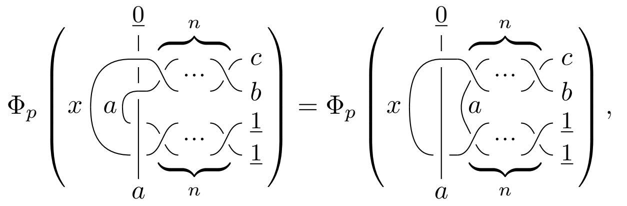


FIGURE 4. Circles with parallel strands

Proof. The first equality follows from the first skein relation in (2). The second equality follows from the first skein relation in (1). From the first skein relation in (2), we have



which implies the last equality. □

Proposition 5.3. *For any diagram D of a $\mathbb{Z}_2$-flowed handlebody-link and any R_p-coloring C of D, there exists a diagram D' and an R_p-coloring C' of D' such that the relation $\Phi_p(D,C) = \Phi_p(D',C')$ is a consequence of the skein relations* (1)–(3) *and that D' represents a split union of a trivial handlebody-link and an unoriented link consisting of circular components whose flows are* 1.

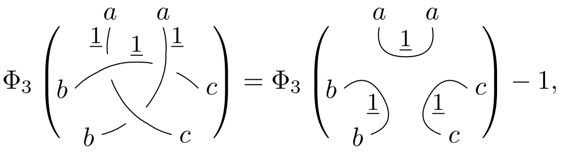
Proof. First we apply the first relation in (7) or (8) to every trivalent vertex of D. Then each trivalent vertex is contained in a trivial handlebody-knot of genus 2. By applying Reidemeister moves, we may separate these trivial handlebody-knots from the rest of D.

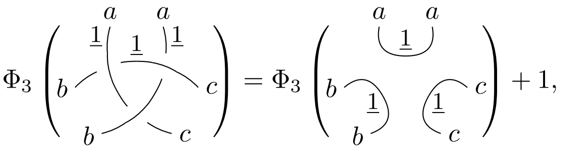
Next, by using the second skein relation in (2), the circular components with flow 0 can be transformed into a trivial link. We then isotope the diagram so that, around each trivial circular component with flow 0, the remaining strands with flow 1 run parallel to the circle, as illustrated in Figure 4.

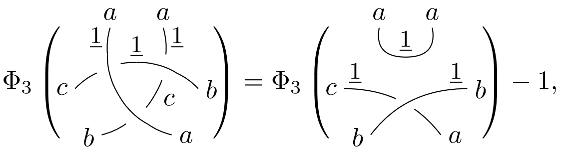
Now we apply Lemma 5.2. The relations in Lemma 5.2 enable us to remove the parallel strands one by one from the inside. Noting that, in the case $n = 1$, we have $b_{i1} = a_i$, we see that all such strands can eventually be removed.

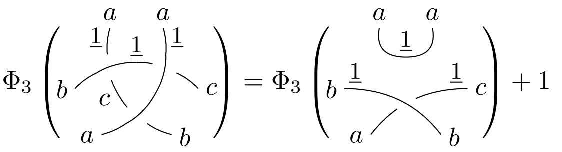
The resulting diagram D' consists of a disjoint union of trivial handlebody-knots of genus 2, trivial circular components with flow 0, and circular components with flow 1, where the trivial handlebody-knots of genus 2 and the trivial circular components with flow 0 form a trivial handlebody-link. This completes the proof. □

