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A Note on Hecke's absolute invariants

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Abstract : Let J_q be Hecke's absolute invariant. The *n-th* Fourier coefficient of J_q is written in the form $a_q(n)r_q^n$, where $a_q(n) \in Q$, $r_q \in R$. We regard $a_q(n)$ as a rational function of q. In § 3, it is proved that $a_q(n)$ is divisible by q^2-4 for $n \geq 2$. This implies the conjecture of J. Raleigh [4]. Prime divisors of the denominator of $a_q(n)$ are also treated in § 4. In the last section, we propose interesting conjectures concerning $a_q(n)$.

§1. Introduction

Let $G(q)$ $(q \geq 3)$ be the Hecke group, which is the properly discontinuous subgroup of SL $(2,R)$ generated by:

$$\begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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where $\lambda_q = 2\cos(\pi/q)$. The set of automorphic forms with respect to $G(q)$ forms a rational function field over $\mathbb{C}$ with a generator $J = J_q(z)$. We choose $J_q(z)$ in this manner:

$$J_q(\rho) = 0,\ J_q(\sqrt{-1}) = 1 \text{ and } J_q(\sqrt{-1}\infty) = \infty, \qquad \text{.......(1)}$$

where $\rho = -\exp(-\pi\sqrt{-1}/q)$. The function $J = J_q(z)$ is known as "Hecke's absolute invariant" of $G(q)$. The Fourier expansion of $J_q(z)$ around $\sqrt{-1}\infty$ is given in the form

$$J_q(z) = \sum_{n \geq -1} a_q(n)\ r_q^{\,n} \exp(2\pi\sqrt{-1}nz/\lambda_q)$$

where $a_q(n) \in \mathbb{Q}$ and $r_q \in R$ (J. Raleigh [4]). By definition, r_q is determined up to a rational factor. For the case q =3,4, 6 and ∞, we put $D_q = 12^3,\ 2^8,\ 2^2 \cdot 3^3$ and 2^6 respectively. Then we have $r_q = 1$ and $a_q(n) \in Z/D_q$. The number r_q is transcendental in other cases (J. Wolfart [5]). The expansion of this type also appears, if we start from general Fuchsian triangle groups (J. Wolfart [6], [7]). Using the value r_q, we can construct an explicit analytic invariant with respect to an inclusion relation of groups (S. Akiyama [1]).

In this note, we are concerned with two aspects of $a_q(n)$ as a rational function of q. First we show the conjecture of J. Raleigh [4], which states

numerator of $a_q(n) \equiv 0 \pmod{(q^2-4)\mathbb{Q}[q]}$ if $n \equiv \pm 2 \pmod 5$,

is valid. Moreover we show

numerator of $a_q(n) \equiv 0 \pmod{(q^2-4)\mathbb{Q}[q]}$ if $n \geq 2$.

Second we consider the denominator of $a_q(n)$. We can see that the possible prime factors that appear in the denominator are bounded by $n+1$. We shall state some conjectures in the final section.

§ 2. Schwarzian derivative and J_q functions

In this section, we give a brief review of the differential equation satisfied by $J_q(z)$. By definition, the correspondence $z \to w = J_q(z)$ induces a conformal mapping between the standard fundamental domain of $G(q)$, which is a circular triangle with vertices ρ, $-\bar{\rho}$ and $\sqrt{-1}\infty$, to the complex plane C. By (1), we see that the boundary of the triangle of vertices ρ, $\sqrt{-1}$ and $\sqrt{-1}\infty$ is mapped to the real axis and the inner part of this triangle is mapped to the upper half plane H. (Using the reflexion principle repeatedly, we can redefine the value of $J_q(z)$ on H.) Let us consider the inverse correspondence $w \to z$. If we restrict $w \in H$, then we can quote the classical result of Schwarz:

Let H *be mapped conformally onto an arbitrary circular triangle whose angles at its vertices* A, B *and* C *are* πa, πb *and* πc, *and let the vertices A, B, C be the images of the point* w =0, 1, ∞ *respectively. Then* $z(w)$ *must be the solution of the differential equation*

$$\{z(w)\,;\,w\} = \frac{1-a^2}{2w^2} + \frac{1-b^2}{2(1-w)^2} + \frac{1-a^2-b^2+c^2}{2w(1-w)} \qquad \text{..........(2)}$$

where $\{z,w\}$ *is the Schwarzian derivative defined by*

$$\{z\,;\,w\} = z'''/z' - \frac{3}{2}(z''/z')^2$$

(See Carathéodory [3], p. 129-135).

In our case, we have $a=1/q$, $b=1/2$, and $c=0$. We can easily check that

$$\{z;\,w\}\,(dw)^2 + \{w\,;\,z\}\,(dz)^2 = 0. \qquad ....(3)$$

Using (2) and (3), we have

$$\frac{3(J'')^2 - 2\,J'J'''}{(J')^4} = \frac{4q^2J^2 - (5q^2-4)\,J + 4\,q^2 - 4}{4q^{2\,2}\,(1-J)^2}. \qquad ....(4)$$

We put

$$J_q(z) = \sum_{n\geq -1} c_n\, x^n \text{ and } x = \exp(2\pi\sqrt{-1}z/\lambda_q). \qquad ....(5)$$

Each c_n is written in the form

$$\frac{P_n(q^2)}{Q_n c_{-1}^{n}\, q^{2(n+1)}}, \qquad ....(6)$$

where $Q_n \in Z$ and $P_n(t)$ is a polynomial in t of degree not larger than $n+1$ whose coefficients are rational integers. To see this, we replace q, z with $1/q$, $z+(\lambda_q/2\pi\sqrt{-1})\log c_{-1}$ respectively and compare both sides of (4). Assume that the greatest common divisor of the coefficients of P_n is coprime to Q_n. Let $q=\infty$. Then J_∞ is algebraically related to J_3;

$$27\,J_3\,J_\infty = (4\,J_\infty - 1)^3.$$

Utilizing the well known expression of J_3 by the λ-function we have

$$2^6 J_\infty = \frac{16}{\lambda(\lambda-1)}$$
$$= x_\infty^{-1} \prod_{n\geq 1} (1 + x_\infty^{2n-1})^{24},$$

where $x_\infty = \exp(\pi\sqrt{-1}z)$. The final expression is deduced from the expression of the λ-function by theta-null series. So every Fourier coefficient of J_∞ is not zero. Thus the degree of P_n is always $n+1$. Putting $r_q = 1/c_{-1}$ then we get the expression mentioned in the introduction. Using the theory of the hypergeometric differential equation, the actual values of r_q are calculated explicitly (See [4]).

§ 3. A conjecture of J. Raleigh

Now we prove the conjecture of J. Raleigh [4] cited in the introduction.

Proposition 1. *For $n \geq 2$, the numerator of $a_q(n)$ is divisible by q^2-4.*

Proof. In fact (4) has a solution for $q=3$, so that $Q_n \neq 0$. Thus the third order differential equation (4) has a solution in formal power series of the form

$$\sum_{n\geq -1} d_n x^n \quad (d_n \in \mathbf{C}(\tau),\, d_{-1} \in \mathbf{C}^*,\, x = \exp(2\pi\sqrt{-1}z/\lambda),\, \lambda \in R^*),$$

where τ is an indeterminate. Note that if we fix d_{-1} then the solution of the differential equation (4) is determined uniquely and d_n does not depend on the choice of $\lambda \in R^*$. If we restrict ourselves the case $\tau = q = 3, 4, \cdots, \infty$ and $\lambda = \lambda_q$, then this power series is equal to J_q and $d_n = c_n$. In the following, we treat $\tau = q$ as a non zero real variable. Put

$$J_2 = d_{-1}/x + 1/2 + x/(16 d_{-1}).$$

Replace J with J_2 in (4). Then we get

$$4q^2\left(3(d_{-1}+x^2/16d_{-1})^2-2(-d_{-1}+x^2/16d_{-1})^2\right)\left(d_{-1}+x/2+x^2/16d_{-1}\right)^2$$
$$\cdot(d_{-1}-x/2+x^2/16d_{-1})^2-(-d_{-1}+x^2/16d_{-1})^4(4q^2(d_{-1}+x/2+x^2/16d_{-1})^2$$
$$-(5q^2-4)(d_{-1}+x/2+x^2/16d_{-1})x+(4q^2-4)x^2\Big) = 0.$$

After a laborious calculation we have

$$\frac{(4d_{-1}+x)^4 \quad (4d_{-1}-x)^6(q^2-4)x}{2^{20} d_{-1}^5} = 0$$

This shows that when $q = \pm 2$ then J_2 is a solution of (4), and $J_2 \in \mathbb{C}(x)$ has no terms in x^n ($n \geq 2$). So if we regard d_n as a rational function of τ, then d_n is divisible by τ^2-4 for $n \geq 2$. This fact shows the validity of the assertion.

§ 4. The denominator of the J_q functions

Using (4), we also have

Proposition 2. *For $n \geq 1$, the prime divisors of Q_n are not larger than $n+1$.*

Proof. The statement is true for $n=1$ because $Q_1 = 1024 = 2^{10}$ (see table I). Rewrite (4) in the following form

$$4q^2\left(3(xJ'')^2 - 2(xJ')(xJ''')\right)(xJ)^2\,(xJ-x)^2$$
$$-(xJ')^4\left(4q^2(xJ)^2 - (5q^2-4)(xJ)\,x + (4q^2-4)x^2\right) = 0 \qquad \ldots(7)$$

where $x = \exp(2\pi\sqrt{-1}z/\lambda_q)$. Use the expansion (5) and regard the left hand side of (7) as a formal power series of x. Multiplying both sides by $(\lambda_q/2\pi\sqrt{-1})^4$, the coefficient of x^n is a polynomial in $q, c_{-1}, c_0, \cdots c_{n-1}$, with rational integer coefficients, which is of degree 1 with respect to c_{n-1}. Hence we write

$$\text{the left hand side of (7)} = \Sigma\ \xi_n x^n,$$

and $\xi_n = c_{n-1}\eta_n + \tau_n$

where η_n, $\tau_n \in Z[q, c_{-1}, \cdots, c_{n-2}]$. Let us calculate η_n explicitly, for $n \geq 3$:

$$\eta_n = 4q^2\left(3(2c_{-1}(n-1)^2) - 2(-c_{-1}(n-1)^3 - c_{-1}(n-1))\right)c_{-1}^4$$
$$+ 8q^2(3c_{-1}^2 - 2c_{-1}^2)2c_{-1}^3 - 4\left(-c_{-1}^3(n-1)\right)\cdot 4q^2c_{-1}^2 - c_{-1}^4 \cdot 8q^2 c_{-1}$$
$$= 8q^2n^3c_{-1}^5.$$

Applying this formula, we can prove the proposition by induction.

§ 5. Some conjectures.

Using the differential equation (4) repeatedly, we get $a_q(n)$ for $n < 15$ by machine calculation. We use the notation (6) and put

$$P_n(t) = \sum_{i=0}^{n+1} R_n(i)t^i \quad (R_n(i) \in Z).$$

The results are listed in the table I and table II. We can make some conjectures from these.

Conjecture 1. $R_n(i)$ is divisible by $2^{2(n+1-i)}$. Moreover, $R_n(0)$ is divisible by 2 exactly $2(n+1)$ times and $R_n(n+1)$ is an odd number.

Conjecture 2.

$$R_n(0)/R_n(1) = 12/(n+1). \qquad (!)$$

These conjectures imply the following interesting assertions. (We use the notation used in the introduction. Thus, for $q = 3, 4, 6$ and ∞, we have $r_q = 1$ and $D_q \cdot a_q(n) \in Z$.)

Proposition 3. *If conjectue 1 is true, we have*

$$\mathrm{ord}_2(D_3 a_3(n)) = \mathrm{ord}_2(D_4 a_4(n)) = \mathrm{ord}_2(D_\infty a_\infty(n)).$$

Proposition 4. *If* $3^4 \nmid R_n(0)$ *and conjecture 2 is true, we have*

$$\mathrm{ord}_3(D_3 a_3(n)) = \mathrm{ord}_3(D_6 a_6(n)).$$

Here we denote by $\mathrm{ord}_p(n)$ the multiplicity of the prime divisor p in the prime factorization of n.

Proof of the propositions 3 and 4.

Note that $C_{-1} = D_q^{-1}$ in the case $q = 3, 4, 6$ and ∞. By (6), we have $\mathrm{ord}_2(a_3(n)) = \mathrm{ord}_2\left(\dfrac{P_n(9)}{Q_n D_3^{-n} 3^{2(n+1)}}\right)$

$$= n\,\mathrm{ord}_2(D_3) + \mathrm{ord}_2(P_n(9)) - \mathrm{ord}_2(Q_n)$$

$$= 6n + \mathrm{ord}_2(R_n(n+1)) - \mathrm{ord}_2(Q_n)$$

$$= 6n - \mathrm{ord}_2(Q_n).$$

$$\mathrm{ord}_2(a_4(n)) = \mathrm{ord}_2\left(\frac{P_n(4)}{Q_n D_4^{-n} 4^{2(n+1)}}\right)$$

$$= n\,\mathrm{ord}_2(D_4) + \mathrm{ord}_2(R_n(0)) - \mathrm{ord}_2(Q_n) - 4(n+1)$$

$$= 8n + 2(n+1) - \mathrm{ord}_2(Q_n) - 4(n+1)$$

$$= 6n - 2 - \mathrm{ord}_2(Q_n).$$

$$\mathrm{ord}_2(a_\infty(n)) = \mathrm{ord}_2\left(\frac{R_n(n+1)}{Q_n D_\infty^{-n}}\right)$$

$$= 6n - \mathrm{ord}_2(Q_n)$$

So $\mathrm{ord}_2(D_3 a_3(n)) = \mathrm{ord}_2(D_4 a_4(n)) = \mathrm{ord}_2(D_\infty a_\infty(n)) = 6(n+1) - \mathrm{ord}_2(Q_n)$. This proves proposition 3. Noting

$$\mathrm{ord}_3(R_n(0)) - \mathrm{ord}_3(R_n(1)) \leq 1,$$

We can show proposition 4 in a similar way. