TOPOLOGICAL PROPERTIES OF TWO-DIMENSIONAL NUMBER SYSTEMS

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ABSTRACT. In the two dimensional real vector space $\mathbb{R}^2$ one can define analogs of the well-known q-adic number systems. In these number systems a matrix M plays the role of the base number q. In the present paper we study the so-called fundamental domain $\mathcal{F}$ of such number systems. This is the set of all elements of $\mathbb{R}^2$ having zero integer part in their "M-adic" representation. It was proved by Kátai and Kőrnyei, that $\mathcal{F}$ is a compact set and certain translates of it form a tiling of the $\mathbb{R}^2$. We construct points, where three different tiles of this tiling coincide. Furthermore, we prove the connectedness of $\mathcal{F}$ and give a result on the structure of its inner points.

1. INTRODUCTION

In this paper we use the following notations: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of real numbers, rational numbers, integers and positive integers, respectively. If $x \in \mathbb{R}$ we will write $\lfloor x \rfloor$ for the largest integer less than or equal to x. λ will denote the 2-dimensional Lebesgue measure. Furthermore, we write ∂A for the boundary of the set A and $\mathrm{int}(A)$ for its interior. $\mathrm{diag}(\lambda_1, \lambda_2)$ denotes a 2×2 diagonal matrix with diagonal elements λ_1 and λ_2.

Let $q \geq 2$ be an integer. Then each positive integer n has a unique q-adic representation of the shape $n = \sum_{k=0}^{H} a_k q^k$ with $a_k \in \{0, 1, \dots, q-1\}$ $(0 \leq k \leq H)$ and $a_H \neq 0$ for $H \neq 0$. These q-adic number systems have been generalized in various ways. In the present paper we deal with analogs of these number systems in the 2-dimensional real vector space, that emerge from number systems in quadratic number fields. The first major step in the investigation of number systems in number fields was done by Knuth [13], who studied number systems with negative bases as well as number systems in the ring of Gaussian integers. Meanwhile, Kátai, Kovács, Pethő and Szabó invented a general notion of number systems in rings of integers of number fields, the so-called *canonical number systems* (cf. for instance [10, 11, 12, 15]). We recall their definition.

Let K be a number field with ring of integers Z_K. For an algebraic integer $b \in Z_K$ define $\mathcal{N} = \{0, 1, \dots, |N(b)| - 1\}$, where $N(b)$ denotes the norm of b over $\mathbb{Q}$. The pair $(b, \mathcal{N})$ is called a *canonical number system* if any $\gamma \in Z_K$ admits a representation of the shape

$$\gamma = c_0 + c_1 b + \cdots + c_H b^H,$$

where $c_k \in \mathcal{N}$ $(1 \leq k \leq H)$ and $c_H \neq 0$ for $H \neq 0$.

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These number systems resemble a natural generalization of q-adic number systems to number fields. Each of these number systems gives rise to a number system in the n-dimensional real vector space. Since we are only interested in the 2-dimensional case, we construct these number systems only for this case. Consider a canonical number system $(b, \mathcal{N})$ in a quadratic number field K with ring of integers Z_K. Let $p_b(x) = x^2 + Ax + B$ be the minimal polynomial of b. It is known, that for bases of canonical number systems $-1 \leq A \leq B \geq 2$ holds (cf. [10, 11, 12]). Now consider the embedding $\Phi : K \to \mathbb{R}^2$, $\alpha_1 + \alpha_2 b \mapsto (\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \in \mathbb{Q}$. Kovacs [14] proved, that $\{1, b\}$ forms an integral basis of Z_K. Thus we have $\Phi(Z_K) = \mathbb{Z}^2$. Furthermore, note that $\Phi(bz) = M\Phi(z)$ with

$$M = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix}.$$

Since the elements of $\mathcal{N}$ are rational integers, for each $c \in \mathcal{N}$, $\Phi(c) = (c, 0)^T$. Summing up we see, that $(M, \Phi(\mathcal{N}))$ forms a number system in the two dimensional real vector space in the following sense (cf. also [8], where some properties of these number systems are studied). Each $g \in \mathbb{Z}^2$ has a unique representation of the form

$$g = d_0 + Md_1 + \ldots + M^H d_H,$$

with $d_k \in \Phi(\mathcal{N})$ $(1 \leq k \leq H)$ and $d_H \neq (0,0)^T$ for $H \neq 0$. These number systems form the object of this paper. In particular, we want to study the so-called *fundamental domains* of these number systems. The *fundamental domain* of a number system $(M, \Phi(\mathcal{N}))$ is defined by

$$\mathcal{F} = \left\{ z \,\middle|\, z = \sum_{j \geq 1} M^{-j} d_j,\ d_j \in \Phi(\mathcal{N}) \right\}.$$

Sloppily spoken, $\mathcal{F}$ contains all elements of $\mathbb{R}^2$, with integer part zero in their "M-adic" representation. In Figure 1 the fundamental domain corresponding to the M-adic representations arising from the Gaussian integer $-1 + i$ is depicted. This so-called "twin dragon" was studied extensively by Knuth in his book [13].

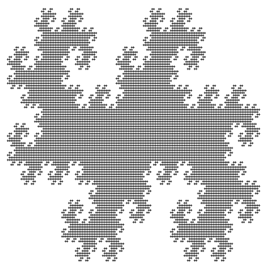

FIGURE 1. The fundamental domain of a number system

Fundamental domains of number systems have been studied in various papers. Kátai and Kőrnyei [9] proved, that $\mathcal{F}$ is a compact set that tesselates the plane in the following way.

$$\bigcup_{g \in \mathbb{Z}^2} (\mathcal{F} + g) = \mathbb{R}^2 \quad \text{where} \quad \lambda((\mathcal{F} + g_1) \cap (\mathcal{F} + g_2)) = 0 \quad (g_1, g_2 \in \mathbb{Z}^2;\ g_1 \neq g_2). \tag{1}$$

Furthermore, we want to mention, that the boundary of $\mathcal{F}$ has fractal dimension. Its Hausdorff and box counting dimension has been calculated by Gilbert [4], Ito [7], Müller-Thuswaldner-Tichy [16] and Thuswaldner [17]. In the present paper we are interested in topological properties of $\mathcal{F}$. Before we give a survey on our results we shall define some basic objects. Let S be the set of all translates of $\mathcal{F}$, that "touch" $\mathcal{F}$, i.e.

$$S := \{g \in \mathbb{Z}^2 \setminus (0,0)^T \,|\, \mathcal{F} \cap (\mathcal{F} + g) \neq \emptyset\}.$$

Then by (1) the boundary of $\mathcal{F}$ has the representation

$$\partial\mathcal{F} = \bigcup_{g \in S} (\mathcal{F} \cap (\mathcal{F} + g)). \tag{2}$$

Hence, the boundary of $\mathcal{F}$ is the set of all elements of $\mathcal{F}$, that are contained in $\mathcal{F} + g$ for a certain $g \neq (0,0)^T$. Of course, $\partial\mathcal{F}$ may contain points, that belong to $\mathcal{F}$ and two other different translates of $\mathcal{F}$. These points we call *vertices* of $\mathcal{F}$. Thus the set of vertices of $\mathcal{F}$ is defined by

$$V := \{z \in \mathcal{F} \,|\, z \in (\mathcal{F} + g_1) \cap (\mathcal{F} + g_2),\ g_1, g_2 \in \mathbb{Z}^2;\ g_1 \neq g_2, g_1 \neq 0, g_2 \neq 0\}.$$

In Section 2 we study the set of vertices of $\mathcal{F}$. It turns out, that, apart from one exceptional case, $\mathcal{F}$ has at least 6 vertices. In some cases we derive that V is an infinite or even uncountable set. In Section 3 we prove the connectedness of $\mathcal{F}$ and show that each element of $\mathcal{F}$, which has a finite M-adic expansion, is an inner point of $\mathcal{F}$.

2. Vertices of the Fundamental Domain $\mathcal{F}$

In this section we give some results on the set of vertices V of $\mathcal{F}$. For number systems emerging from Gaussian integers, similar results have been established with help of different methods in Gilbert [3]. We start with the definition of useful abbrevations. Let

$$g = M^{-H_1} d_{-H_1} + \cdots + M^{H_2} d_{H_2} \tag{3}$$

be the M-adic representation of g. Note, that the digits d_j $(-H_1 \leq j \leq H_2)$ are of the shape $d_j = (c_j, 0)^T \in \Phi(\mathcal{N})$. Thus for the expansion (3) we will write

$$g = c_{H_2} c_{H_2-1} \ldots c_1 c_0 . c_{-1} \ldots c_{H_1}.$$

If the string $c_1 \ldots c_H$ occurs j times in an M-adic representation, then we write $[c_1 \ldots c_H]_j$. If a representation is ultimately periodic, i.e. a string $c_1 \ldots c_H$ occurs infinitely often, we write $[c_1 \ldots c_H]_\infty$. First we show, that for $A > 0$ any fundamental domain $\mathcal{F}$ contains at least 6 vertices.

Theorem 2.1. *Let $(M, \Phi(\mathcal{N}))$ be a number system in $\mathbb{R}^2$, which is induced by the base b of a canonical number system. Let $p_b(x) = x^2 + Ax + B$ with $A > 0$ be the minimal polynomial of b. Then the set of vertices V of the fundamental domain $\mathcal{F}$ of this number system contains the points*

$$\begin{aligned} P_1 &= 0.[0(A-1)(B-1)]_\infty, & P_2 &= 0.[(A-1)(B-1)0]_\infty, \\ P_3 &= 0.[0(B-1)(B-A)]_\infty, & P_4 &= 0.[(B-1)(B-A)0]_\infty, \\ P_5 &= 0.[(B-1)0(A-1)]_\infty, & P_6 &= 0.[(B-A)0(B-1)]_\infty. \end{aligned}$$

Depending on the cases $A = 1$, $1 < A < B$ and $A = B$, the points P_j $(1 \leq j \leq 6)$ belong to the following translates $\mathcal{F} + w$ of $\mathcal{F}$.

	values of w for $1 < A < B$	*values of w for $A = B$*
P_1	$0,\ 1,\ 1A$	$0,\ 1,\ 1(A-1)10$
P_2	$0,\ 1(A-1),\ 1A(B-1)$	$0,\ 1(A-1),\ 1(A-1)10(A-1)$
P_3	$0,\ 1A,\ 1(A-1)$	$0,\ 1(A-1),\ 1(A-1)10$
P_4	$0,\ 1A(B-1),\ 1(A-1)(B-A)$	$0,\ 1A(A-1),\ 1(A-1)0$
P_5	$0,\ 1(A-1)(B-A+1),\ 1$	$0,\ 1(A-1)1,\ 1$
P_6	$0,\ 1(A-1)(B-A),\ 1(A-1)(B-A+1)$	$0,\ 1(A-1)1,\ 1(A-1)0$

The case $A = 1$ is very similar to the case $1 < A < B$; just replace the representation $1(A-1)(B-A+1)$ by $11(B-1)0$ in the above table.

Remark 2.1. *Note, that we have $0 < A \leq B \geq 2$. Hence the digits of the 6 points indicated in Theorem 2.1 are all admissible.*

Proof of the theorem. We will prove that each of the 6 points $P_1, \dots, P_6$ is contained in three different translates of $\mathcal{F}$, as indicated in the statement of the theorem. First we consider the point P_1. Write $\overline{x} = -x$. By using $b^2 + Ab + B = 0$, we see that

$$0.1(A-1)(B-A)\overline{B} = 0.1[(A-1)(B-A)\overline{(B-1)}]_\infty = 0 \tag{4}$$

are formal representations of zero. Adding the second representation for 0 given in (4) twice, we have

$$\begin{aligned} P_1 &= 0.[0(A-1)(B-1)]_\infty + 1.[(A-1)(B-A)\overline{(B-1)}]_\infty \\ &= 1.[(A-1)(B-1)0]_\infty \\ &= 1.[(A-1)(B-1)0]_\infty + 1(A-1).[(B-A)\overline{(B-1)}(A-1)]_\infty \\ &= 1A.[(B-1)0(A-1)]_\infty. \end{aligned}$$

For $A < B$ this yields

$$P_1 \in \mathcal{F} \cap (\mathcal{F} + 1) \cap (\mathcal{F} + 1A).$$

For $A = B$ the last expansion $1A.[(B-1)0(A-1)]_\infty$ is not admissible since $A > B - 1$. In order to settle this case we use the first representation of zero given in (4) to get $1A = 1B = 1B + 1(B-1)0\overline{B} = 1(A-1)10$. As a result, we have

$$P_1 \in \mathcal{F} \cap (\mathcal{F} + 1) \cap (\mathcal{F} + 1(A-1)10)$$

for $A = B$. Since $P_2 = MP_1$, we get the desired results also for P_2. Now we treat

$$P_3 = 0.[0(B-1)(B-A)]_\infty.$$

In the same way as before, we get, using both representations of zero in (4)

$$\begin{aligned} P_3 &= 0.[0(B-1)(B-A)]_\infty + 1A.B - 0.1[(A-1)(B-A)\overline{(B-1)}]_\infty \\ &= 1A.[(B-1)(B-A)0]_\infty \\ &= 1A.[(B-1)(B-A)0]_\infty - 1.[(A-1)(B-A)\overline{(B-1)}]_\infty \\ &= 1(A-1).[(B-A)0(B-1)]_\infty, \end{aligned}$$

which implies

$$P_3 \in \mathcal{F} \cap (\mathcal{F} + 1A) \cap (\mathcal{F} + 1(A-1))$$

for $A < B$ and

$$P_3 \in \mathcal{F} \cap (\mathcal{F} + 1(A-1)10) \cap (\mathcal{F} + 1(A-1))$$

for $A = B$. Since $\mathcal{F}$ permits an involution $\varphi : x \to \sum_{j\geq 1} M^{-j}(B-1,0)^T - x$, $\mathcal{F}$ is symmetric with respect to the center $\frac{1}{2}\sum_{j\geq 1} M^{-j}(B-1,0)^T$. For $w \in \mathbb{Z}^2$ this map sends each $\mathcal{F}+w$ to $\mathcal{F}-w$. Thus we have

$$\begin{aligned}
\varphi(\mathcal{F}+1) &= \mathcal{F} + 1A(B-1),\\
\varphi(\mathcal{F}+1(A-1)) &= \begin{cases} \mathcal{F}+1(A-1)(B-A+1) & \text{for } A>1,\\ \mathcal{F}+11(B-1)0 & \text{for } A=1,\end{cases}\\
\varphi(\mathcal{F}+1A) &= \mathcal{F}+1(A-1)(B-A),
\end{aligned}$$

for $A < B$ and

$$\begin{aligned}
\varphi(\mathcal{F}+1) &= \mathcal{F}+1(A-1)10(A-1),\\
\varphi(\mathcal{F}+1(A-1)) &= \mathcal{F}+1(A-1)1,\\
\varphi(\mathcal{F}+1(A-1)10) &= \mathcal{F}+1(A-1)0,
\end{aligned}$$

for $A = B$. Furthermore, it is easy to see, that $\varphi(P_1) = P_4$, $\varphi(P_2) = P_5$ and $\varphi(P_3) = P_6$ Thus also P_4, P_5 and P_6 are vertices of $\mathcal{F}$ that are contained in the translates of $\mathcal{F}$ indicated in the statement of the theorem. □

In the case $A = 0$ it is easy to see that $\mathcal{F}$ is a square. It has exactly 4 vertices. These are the "usual" vertices of the square. Thus we only have to deal with the case $A = -1$. We will folmulate the corresponding result as a corollary.

Corollary 2.1. *Let the same settings as in Theorem 2.1 be in force, but assume now, that* $A = -1$. *Then the following table gives* 6 *points* P_j $(1 \leq j \leq 6)$, *that are contained in the set of vertices* V *of* $\mathcal{F}$. *Furthermore, we give the translates* $\mathcal{F}+w$, *to which* P_j *belongs.*

P_j	*translates* w, *for which* $P_j \in \mathcal{F}+w$
$0.[0(B-1)(B-1)(B-1)00]_\infty$	$0, 10(B-1), 10(B-1)(B-1)$
$0.[000(B-1)(B-1)(B-1)]_\infty$	$0, 1, 10$
$0.[00(B-1)(B-1)(B-1)0]_\infty$	$0, 10, 10(B-1)$
$0.[(B-1)000(B-1)(B-1)]_\infty$	$0, 1, 10(B-1)(B-1)1$
$0.[(B-1)(B-1)000(B-1)]_\infty$	$0, 10(B-1)(B-1), 10(B-1)(B-1)1$
$0.[(B-1)(B-1)(B-1)000]_\infty$	$0, 10(B-1)(B-1)0, 10(B-1)(B-1)1$

Proof. Let $M_1 = \begin{pmatrix} 0 & -B \\ 1 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & -B \\ 1 & 1 \end{pmatrix}$ be bases of number systems in $\mathbb{R}^2$and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the fundamental domains corrresponding to M_1 and M_2, respectively. We know the vertices of $\mathcal{F}_1$ from Theorem 2.1 and will construct the vertices of $\mathcal{F}_2$ from it. To this matter let $M_1 = G_1 \mathrm{diag}(b_1, b_2) G_1^{-1}$. It is easy to see, that then $M_2 = G_2 \mathrm{diag}(-b_1, -b_2) G_2^{-1}$ with $G_2 = \mathrm{diag}(-1,1)G_1$. Now suppose, that $\sum_{k\geq 1} M_1^{-k} a_j \in \mathcal{F}_1 \cap \mathcal{F}_1 + (v_1, v_2)^T \cap \mathcal{F}_1 +$

$(w_1, w_2)^T$ with $v_1, v_2, w_1, w_2 \in \mathbb{Z}$ is a vertex of $\mathcal{F}_1$. Using the fact, that $G_1^{-1} a_k = -G_2^{-1} a_k$ for $a_k \in \mathcal{M}$ and setting $d = 0.[0(B-1)]_\infty$ we easily derive that

$$Q := \sum_{k \geq 1} (-1)^{k+1} M_2^{-k} a_k + d \in \operatorname{diag}(-1,1)(\mathcal{F}_1 \cap \mathcal{F}_1 + (v_1, v_2)^T \cap \mathcal{F}_1 + (w_1, w_2)^T) + d \tag{5}$$

Observe, that by the selection of d, Q has an admissible M_2-adic representation with integer part zero. Thus $Q \in \mathcal{F}_2$. Since any element of $\mathcal{F}_2$ can be constructed from elements of $\mathcal{F}_1$ in the same way we conclude, that $\mathcal{F}_2 = \operatorname{diag}(-1,1)\mathcal{F}_1 + d$. But with that (5) reads $Q \in \mathcal{F}_2 \cap \mathcal{F}_2 + (-v_1, v_2)^T \cap \mathcal{F}_2 + (-w_1, w_2)^T$. Thus Q is a vertex of $\mathcal{F}_2$. The representations in the table above, can now easily be obtained from the results for $A = 1$ in Theorem 2.1. $\square$

The following corollary is an immediate consequence of Theorem 2.1 and Corollary 2.1.

Corollary 2.2. *For $1 < A < B$ we have*

$$S \supset \{1, 1A, 1(A-1), 1A(B-1), 1(A-1)(B-A), 1(A-1)(B-A+1)\},$$

for $A = B$

$$S \supset \{1, 1(A-1)10, 1(A-1), 1(A-1)10(A-1), 1(A-1)0, 1(A-1)1\},$$

while for $A = 1$

$$S \supset \{1, 10, 10(B-1), 10(B-1)(B-1), 10(B-1)(B-1)0, 10(B-1)(B-1)1\}$$

holds.

Remark 2.2. *Note, that "$\supset$" may be replaced by "$=$" in Corollary 2.2 if $2A < B+3$. This is shown for the Gaussian case in* [16]. *For arbitrary quadratic number fields this fact can be proved in a similar way.*

Theorem 2.2. *Let the same settings as in Theorem 2.1 be in force. If $2A = B+3$ then $\mathcal{F}$ has infinitely many vertices.*

Proof. Set $K = B - A + 1 = \frac{B-1}{2}$. Then, using $b^2 + Ab + B = 0$, we get $(j \geq 0)$

$$\begin{aligned} 0 &= \sum_{k=2}^{\infty} (-1)^k \left(M^{-k+2}(1,0)^T + M^{-k+1}(A,0)^T + M^{-k}(B,0)^T \right) \\ &= 1.(A-1)[K\overline{K}]_\infty. \end{aligned} \tag{6}$$

Here we set $\overline{x} = -x$, as before. We will show, that the points

$$Q_j = 1A.[(B-1)0(A-1)]_{2j}(B-1)0[K]_\infty \quad (j \in \mathbb{N}) \tag{7}$$

are vertices of $\mathcal{F}$. Therefore we need the representation (6). With help of this representation we define the following representations of zero.

$$\begin{aligned} N_1 &:= 1(A-1).[K\overline{K}]_\infty = 0, \\ N_2 &:= 1.(A-1)[K\overline{K}]_\infty = 0, \\ X_j &:= 0.[0]_j 1AB = 0 \quad (j \geq 0). \end{aligned}$$

In the sequel we write kX_j ($k \in \mathbb{Z}$) if we want to multiply each digit of the representation X_j by k. Furthermore, addition and subtraction of representations is always meant digit-wise. After these definitions we define the following, more complicated representations of zero.

$$\begin{aligned}
Z_1(j) &:= N_1 + \sum_{k=1}^{j} (X_{6k-1} - 2X_{6k-2} + 2X_{6k-3} - X_{6k-4}) + (1.AB) - 2(1A.B) \\
&= \overline{1}\overline{A}.[(\overline{B-1})(A-1)(B-A)]_{2j}(\overline{B-1})(A-1)[K\overline{K}]_\infty, \\
Z_2(j) &:= N_2 + \sum_{k=1}^{j} (-X_{6k-3} + 2X_{6k-4} - 2X_{6k-5} + X_{6k-6}) - (1A.B) \\
&= \overline{1}(\overline{A-1}).(\overline{B-A})[(B-1)(\overline{A-1})(\overline{B-A})]_{2j-1}(B-1)(\overline{A-1})\overline{K}[K\overline{K}]_\infty.
\end{aligned}$$

Finally, we observe, that for $j \in \mathbb{N}$

$$\begin{aligned}
Q_j &= Q_j + Z_1(j) \\
&= 0.[0(A-1)(B-1)]_{2j}0(A-1)[(B-1)0]_\infty \\
&= Q_j + Z_2(j) \\
&= 1.(A-1)[(B-1)0(A-1)]_{2j-1}(B-1)0KK[0(B-1)]_\infty,
\end{aligned}$$

and this implies $Q_j \in V$. It remains to show, that the elements Q_j, $j \geq 1$, are pairwise different. This follows from the following observation. Select $k \in \mathbb{N}$ arbitrary and let $j_1, j_2 \leq k$. Suppose, that Q_{j_1} and Q_{j_2} are represented by the representation (7) for $j = j_1$ and $j = j_2$, respectively. Then $Q_{j_1} = Q_{j_2}$ if and only if $M^{6k+2}Q_{j_1} = M^{6k+2}Q_{j_2}$. For $k \geq \max(j_1, j_2)$, $M^{6k+2}Q_{j_1}$ and $M^{6k+2}Q_{j_2}$ have the same digit string $[0(B-1)]_\infty$ after the comma. Hence, they can only be equal, if their integer parts are equal. But since $(M, \Phi(\mathcal{N}))$ is a number system, this can only be the case, if the digit strings of their integer parts are the same. This implies $j_1 = j_2$. So we have proved, that the points Q_j are pairwise different for $j \leq k$. Since k can be selected arbitrary, the result follows. Thus we found infinitely many different vertices of $\mathcal{F}$. □

Theorem 2.3. *Let the same settings as in Theorem 2.1 be in force. If $2A > B + 3$ then $\mathcal{F}$ has uncountably many vertices.*

Proof. Set $K = B - A + 1$ and $\xi = \lfloor (B-1)/2 \rfloor$. As $\xi + K, \xi - K \in \mathcal{N}$, by using (6), we see that

$$\begin{aligned}
0.[\xi]_\infty &= 1(A-1).[(\xi+K)(\xi-K)]_\infty \\
&= 1(A-1)K.[(\xi-K)(\xi+K)]_\infty.
\end{aligned}$$

Thus $0.[\xi]_\infty$ is a vertex of $\mathcal{F}$. Fix an integer k, such that all eigenvalues of M^k are greater than 2 (such an integer exists, since the eigenvalues of M are all greater than 1). This implies, that the representations $0.c_1[0]_k c_2[0]_k c_3[0]_k c_4 \ldots$, $c_j \in \{0,1\}$ ($j \geq 1$) represent pairwise different elements of $\mathbb{R}^2$ for different $\{0,1\}$ sequences $\{c_j\}_{j\geq 1}$. Because $\xi + K < B - 1$, each of the uncountably many representations

$$0.[\xi]_\infty + 0.c_1[0]_k c_2[0]_k c_3[0]_k c_4 \ldots \qquad (c_j \in \{0,1\},\, j \geq 1)$$

corresponds to a vertex of $\mathcal{F}$. Since they are pairwise different, the theorem is proved. □

3. Connectedness and Inner Points of the Fundamental Domain $\mathcal{F}$

In this section we will show, that the fundamental domain $\mathcal{F}$ is arcwise connected. To establish this result, we will apply a general theorem due to Hata (cf. [5, 6]) which assures arcwise connectedness for a large class of sets. The second result of this section is devoted to the structure of the inner points of $\mathcal{F}$. In particular, we prove, that each point with finite M-adic representation is an inner point of $\mathcal{F}$. In this section we will use the notation

$$\mathcal{F}_k := \left\{ z \,\middle|\, z = \sum_{j=1}^{k} M^{-j} a_j,\ a_j \in \Phi(\mathcal{N}) \right\} \qquad (k \in \mathbb{N}).$$

We start with the connectedness result.

Theorem 3.1. *Let $(M, \Phi(\mathcal{N}))$ be a number system in $\mathbb{R}^2$, which is induced by the base b of a canonical number system in a quadratic number field. Then the fundamental domain $\mathcal{F}$ of $(M, \Phi(\mathcal{N}))$ is arcwise connected.*

Proof. It is an easy consequence of the definition of $\mathcal{F}$, that

$$\mathcal{F} = \bigcup_{g \in \Phi(\mathcal{N})} M^{-1}(\mathcal{F} + g). \tag{8}$$

Furthermore, Theorem 2.1 implies that $\mathcal{F} \cap (\mathcal{F} + (1,0)^T) \neq \emptyset$. Thus the sets contained in the union of (8) form a *chain* in the sense that $(\mathcal{F} + g) \cap (\mathcal{F} + (g + (1,0)^T)) \neq \emptyset$ for $g \in \Phi(\mathcal{N}) \setminus (B-1,0)^T$. Thus $\mathcal{F}$ fulfills the conditions being necessary for the application of a theorem of Hata, namely [5, Theorem 4.6]. This theorem yields the arcwise connectedness of $\mathcal{F}$. □

Now we prove the result on the inner points of $\mathcal{F}$. Note, that the existence of inner points is an immediate consequence of [9, Theorem 1].

Theorem 3.2. *Let $(M, \Phi(\mathcal{N}))$ be a number system in $\mathbb{R}^2$, which is induced by the base b of a canonical number system in a quadratic number field. Then for each $k \in \mathbb{N}$ we have*

$$\mathcal{F}_k \subset \operatorname{int}(\mathcal{F}).$$

Proof. First we will show, that 0 is an inner point of $\mathcal{F}$. Suppose, that 0 is contained in the boundary of $\mathcal{F}$. Then by (2) there exists a representation of zero of the shape

$$0 = c_{H_1} c_{H_1 - 1} \dots c_1 c_0 . c_{-1} c_{-2} \dots. \tag{9}$$

This representation implies $0 \in \mathcal{F} + c_{H_1} c_{H_1-1} \dots c_1 c_0$. If we multiply (9) by M^j for $j \in \mathbb{N}$ arbitrary, we conclude, that $0 \in \mathcal{F} + c_{H_1} c_{H_1-1} \dots c_1 c_0 c_{-1} \dots c_{-j}$ for each $j \in \mathbb{N}$. Hence, 0 is contained in infinitely many different translates of $\mathcal{F}$. But since $\mathcal{F}$ is a compact set this is a contradiction to (1). Thus $0 \in \operatorname{int}(\mathcal{F})$.

Now fix $k \in \mathbb{N}$ and $g \in \mathcal{F}_k$. Then $0 \in \mathrm{int}(\mathcal{F})$ implies, that $g \in \mathrm{int}(M^{-k}\mathcal{F} + g)$. The result now follows from the representation

$$\mathcal{F} = \bigcup_{g \in \mathcal{F}_k} (M^{-k}\mathcal{F} + g).$$

□

There is a direct alternative proof of this theorem by using the methods of [1] and [2]. In these papers a similar result for the tiling generated by Pisot number systems is shown.

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