Lemma 5.4. *For mutually distinct elements $a, b, c \in R_3$, the relations*



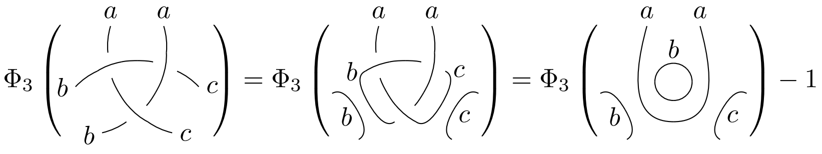


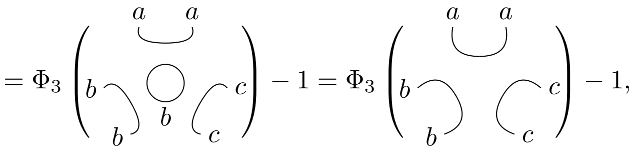




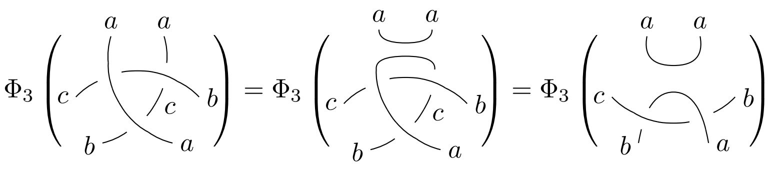
are consequences of the first skein relation in (1) *and the skein relation* (3).

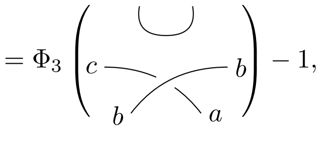
Proof. In this proof, we omit the flows that are all 1. We have





where the first and last equalities follow from the first skein relation in (1) and the second equality follows from the skein relation (3). In a similar manner, we have the second relation. We have





where the first equality follows from the first skein relation in (1) and the last equality follows from the skein relation (3). In a similar manner, we have the last relation. ☐

Theorem 5.5. *Let F be an invariant of R_3-colored diagrams of $\mathbb{Z}_2$-flowed handlebody-links with values in $\mathbb{Z}_3$. Assume that:*

1. *F satisfies the skein relations (1)–(3);*
2. *$F(D,C) = 0$ whenever D is a diagram of a $\mathbb{Z}_2$-flowed trivial handlebody-link.*

Then $F(D,C) = \Phi_3(D,C)$ for every R_3-colored diagram (D,C).

In particular, the skein relations (1)–(3) are defining skein relations of Φ_3, with the condition that Φ_3 vanishes for any $\mathbb{Z}_2$-flowed trivial handlebody-link diagram.

Proof. Let D be a diagram of a $\mathbb{Z}_2$-flowed handlebody-link, and let C be an R_3-coloring of D. By Proposition 5.3, we may assume that D is a split union of a diagram of a trivial handlebody-link and a diagram of an unoriented link consisting of circular components whose flows are 1.

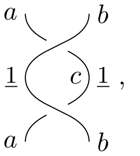
By applying the second relation in (7), we may assume that D has no crossings whose arcs have the same color. We also assume, by applying Reidemeister moves, that each non-splittable sublink of L lies in a disk in D. In what follows, we work under these two assumptions, applying the second relation in (7) and Reidemeister moves as necessary at each step.

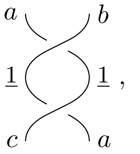
We consider the diagram corresponding to a non-splittable sublink of L. The diagram contains a digon or a triangle, since we have

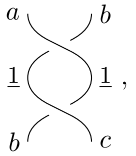
$$3n_1 + 2n_2 + n_3 = 8 + \sum_{i=1}^{\infty} i\, n_{i+4}$$

by an argument using the Euler characteristic, where n_i denotes the number of i-gon in the projection of the sublink when the diagram is regarded as lying on S^2. Here, note that $n_1 = 0$ by the first assumption.

If there is a digon, the number of crossings can be reduced by applying the second Reidemeister move or the skein relation (6), since the digon is one of the following types:



,


,


,

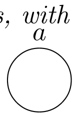
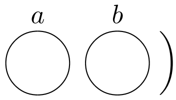
where a, b, c are mutually distinct elements in R_3. If there is a triangle, the number of crossings can be reduced by using the relations in Lemma 5.4, where we remark that we have no crossings whose arcs have the same color by the first assumption. Repeating this procedure, we obtain a trivial handlebody-link diagram. This completes the proof. □

Theorem 5.6. *Let F be an invariant of R_3-colored diagrams of unoriented links with values in $\mathbb{Z}_3$. Assume that:*

1. *F satisfies the skein relations in Proposition 4.2;*
2. *F vanishes on trivial knots and on trivial 2-component links with distinct colors.*

Then $F(D,C) = \Phi_3(D,C)$ for every R_3-colored diagram (D,C).

In particular, the skein relations in Proposition 4.2 are defining skein relations of Φ_3 for unoriented links, with

Φ_3 () $= \Phi_3$ () $= 0$

for distinct elements $a, b \in R_3$.

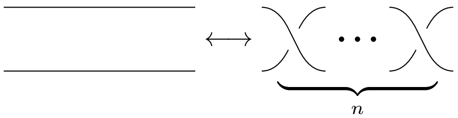


Figure 5. An n-move

Proof. Let D be a diagram of an unoriented link, and let C be an R_3-coloring of D. The proof is almost the same as that of Theorem 5.5. In the present setting we do not need to use Proposition 5.3, so we can transform D into a link diagram $D(a_1, \dots, a_n)$ without crossings only by applying the skein relations in Proposition 4.2, where n is the number of components of the link and $a_1, \dots, a_n$ are the colors of the components. By applying the skein relation (5), $D(a_1, \dots, a_n)$ can be transformed into $D(a)$, $D(a, b)$ or $D(a, b, c)$, where a, b, c are mutually distinct elements of R_3. Since $D(a, b, c)$ can be transformed into $D(a, b, b)$ by using Reidemeister moves, and $D(a, b, b)$ can be transformed into $D(a, b)$ by applying the skein relation (5), the proof is complete. □

6. On n-MOVES

An *n-move* is a local transformation of a handlebody-link diagram that adds or removes n half-twists between two parallel strands (see Figure 5). Two handlebody-links are said to be *n-equivalent* if their diagrams are related by a finite sequence of n-moves and Reidemeister moves. For handlebody-links H and H', we define the *n-move distance* $d_n(H, H')$ to be the minimum number of n-moves required to transform a diagram of H into a diagram of H', allowing Reidemeister moves in between. If H and H' are not n-equivalent, we set $d_n(H, H') = \infty$.

For classical links, n-moves have been studied extensively. A 2-move is just a crossing change. The Montesinos–Nakanishi 3-move conjecture was shown to hold for several classes of links [4, 15, 16], but it is false in general [6]. The analogous 4-move problem for knots remains open, and possible counterexamples have been discussed in [3, 14]. For 5-moves, see for example [5, 8, 13, 18].

For handlebody-links, related problems have also been studied. Quandle colorings were used in [10] to estimate unknotting numbers of handlebody-knots, and crossing numbers were used in [1]. Murao obtained estimates of Gordian distances between handlebody-knots in [11] and gave examples of handlebody-links that are not 4-equivalent to trivial ones in [12]. The purpose of this section is to show that the invariant Φ_p gives lower bounds for certain n-move distances.

Let S be a subset of $\mathbb{Z}_p$ containing 0 and a nonzero element. For $a \in \mathbb{Z}_p$ and $r \in \mathbb{Z}_{\geq 0}$, we set

$$N_S(a; r) := \{a + \varepsilon_1 a_1 + \cdots + \varepsilon_r a_r \in \mathbb{Z}_p \mid a_i \in S, \varepsilon_i \in \{\pm 1\}\},$$

where we note that $N_S(a; 0) = \{a\}$. Then we have

$$\begin{aligned}
&r_1 \leq r_2 \Rightarrow N_S(a; r_1) \subset N_S(a; r_2),\\
&b \in N_S(a; r) \Leftrightarrow a \in N_S(b; r),\\
&b \in N_S(a; r_1), c \in N_S(b; r_2) \Rightarrow c \in N_S(a; r_1 + r_2)
\end{aligned}$$

for $a, b, c \in \mathbb{Z}_p$ and $r, r_1, r_2 \in \mathbb{Z}_{\geq 0}$. For $a, b \in \mathbb{Z}_p$, we define

$$d_S(a, b) := \min\{r \in \mathbb{Z}_{\geq 0} \mid b \in N_S(a; r)\}.$$

Then d_S is a metric on $\mathbb{Z}_p$.

For a multiset $\{a_1, \dots, a_n\}$ and a multiset $\{S_1, \dots, S_n\}$ of sets, we write

$$\{a_1, \dots, a_n\} \mathrel{\widehat{\in}} \{S_1, \dots, S_n\}$$

if there exists a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that

$$a_{\sigma(i)} \in S_i \quad (i = 1, \dots, n).$$

For a multiset $\{a_1, \dots, a_n\}$ ($a_i \in \mathbb{Z}_p$) and $r \in \mathbb{Z}_{\geq 0}$, we define a multiset $\widehat{N}_S(\{a_1, \dots, a_n\}; r)$ by

$$\widehat{N}_S(\{a_1, \dots, a_n\}; r) = \{N_S(a_i; r) \mid i \in \{1, \dots, n\}\}.$$

For multisets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ ($a_i, b_i \in \mathbb{Z}_p$), we then define

$$\begin{aligned} &\widehat{d}_S(\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}) \\ &:= \min\{r \in \mathbb{Z}_{\geq 0} \mid \{b_1, \dots, b_n\} \mathrel{\widehat{\in}} \widehat{N}_S(\{a_1, \dots, a_n\}; r)\}. \end{aligned}$$

For $n \in \mathbb{Z}_{>0}$, we set $S_{n,p} := \{na^2 \mid a \in \mathbb{Z}_p\}$. The following simple example will be used later:

$$\widehat{d}_{S_{4,5}}(\{0, \dots, 0\}, \{0, \dots, 0, 2, \dots, 2, 3, \dots, 3\}) = 2,$$

because

$$\begin{aligned} &\widehat{N}_{S_{4,5}}(\{0, \dots, 0\}; 0) = \{\{0\}, \dots, \{0\}\}, \\ &\widehat{N}_{S_{4,5}}(\{0, \dots, 0\}; 1) = \{\{0, 1, 4\}, \dots, \{0, 1, 4\}\}, \\ &\widehat{N}_{S_{4,5}}(\{0, \dots, 0\}; 2) = \{\{0, 1, 2, 3, 4\}, \dots, \{0, 1, 2, 3, 4\}\}. \end{aligned}$$

Lemma 6.1. *Let k be a positive integer not divisible by p. Let (D, C) be an R_p-colored diagram of a $\mathbb{Z}_2$-flowed handlebody-link. Let (D', C') be an R_p-colored diagram obtained from (D, C) by applying $4kp$-moves n times, allowing Reidemeister moves. We then have*

$$\Phi_p(D', C') \in N_{S_{4k,p}}(\Phi_p(D, C); n).$$

That is,

$$n \geq d_{S_{4k,p}}(\Phi_p(D, C), \Phi_p(D', C')).$$

Proof. By the skein relations (3) and (4), a single $4kp$-move changes the value of $\Phi_p(D, C)$ by $\pm 4ka^2$ for some $a \in \mathbb{Z}_p$. Hence, after n applications of $4kp$-moves, we have

$$\Phi_p(D', C') \in N_{S_{4k,p}}(\Phi_p(D, C); n).$$

The stated inequality follows immediately from the definition of $d_{S_{4k,p}}$. □

Let H be a handlebody-link, and let D be a diagram of H. For a $\mathbb{Z}_2$-flow φ of H, we write D_φ for the diagram D equipped with φ. We set

$$\mathrm{ColDiag}_p(D) := \{(D_\varphi, C) \mid \varphi \in \mathrm{Flow}(H; \mathbb{Z}_2), C \in \mathrm{Col}_p(D_\varphi)\}.$$

We define the multiset

$$\Phi_p(H) := \{\Phi_p(D_\varphi, C) \mid (D_\varphi, C) \in \mathrm{ColDiag}_p(D)\}.$$

Then $\Phi_p(H)$ is an invariant of H (see [7]).

Theorem 6.2. *Let k be a positive integer not divisible by p. For handlebody-links H and H', we have*

$$d_{4kp}(H, H') \geq \widehat{d}_{S_{4k,p}}(\Phi_p(H), \Phi_p(H')).$$

Proof. Put $n := d_{4kp}(H, H')$. Let D and D' be diagrams of H and H', respectively. Take a sequence

$$D = D_1 \leftrightarrow D_2 \leftrightarrow \cdots \leftrightarrow D_m = D'$$

in which exactly n moves are $4kp$-moves and all remaining moves are Reidemeister moves. Let φ be a $\mathbb{Z}_2$-flow of H, and set $\varphi_1 := \varphi$. Then there exists a unique sequence of $\mathbb{Z}_2$-flows $\varphi_2, \dots, \varphi_m$ such that, for each $i = 1, \dots, m-1$, the $\mathbb{Z}_2$-flows

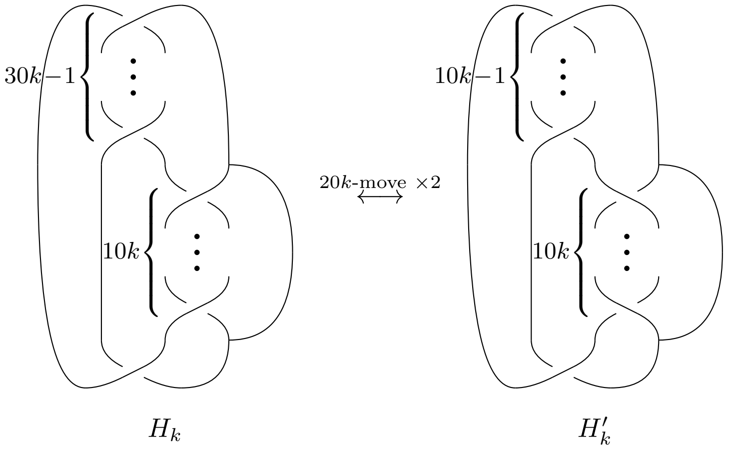


Figure 6. H_k and H'_k

φ_i and φ_{i+1} coincide outside the disk where the move $D_i \leftrightarrow D_{i+1}$ is performed. Write $\varphi' := \varphi_m$. We equip each $\mathbb{Z}_2$-flow φ_i to the diagram D_i. Let C be an R_p-coloring of D, and set $C_1 := C$. Then there exists a unique sequence of R_p-colorings $C_2, \ldots, C_m$ such that, for each $i = 1, \ldots, m-1$, the R_p-colorings C_i and C_{i+1} coincide outside the disk where the move $D_i \leftrightarrow D_{i+1}$ is performed. Write $C' := C_m$. This yields a bijection $\sigma : \mathrm{ColDiag}_p(D) \to \mathrm{ColDiag}_p(D')$ given by $\sigma(D_\varphi, C) = (D'_{\varphi'}, C')$.

By Lemma 6.1, we have

$$\Phi_p(\sigma(D_\varphi, C)) \in N_{S_{4k,p}}(\Phi_p(D_\varphi, C); n)$$

for any $(D_\varphi, C) \in \mathrm{ColDiag}_p(D)$. Therefore,

$$\Phi_p(H') \mathrel{\widehat{\in}} \widehat{N}_{S_{4k,p}}(\Phi_p(H); n),$$

and hence

$$d_{4kp}(H, H') \geq \widehat{d}_{S_{4k,p}}(\Phi_p(H), \Phi_p(H')).$$

□

Proposition 6.3. *Let k be an odd integer not divisible by 5. Let H_k and H'_k be the handlebody-links represented in Figure 6. Then $d_{20k}(H_k, H'_k) = 2$.*

Proof. The handlebody-links H_k and H'_k can be transformed into each other by two $20k$-moves. This implies that $d_{20k}(H_k, H'_k) \leq 2$. On the other hand, by Theorem 6.2, we have

$$d_{20k}(H_k, H'_k) \geq \widehat{d}_{S_{4k,5}}(\Phi_5(H_k), \Phi_5(H'_k)) = 2,$$

since

$$\Phi_5(H_k) = \{\underbrace{0, \ldots, 0}_{80}, \underbrace{2k, \ldots, 2k}_{10}, \underbrace{-2k, \ldots, -2k}_{10}\}, \qquad \Phi_5(H'_k) = \{\underbrace{0, \ldots, 0}_{100}\}.$$

Hence $d_{20k}(H_k, H'_k) = 2$. □

A handlebody-link whose components all have genus 1 may be identified with an unoriented link. The following lemma and theorem are the unoriented-link versions of Lemma 6.1 and Theorem 6.2.

Lemma 6.4. *Let k be a positive integer not divisible by p. Let (D, C) be an R_p-colored diagram of an unoriented link. Let (D', C') be an R_p-colored diagram obtained from (D, C) by applying kp-moves n times, allowing Reidemeister moves. We then have*

$$\Phi_p(D', C') \in N_{S_{k,p}}(\Phi_p(D, C); n).$$

That is,

$$n \geq d_{S_{k,p}}(\Phi_p(D, C), \Phi_p(D', C')).$$

Proof. By the skein relation (6), a single kp-move changes the value of $\Phi_p(D, C)$ by $\pm ka^2$ for some $a \in \mathbb{Z}_p$. Hence, after n applications of kp-moves, we have

$$\Phi_p(D', C') \in N_{S_{k,p}}(\Phi_p(D, C); n).$$

The stated inequality follows immediately from the definition of $d_{S_{k,p}}$. □

Let L be an unoriented link, and let D be a diagram of L. We define the multiset

$$\Phi_p(L) := \{\Phi_p(D, C) \mid C \in \mathrm{Col}_p(D)\}.$$

Then $\Phi_p(L)$ is an invariant of L (see Remark 3.3).

Theorem 6.5. *Let k be a positive integer not divisible by p. For unoriented links L and L', we have*

$$d_{kp}(L, L') \geq \widehat{d}_{S_{k,p}}(\Phi_p(L), \Phi_p(L')).$$

Proof. Put $n := d_{kp}(L, L')$. Let D and D' be diagrams of L and L', respectively. Take a sequence

$$D = D_1 \leftrightarrow D_2 \leftrightarrow \cdots \leftrightarrow D_m = D'$$

in which exactly n moves are kp-moves and all remaining moves are Reidemeister moves. Let C be an R_p-coloring of D, and set $C_1 := C$. Then there exists a unique sequence of R_p-colorings $C_2, \ldots, C_m$ such that, for each $i = 1, \ldots, m-1$, the R_p-colorings C_i and C_{i+1} coincide outside the disk where the move $D_i \leftrightarrow D_{i+1}$ is performed. Write $C' := C_m$. This yields a bijection $\sigma : \mathrm{Col}_p(D) \to \mathrm{Col}_p(D')$ given by $\sigma(C) = C'$.

By Lemma 6.4, we have

$$\Phi_p(D', \sigma(C)) \in N_{S_{k,p}}(\Phi_p(D, C); n)$$

for any $C \in \mathrm{Col}_p(D)$. Therefore,

$$\Phi_p(L') \mathrel{\widehat{\in}} \widehat{N}_{S_{k,p}}(\Phi_p(L); n),$$

and hence

$$d_{kp}(L, L') \geq \widehat{d}_{S_{k,p}}(\Phi_p(L), \Phi_p(L')).$$

□

Proposition 6.6. *Let k be an odd integer not divisible by 5. Let L_k and L'_k be the links represented in Figure 7. Then $d_{5k}(L_k, L'_k) = 2$.*

Proof. The links L_k and L'_k can be transformed into each other by two $5k$-moves. This implies that $d_{5k}(L_k, L'_k) \leq 2$. On the other hand, by Theorem 6.5, we have

$$d_{5k}(L_k, L'_k) \geq \widehat{d}_{S_{k,5}}(\Phi_5(L_k), \Phi_5(L'_k)) = 2,$$

since

$$\Phi_5(L_k) = \{\underbrace{0, \ldots, 0}_{5}, \underbrace{2k, \ldots, 2k}_{10}, \underbrace{-2k, \ldots, -2k}_{10}\}, \qquad \Phi_5(L'_k) = \{\underbrace{0, \ldots, 0}_{25}\}.$$

Hence $d_{5k}(L_k, L'_k) = 2$. □

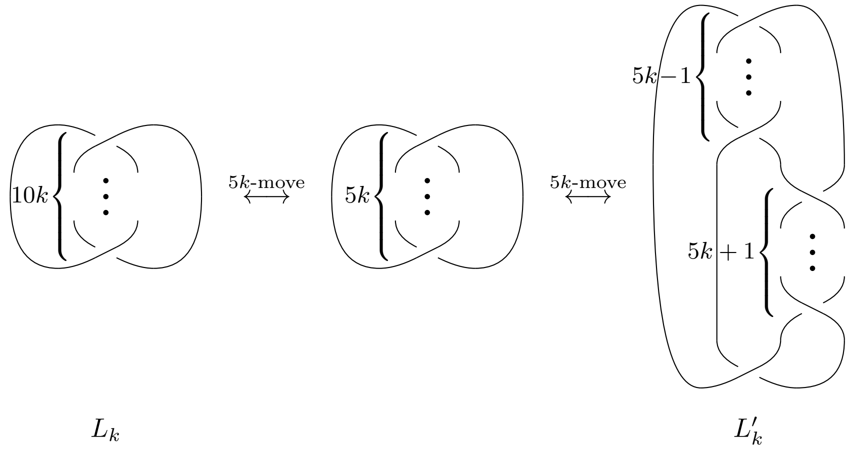


FIGURE 7. L_k and L'_k

ACKNOWLEDGMENTS

The authors would like to thank Ayumu Inoue for his valuable comments. The first author was supported by JSPS KAKENHI Grant Number 21K03217. The second author was supported by JSPS KAKENHI Grant Number 22K13917. The third author was supported by JSPS KAKENHI Grant Number 21K03233.

REFERENCES

[1] Y. Akimoto, *On inequalities between unknotting numbers and crossing numbers of spatial embeddings of trivializable graphs and handlebody-knots*, J. Knot Theory Ramifications **31** (2022), 2150079, 23 pp.

[2] S. Asami and S. Satoh, *An infinite family of non-invertible surfaces in* 4*-space*, Bull. London Math. Soc. **37** (2005), no. 2, 285–296.

[3] N. A. Askitas, *On* 4*-equivalent tangles*, Kobe J. Math. **16** (1999), 87–91.

[4] Q. Chen, *The* 3*-move conjecture for* 5*-braids*, Knots in Hellas 98 (Delphi), 36–47, Ser. Knots Everything **24**, World Sci. Publ., River Edge, NJ (2000).

[5] M. K. Dąbkowski, M. Ishiwata and J. H. Przytycki, 5*-move equivalence classes of links and their algebraic invariants*, J. Knot Theory Ramifications **16** (2007), no. 10, 1413–1449.

[6] M. K. Dąbkowski and J. H. Przytycki, *Burnside obstructions to the Montesinos-Nakanishi* 3*-move conjecture*, Geom. Topol. **6** (2002), 355–360.

[7] A. Ishii, *Moves and invariants for knotted handlebodies*, Algebr. Geom. Topol. **8** (2008), 1403–1418.

[8] A. Ishii and K. Oshiro, *Derivatives with Alexander pairs for quandles*, Fund. Math. **259** (2022), no. 1, 1–31.

[9] A. Ishii and K. Oshiro, *Normalized quandle twisted Alexander invariants*, Internat. J. Math. 35 (2024), no. 5, Paper No. 2450011.

[10] M. Iwakiri, *Unknotting numbers for handlebody-knots and Alexander quandle colorings*, J. Knot Theory Ramifications **24** (2015), 1550059, 13 pp.

[11] T. Murao, *The Gordian distance of handlebody-knots and Alexander biquandle colorings*, J. Math. Soc. Japan **70** (2018), 1247–1267.

[12] T. Murao, 4*-move inequivalent handlebody-links and* f*-twisted Alexander matrices*, Osaka J. Math. **61** (2024), no. 4, 601–616.

[13] J. H. Przytycki, t_k*-moves on links*, Braids (Santa Cruz, CA, 1986), Contemp. Math., **78**, American Mathematical Society, Providence, RI, 1988, 615–656.

[14] J. H. Przytycki, *The* t_3*,* $\bar{t}_4$ *moves conjecture for oriented links with matched diagrams*, Math. Proc. Cambridge Philos. Soc. **108** (1990), no. 1, 55–61.

[15] J. H. Przytycki, *Elementary conjectures in classical knot theory*, Quantum topology, 292–320, Ser. Knots Everything **3**, World Sci. Publ., River Edge, NJ (1993).

[16] J. H. Przytycki and T. Tsukamoto, *The fourth skein module and the Montesinos-Nakanishi conjecture for* 3*-algebraic links*, J. Knot Theory Ramifications **10** (2001), 959–982.

[17] S. Satoh, *A note on the shadow cocycle invariant of a knot with a base point*, J. Knot Theory Ramifications **16** (2007), no.7, 959–967.
[18] A. Stoimenow, 5-*moves and Montesinos links*, J. Math. Soc. Japan **59** (2007), no. 3, 729–749.