# PISOT NUMBERS AND GREEDY ALGORITHM

Shigeki Akiyama

Niigata University

Abstract. We study the greedy expansion of real numbers in Pisot number base. We will show a certain criterions of finiteness, periodicity, and purely periodicity. Further, it is proved that every sufficiently small positive rational numbers has purely periodic greedy expansion in Pisot unit base under a certain finiteness condition.

## 1. Introduction

Let $\beta$ be the fixed real number greater than 1 and $x$ be a positive real number. Then the expansion of the form $x = \sum_{N_0 \le i}^{\infty} a_i \beta^{-i}$ is said to be a *greedy expansion* if

$$|x - \sum_{N_0 \le i \le N} a_i \beta^{-i}| < \beta^{-N}, \tag{G}$$

holds for every $N$ and $a_i$ is a non negative integer with $0 \le a_i < \beta$. It is a natural generalization of binary or decimal expansion to the expansion in real bases. In this note, we say '$\beta$-expansion' by the algorithm:

$$u_n = \{\beta u_{n-1}\}, v_n = [\beta u_{n-1}],$$

with $u_0 = x$. It coincides with the greedy expansion when $0 \le x < 1$. Here we denote by $[x]$ the maximal integer not exceeds $x$ and by $\{x\}$ the value $x - [x]$. It seems that this $\beta$-expansion is more popular than greedy expansion in the field of symbolic dynamics.

Now let $K$ be the finite extention of the rational number field $\mathbb{Q}$. Denote by $\sigma$, the embedding of $K$ into $\mathbb{C}$. The algebraic integer $\gamma > 1$ in $K$ is called *Pisot number* if $|\gamma^\sigma| < 1$ for any embedding $\sigma$ besides identity. And if the integer $\gamma > 1$ of $K$ satisfies $|\gamma^\sigma| \le 1$ for any embedding $\sigma$ besides identity and the equality holds for at least one $\sigma$, then $\gamma$ is called *Salem number*.

In this note, we will give new criterions of finiteness, periodicity and purely periodicity of the greedy expansion of rational numbers in base $\beta$ when $\beta$ is a Pisot or Salem number. First we note:

**Theorem (A.Bertrand [2], K.Schmidt [5]).** *Let $\beta$ be a Pisot number. Then positive real $x$ has a periodic greedy expansion in base $\beta$ if and only if $x \in \mathbb{Q}(\beta)$.*

It establishes a remarkable analogy as in the case of the expansion in a rational integer base. Moreover we see

**Theorem (K.Schmidt [5]).** *Let $\beta$ be a fixed real number. If any $x \in \mathbb{Q} \cap [0,1)$ has a periodic greedy expansion in base $\beta$, then $\beta$ must be a Pisot or Salem number.*

In [5], K.Schmidt also conjectured that if $\beta$ is a Salem number then any element of $\mathbb{Q}(\beta) \cap [0,1)$ has periodic greedy expansion in base $\beta$. D.W. Boyd [3] actually showed that the $\beta$-expansion of 1 by Salem numbers of degree 4 is periodic, by giving explicit digits of them.

In section 2, we study two properties of a real number $\beta > 1$:

(F) For every integer $x > 0$ in $\mathbb{Q}(\beta)$, $x$ has finite greedy expansion in base $\beta$.

and

(F') For every integer $x > 0$ in $\mathbb{Z}[\beta]$, $x$ has finite greedy expansion in base $\beta$.

It is obvious that (F) is stronger than (F'). The property (F') implies that $\beta$ is an algebraic integer. Extensive study on the property (F') can be found in [4]. Theorem 2 of [4] provides us many examples of Pisot numbers which satisfies (F'). Let $\beta$ be a fixed Pisot number. We will show in Theorem 2 that there exists a finite algorithm to determine whether $\beta$ has the property (F) (resp. (F')) or not. Remark that there exists considerable difference between (F) and (F'). For instance, there exist a lot of real quadratic units which do not satisfy (F).

In section 3, we state certain criterions of periodicity and purely periodicity. These criterions are rather easily proved but the author could not find them in the literature. Especially, by Proposition 4, one can determine that the greedy expansion of a rational $x \in (0,1)$ in base $\beta = \sqrt[n]{r}$ is periodic or not.

The main purpose of this note is to give a sufficient condition of purely periodicity:

**Theorem 1.** *Let $\beta$ be a Pisot unit with property* (F')*. Then there exists a positive constant $c$ such that every $x \in \mathbb{Q} \cap [0,c]$ has purely periodic greedy expansion in base $\beta$.*

Here we call $\beta$ 'a Pisot unit' if $\beta$ is a Pisot number and is also a unit of the integer ring of $\mathbb{Q}(\beta)$. This result gives us an analogy of the fact that 'every rational $x \in (0,1)$ whose denominator is coprime to 10 is purely periodic in decimal base'. We should also mention that K. Schmidt [5] has proved this analogy for the case of any real quadratic unit with norm $-1$. We might hope that this analogy will be true for non unit bases. See the conjecture in section 4.

According to the proof, the constant $c$ can be explicitly calculated if we fix a Pisot unit. We will show some examples with concrete constants $c$. For instance, let $\theta > 1$ be the root of $\theta^3 - \theta - 1 = 0$, which is the smallest Pisot number, then we can take $c = 0.434$. But we also see that $c$ can not be taken larger than $9/13$. The problem of determining the supremum of $c$ seems to be interesting.

## 2. Finite greedy expansions

First, we state two easy general criterions:

**Proposition 1.** *A real algebraic integer $\beta > 1$ has a different positive real conjugate $\eta$, then $\beta$ does not have the property* (F').

*Proof.* First, we assume that $\eta < \beta$ and $\beta$ has property (F'). Take rational integers $P, Q$ that $\eta < P/Q < \beta$ Then by using (F'), the greedy expansion of $Q\beta - P$ is finite:

$$Q\beta - P = \sum_{N_0 \le i \le N} a_i \beta^{-i}.$$

Conjugating both sides, we have

$$0 > Q\eta - P = \sum_{N_0 \le i \le N} a_i \eta^{-i} > 0,$$

which is a contradiction. The case $\beta < \eta$ is similar if we consider $P - Q\beta$. □

As each Salem number is known to be a root of the reciprocal polynomial, we see that Salem numbers can not have property (F').

**Proposition 2.** *When $\beta$ is a unit of the integer ring of $\mathbb{Q}(\beta)$ and has property* (F)*, the ring $\mathbb{Z}[\beta]$ coincides with the whole integer ring of $\mathbb{Q}(\beta)$.*

*Proof.* Let $f(X) = X^k \pm 1 + \sum_{j=1}^{k-1} b_j X^j$ be the minimal polynomial of $\beta$. Take an arbitrary integer $x$ of $\mathbb{Q}(\beta)$. Then by (F), we have

$$x = \sum_{N_0 \le i \le N} a_i \beta^{-i}.$$

By using the relation $\beta^k \pm 1 + \sum_{j=1}^{k-1} b_j \beta^j = 0$ recursively, we see that $x$ belongs to $\mathbb{Z}[\beta]$. □

Hereafter in this section, we concentrate on the case that $\beta$ is a Pisot number of degree $k$, Denote by $\beta^{(0)} = \beta, \beta^{(1)}, \dots, \beta^{(k-1)}$ its conjugates. Let $p$ be a non negative integer and define $M_j(p)$ $(j = 1, 2, \dots k-1)$ by an upper bound of

$$\left|\sum_{i=0}^{p} a_{p-i} (\beta^{(j)})^i\right|,$$

where $\sum_{i=0}^{p} a_i \beta^{-i}$ runs through finite greedy expansions of length at most $p+1$. Let $M_j$ be an upper bound of $M_j(p)$ $(p = 1, 2, \dots)$. One may take $M_j = [\beta]/(1 - |\beta^{(j)}|)$. Here $[x]$ is the greatest integer not exceeds x. We can also take a better bound in practice:

$$M_j = \frac{M_j(p)}{1 - |\beta^{(j)}|^{p+1}}.$$

Let $b_j$ $(j = 0, 1, \dots k-1)$ be positive real numbers and $C = C(b_0, b_1, \dots, b_{k-1})$ be the set of algebraic integers $x$ in $\mathbb{Q}(\beta)$ such that

$$|x^{(j)}| \le b_j.$$

Then we see that $C$ is a finite set.

**Theorem 2.** *Let $\beta$ be a Pisot number. Then $\beta$ has the property* (F) *(resp.* (F')*) if and only if every element of $C = C(1, M_1, M_2, \dots, M_{k-1})$ (resp. $C \cap \mathbb{Z}[\beta]$ ) has finite greedy expansion in base $\beta$.*

*Proof.* First we prove the case (F). As the set $C$ is finite, there exist $\epsilon > 0$ such that

$$C(1, M_1, M_2, \dots M_{k-1}) = C(1, M_1 + \epsilon, M_2 + \epsilon, \dots M_{k-1} + \epsilon).$$

Thus we shall prove the same statement for $C(1, M_1+\epsilon, M_2+\epsilon, \dots M_{k-1}+\epsilon)$ for any $\epsilon > 0$. Let $x$ be any positive real algebraic integer of $\mathbb{Q}(\beta)$ and q be a sufficiently large integer such that $|x^{(j)}(\beta^{(j)})^q| < \epsilon$ $(j = 1, 2, \dots, k-1)$. Consider the greedy expansion:

$$x\beta^q = \sum_{N_0 \le i}^{\infty} a_i \beta^{-i}.$$

Put $y = x\beta^q - \sum_{N_0 \le i \le 0} a_i \beta^{-i}$. If $N_0 > 0$ then $y = x\beta^q$, because the summand is empty. Then we see that

$$\begin{aligned} |y| &< 1 \\ |y^{(j)}| &< \epsilon + M_j \quad (j = 1, 2, \dots, k-1). \end{aligned}$$

By the assumption, there exists non negative integer $N$ so that $y$ has finite greedy expansion:

$$y = \sum_{0 < i \le N} a_i \beta^{-i}$$

This shows that the greedy expansion of $x$ is

$$x = \sum_{N_0 \le i \le N} a_i \beta^{-i-q},$$

which is actually finite. The proof of the case (F') is similar. □

Consider the case of real quadratic units. We only have to treat the case of negative norm by Proposition 1.

**Corollary 1.** *Let $\varepsilon$ be a real quadratic unit with norm $-1$. Then $\varepsilon$ has property* (F) *if and only if $\mathbb{Z}[\varepsilon]$ coincides with the whole integer ring of $\mathbb{Q}(\varepsilon)$.*

*Proof.* Necessity was proved in Proposition 2. We will prove the sufficiency. We use Theorem 2 with $k = 2$. By the definition, $M_1$ can be taken $[\varepsilon]/(1 - \varepsilon^{-2})$, which is best possible. As a integer in $\mathbb{Q}(\varepsilon)$ has the form $x + y\varepsilon$, we have to show the finiteness for the integer which satisfies:

$$\begin{aligned} |x + y\varepsilon| &< 1 \\ |x - y/\varepsilon| &< M_1 \end{aligned}$$

These inequalities imply that $y = -1, 0, 1$. Thus we only have to consider the integers $\varepsilon - [\varepsilon]$ and $1 - \varepsilon + [\varepsilon]$. The finiteness of these two cases follows from the relation $\varepsilon + \varepsilon^{-1} = [\varepsilon] + 1$. □

## 3. Criterions of periodicity

In this section, we show some useful criterions of periodicity and those of purely periodicity. Let $\beta > 1$ be an arbitrary algebraic number of degree $k > 1$. Denote by $\beta^{(j)}$ $(j = 0, 1, \dots, k-1)$ its conjugates, as in the previous section.

**Proposition 3.** *Assume that there exist a conjugate $\beta^{(j)}$ $(j = 1, 2, \dots k-1)$ with $|\beta^{(j)}| \geq \beta$ and $\beta^{(j)}/\beta$ is not a root of unity. Then the greedy expansion of any rational number $x \in (0, 1)$ is not periodic.*

*Proof.* Assume that $x$ has a periodic greedy expansion $\sum_{0 \leq i} a_i \beta^{-i}$. Then the right hand side is rewritten as a rational function of $\beta$, which admits Galois actions. By the assumption $|\beta^{(j)}| \geq \beta > 1$, we easily see

$$x = \sum_{0 \leq i} a_i (\beta^{(j)})^{-i}.$$

But we also see

$$|\sum_{0 \leq i} a_i (\beta^{(j)})^{-i}| < \sum_{0 \leq i} a_i \beta^{-i},$$

which gives a contradiction. Note that the inequality is true even when $|\beta^{(j)}| = \beta$. To see this, we have to notice that the expansion has at least two non zero terms and these terms can not have the same argument. $\square$

Here we essentially used the fact that the periodic greedy expansion admits Galois actions. One may derive lots of similar criterions in this manner. Especially, it is an easy task to show the non periodicity of a fixed element $x \in \mathbb{Q}(\beta)$ in most cases. It seems well worth to mention:

**Proposition 4.** *Let $\beta = \sqrt[n]{r}$ with a positive integer $r > 1$. Then a rational number $x$ has periodic greedy expansion in base $\beta$ if and only if $x = \sum_{0 \leq i} a_i r^{-i}$ with $a_i \in \mathbb{Z} \cap [0, \beta)$ and the coefficient sequence $a_i$ $(i = 0, 1, \dots)$ is eventually periodic.*

*Proof.* The sufficiency is obvious, as $\sum_{0 \leq i} a_i \beta^{-ni}$ is a greedy expansion of $x$. Assume that $x$ has a periodic greedy expansion $x = \sum_{0 \leq i} a_i \beta^{-i}$. In the similar manner as in the proof of Proposition 3, if $x$ has periodic greedy expansion, we see

$$x = \sum_{0 \leq i} a_i (\beta^{(j)})^{-i}$$

for $j = 0, 1, \dots, k-1$. By using $\sum_j (\beta^{(j)})^s = 0$ for $s \not\equiv 0 \pmod{n}$, we see

$$x = \sum_{i \equiv 0 \pmod{n},\ i \geq 0} a_i (\beta^{(j)})^{-i}.$$

If $j = 0$, then the right hand side must be a greedy expansion. In other words, we have shown that $a_i = 0$ when $i$ is not a multiple of $n$, which shows the necessity. $\square$

For example, let $\beta = \sqrt{2}$. Then we see that $1/2$ and $1/3 = 1/(4-1) = 1/4 + 1/16 + 1/64 + 1/256 + \dots$ have periodic greedy expansion in base $\beta$ but $1/5 = \beta^{-5} + \beta^{-11} + \beta^{-20} + \beta^{-26} + \dots$ does not have periodic expansion by this proposition.

Now let $\beta$ be a Pisot number of degree $k > 1$. Then we see that every $x \in \mathbb{Q}(\beta)$ has periodic greedy expansion. We state an easy criterion of purely periodicity.

**Proposition 5.** *Assume that there exist a real conjugate $\beta^{(j)} \in (0,1)$. Then every rational $x \in (0,1)$ does not have a purely periodic expansion.*

*Proof.* Assume that $x$ has a purely periodic expansion. Then we have

$$x = \frac{\sum_{i=1}^{N} a_i \beta^{-i}}{1-\beta^{-N}} = \frac{\sum_{i=0}^{N-1} a_{N-i}\beta^i}{\beta^N - 1}$$

Thus we see

$$0 > ((\beta^{(j)})^N - 1)x = \sum_{i=0}^{N-1} a_{N-i}(\beta^{(j)})^i > 0,$$

which is absurd. $\square$

This fact should be compared with Proposition 1 and Proposition 3. Combining the results of K. Schmidt [5], we have

**Corollary.** *The greedy expansion of the rational number $x \in (0,1)$ in a real quadratic unit base is purely periodic if and only if the norm of the unit is $-1$.*

In preparing this paper, the author got to know that M. Hama and T.Imahashi [6] completely determined the set consists of elements which has purely periodic expansion in real quadratic unit base, by making concrete 'natural extention' for this case. Of course, their result implies this corollary.

## 4. Proof of Theorem 1

Let $\beta$ be a Pisot unit of degree $k$. We use the same notation as in section 1, such as $\beta^{(j)}, M_j$ $(j = 0, 1, \ldots, k-1)$ and $C = C(b_0, b_1, \ldots, b_{k-1})$. Now we show

*Proof of Theorem 1.* Assume that $\beta$ has property (F'). Consider the greedy expansion of a rational number $x > 0$ in base $\beta$. We assume that $x$ does not have a purely periodic expansion. Our purpose it to prove that $x > c$ with a constant $c$ which depends only on $\beta$. We first see that there exist infinitely many $N$ such that $x(\beta^N - 1) \in \mathbb{Z}[\beta]$. Here we used the fact the $\beta \pmod{\mathbb{Z}[\beta]}$ is the unit of the residue ring $\mathbb{Z}[\beta]/q\mathbb{Z}[\beta]$, where $q$ is the denominator of $x$. By using (F'), we have

$$(\beta^N - 1)x = \sum_{0 \le i \le n} a_i \beta^i + \sum_{-m \le i < 0} a_i \beta^i, \tag{K}$$

with a positive integer $m$ and $a_{-m} > 0$. In fact, if not we have an expansion:

$$\begin{aligned} x =& \frac{a_n\beta^{n-N} + a_{n-1}\beta^{n-1-N} + a_{n-2}\beta^{n-2-N} + \cdots + a_0\beta^{-N}}{1-\beta^{-N}} \\ =& a_n\beta^{n-N} + a_{n-1}\beta^{n-1-N} + a_{n-2}\beta^{n-2-N} + \cdots + a_0\beta^{-N} + \\ & a_n\beta^{n-2N} + a_{n-1}\beta^{n-1-2N} + a_{n-2}\beta^{n-2-2N} + \cdots + a_0\beta^{-2N} + \\ & a_n\beta^{n-3N} + a_{n-1}\beta^{n-1-3N} + a_{n-2}\beta^{n-2-3N} + \cdots + a_0\beta^{-3N} + \ldots. \end{aligned} \tag{D}$$

If we choose $c$ small, then $n - N$ must be small and this expansion satisfies (G). Then this expansion itself is a purely periodic greedy expansion. We denote the upper bound of $c$ determined by this condition, by $c_0$, which clearly depends only on $\beta$.

Conjugating both size of (K), we have

$$((\beta^{(j)})^N - 1)x = \sum_{0\le i\le n} a_i(\beta^{(j)})^i + \sum_{-m\le i<0} a_i(\beta^{(j)})^i,$$

with $j = 1, 2, \dots, k-1$ and a positive integer $m$ and $a_{-m} > 0$.

It seems convenient for the reader to sketch here the idea of the proof. We may choose $N$ large, then the left hand side converges to $-x$ which has small absolute value when $c$ is small. But the right hand side should be large because $|a_{-m}(\beta^{(j)})^{-m}|$ is large.

Now we will show that there exists a positive constant $c_1$ and a non negative integer $p$ that for any $d_i$ $(i = 0, 1, \dots p)$ which admits a greedy expansion $\sum_{i=0}^{p} d_i\beta^{-i}$ with $d_p \neq 0$ , there exist $j \in \mathbb{Z} \cap [1, k-1]$ so that

$$|\sum_{i=0}^{p} d_{p-i}(\beta^{(j)})^i| - |\beta^{(j)}|^{p+1}M_j > c_1|\beta^{(j)}| \tag{K2}$$

holds. Once we show this fact, take a positive $c$ smaller than the $c_0$ and $c_1$. Then we see that $x > c$ which completes the proof. In fact, there exist $j \in \mathbb{Z} \cap [1, k-1]$ such that

$$\begin{aligned}
|((\beta^{(j)})^N - 1)x| &= \sum_{-m\le i\le n} a_i(\beta^{(j)})^i \\
&= |\beta^{(j)}|^{-m} |\sum_{0\le i\le n+m} a_{i-m}(\beta^{(j)})^i| \\
&\ge |\beta^{(j)}|^{-1} \left( |\sum_{0\le i\le p} a_{i-m}(\beta^{(j)})^i| - |\beta^{(j)}|^{p+1}M_j \right) \\
&> c_1.
\end{aligned}$$

(Here we can take sufficiently large $N$ such that $n > p$.) As one can take sufficiently large N, this inequality implies that $x \ge c_1$.

Now we will show (K2). It suffice to show that for any positive constant $\varepsilon$ there exist a non negative integer $p$, and for any $d_i$ $(i = 0, 1, \dots p)$ which admits a greedy expansion $\sum_{i=0}^{p} d_i\beta^{-i}$ with $d_p \neq 0$, there exist $j \in \mathbb{Z} \cap [1, k-1]$ such that

$$|\sum_{i=0}^{p} d_{p-i}(\beta^{(j)})^i| > (1+\varepsilon)|\beta^{(j)}|^{p+1}M_j.$$

In fact, one can take $c_1 = \min_{j=1}^{k-1} \varepsilon|\beta^{(j)}|^p M_j$ to suffice (K2). Assume the contrary, then there exists a positive constant $\varepsilon$, and for any non negative integer $p$, there exist a pair $d_i$ $(i = 0, 1, \dots p)$ satisfying above conditions and we have

$$|\sum_{i=0}^{p} d_{p-i}(\beta^{(j)})^i| \le (1+\varepsilon)|\beta^{(j)}|^{p+1}M_j,$$

for all $j = 1, 2, \dots, k-1$. Then we see that the algebraic integers

$$\sum_{i=0}^{p} d_{p-i}\beta^{i-p} = \sum_{i=0}^{p} d_i\beta^{-i}$$

must lie in $C = C(|\beta|, (1+\varepsilon)|\beta^{(1)}|M_1, (1+\varepsilon)|\beta^{(2)}|M_2, \dots (1+\varepsilon)|\beta^{(k-1)}|M_{k-1}) \cap \mathbb{Z}[\beta]$. As $d_p \neq 0$, the integers of $\mathbb{Q}(\beta)$ expressed by greedy expansion: $\sum_{i=0}^{p} d_i\beta^{-i}$ must be distinct. As we can take infinitely many $p$, this contradicts with the fact that $C \cap \mathbb{Z}[\beta]$ is a finite set. □

Note that, we can show that for any positive $x \in \mathbb{Q}(\beta)$ provided with

$$|x^{(j)}| \leq c,\ (j = 0, 1, \dots, k-1)$$

must have purely periodic expansion in a similar manner.

For the non unit Pisot numbers, we have

**Proposition 6.** *Let $\beta$ be a Pisot number which is not a unit. Then there exists arbitrary small positive rational $x$ which does not have purely periodic greedy expansion.*

*Proof.* Let $x = 1/N_{\mathbb{Q}(\beta)/\mathbb{Q}}(\beta)^n$ $(n = 1, 2, \dots)$. Then if $x$ has purely periodic expansion we see

$$x = \frac{\sum_{i=0}^{n-1} a_i\beta^i}{1-\beta^n}$$

Thus we have

$$x(1-\beta^n) = \sum_{i=0}^{n-1} a_i\beta^i,$$

but the left hand side can not be an algebraic integer. □

But numerical evidence supports the following

**Conjecture.** *Let $\beta$ be a Pisot number with the property* (F'). *Then there exist a positive constant $c$ so that each $x \in \mathbb{Q}(\beta) \cap [0, c]$ whose denominator is coprime to $\beta$ has a purely periodic greedy expansion in base $\beta$.*

Here we should remark about the numerical experiment of this conjecture. If we consider the Pisot number $\beta$ whose irreducible polynomial is of the form:

$$x^n - a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0$$

with $a_{i+1} \mid a_i$ for $i = 0, 1, \dots, n-2$. Here $a_i$ are positive integers. It is proved in Theorem 2 of [4], that $\beta$ has property (F'). Then it seems that one can take pretty large $c$, for example $c = 1$(!?) for $\beta = 2 + \sqrt{6}$. But in the general case, the conjecture seems not so convincing. For example, consider the Pisot number which is a root of $x^4 - 3x^3 + x - 2$. One can show the property (F) for this Pisot number by Theorem 2. But we see that $1/3^{12}$ is *not* purely periodic. We can also observe that, the length of non periodic part is decreasing if we choose smaller $x$, which suggest us the truth of the conjecture.

## 5. Some examples

Let us denote by $\gamma = \gamma(\beta)$ the supremum of $c$ that every element in $\mathbb{Q} \cap [0, c]$ must have purely periodic expansion in base $\beta$. Theorem 1 not only assures the existence of $\gamma > 0$ but also gives an effective algorithm to obtain a concrete lower bound of $\gamma$ for a fixed Pisot unit $\beta$. But actually, one can derive a better bound. We treat some examples in this section.

Let $\theta$ be a positive root of $x^3 - x - 1 = 0$. This $\theta$ is known to be the smallest Pisot number. (See Theorem 3.5 in [1].) We can confirm by Theorem 2 that $\theta$ has the property (F). We will show

**Claim 1.** $0.4342 \le \gamma(\theta) \le 0.6924.$

*Proof.* By numerical evaluation, we have

$$\begin{aligned} \theta =& \theta^{(0)} \approx 1.324717957 \\ \theta' =& \theta^{(1)} \approx -0.6623589786 + 0.562279512I \\ & \theta^{(2)} \approx -0.6623589786 - 0.562279512I. \end{aligned}$$

Thus the coefficients $a_i$ of the greedy expansion is 0 or 1. Noting $\theta^5 - \theta^4 - 1 = 0$, we see that

$$\text{(R)} \qquad a_i = 1 \Longrightarrow a_{i+1} = a_{i+2} = a_{i+3} = a_{i+4} = 0.$$

Conversely if the coefficients $a_i$ satisfies (R) then $\sum a_i \theta^{-i}$ is a greedy expansion. By computer calculation, the minimum of $\sum_{i=0}^{29} |a_{29-i}(\theta')^i|$ under conditions $a_{29} \neq 0$ and (R) is

$$|1 + (\theta')^9 + (\theta')^{14} + (\theta')^{22} + (\theta')^{27}| \approx 0.5289615027.$$

On the other hand, we can take $M_1 = 1/(1 - |\theta'|^5)$, as $M_1(4) = 1$. Thus we obtain $c_1$ of (K2):

$$\begin{aligned} c_1|\theta'| &= |1 + (\theta')^9 + (\theta')^{14} + (\theta')^{22} + (\theta')^{27}| - |\theta'|^{30} M_1 \\ c_1 &\approx 0.5752415728 \end{aligned}$$

By using (R), one can take $c_0 < \theta^{-4} \approx 0.3247179572$. Thus we have shown that $\gamma(\theta) > 0.324$. With a little more consideration, we can proceed further. By the definition of $c_1$, we see that

$$((\theta')^N - 1)x = \sum_{0 \le i \le n} a_i (\theta')^i,$$

if $x \le c_1$. Reviewing the proof of Theorem 1, we see if $x \le c_1|\theta'|^{1+s}$ for a non negative integer $s$, then $a_i = 0$ for $i = 0, 1, \dots, s$. Then the restriction on $c_0$ is weakened to $c_0 < \theta^{s-3}$ to get the greedy expansion (D). In the above case, take $s = 1$ then $c \le c_1|\theta'|^2 \approx 0.434237016$ and $c < \theta^{-2} = 0.569840291$. Thus we can take $c = 0.43423$ and we have $0.4342 < \gamma(\theta)$.

On the contrary, we have a non periodic expansion:

$$\frac{9}{13} = \theta^{-2} + \theta^{-8} + \frac{1}{1-\theta^{-183}} \times$$
$$(\theta^{-15} + \theta^{-22} + \theta^{-30} + \theta^{-38} + \theta^{-52} + \theta^{-67} + \theta^{-72} + \theta^{-90} + \theta^{-95} + \theta^{-100}$$
$$+\theta^{-132} + \theta^{-137} + \theta^{-144} + \theta^{-158} + \theta^{-163} + \theta^{-168} + \theta^{-176} + \theta^{-181} + \theta^{-188})$$

□

Now we treat another example. Let $\eta$ be the positive root of $x^4 - 4x^3 - 2x - 1$. Then we can also confirm (F) for $\eta$ by Theorem 2. Main difference from $\theta$ is that we have to consider two places of $\mathbb{Q}(\eta)$ simultaneously. This difference causes the bad lower bound of $\gamma(\eta)$.

**Claim 2.** $0.00365 \le \gamma(\eta) \le 0.05883$.

*Proof.* We have

$$\begin{aligned} \eta =& \eta^{(0)} \approx 4.131358944 \\ \eta' =& \eta^{(1)} \approx -0.3799141256 \\ \eta" =& \eta^{(2)} \approx 0.1242775907 + 0.7884641106I \\ & \eta^{(3)} \approx 0.1242775907 - 0.7884641106I. \end{aligned}$$

Thus the coefficients $a_i \in \{0,1,2,3,4\}$. We easily see the rule of lexicographical ordering:

$$\text{(R2)} \qquad \begin{aligned} a_i = 4 &\Longrightarrow a_{i+1} = 0, \\ a_i = 4, a_{i+1} = 0 &\Longrightarrow a_{i+2} \le 2, \\ a_i = 4, a_{i+1} = 0, a_{i+2} = 2 &\Longrightarrow a_{i+3} = 0. \end{aligned}$$

Conversely if $a_i$ satisfies (R2), then $\sum a_i \eta^{-i}$ is a greedy expansion. As $\eta' < 0$, we have $M_1 = (4 + 2(\eta')^2)/(1-(\eta')^4)$ which is actually attained by

$$4 + 2(\eta')^2 + 4(\eta')^4 + 2(\eta')^6 + 4(\eta')^8 + 2(\eta')^{10} + \dots.$$

Note that the coefficients satisfies (R2). One can show that $M_2(7)$ is attained by $|4 + 3\eta" + 4(\eta")^4 + 3(\eta")^5|$. Thus we get $M_2 = M_2(7)/(1 - |\eta"|^8)$. Now we compute

$$\min_{a_i}\left(\max\left(\frac{|\sum_{i=0}^{10} a_{10-i}(\eta')^i| - |\eta'|^{11}M_1}{|\eta'|}, \frac{|\sum_{i=0}^{10} a_{10-i}(\eta")^i| - |\eta"|^{11}M_2}{|\eta"|}\right)\right),$$

where $a_i \in \{0,1,2,3,4\}$, $i = 0,1,2,\dots,10$ is taken under the conditions (R2) and $a_{10} \ne 0$. The minimum is obtained when

$$(a_0, a_1, \dots, a_{10}) = (1,3,2,3,1,0,0,4,3,3,0),$$

and its value is

$$c_1 = \frac{|\sum_{i=0}^{10} a_{10-i}(\eta')^i| - |\eta'|^{11} M_1}{|\eta'|} \approx 0.003655885.$$

We see that $0 < c_0 = \eta^{-1} - \epsilon$ for any positive $\epsilon$. The trick used in the case of $\theta$ can not work as $c_0 > c_1$. Thus we have $0.00365 \leq \gamma(\eta)$. On the other hand, we see that the greedy expansion:

$$\begin{aligned} 1/17 = &0\overline{1000102321003024000122010332022400212114000330321230} \\ &\overline{2303000400023031040022401212222300334002323 1} \end{aligned}$$

is not purely periodic. Here we express the greedy expansion in base $\eta$ by its coefficients, similarly as decimal expansion. Overlined part is the period of the expansion. $\square$

We finish this article by proposing a

**Problem.** *Is there a constant $\delta > 0$ depending only on the degree of the Pisot unit $\beta$ with* (F'), *that $\gamma(\beta) > \delta$ ? Can we take the absolute constant $\delta > 0$ ?*

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Department of Mathematics, Faculty of Science, Niigata University, Ikarashi 2-8050, Niigata, 950-21, JAPAN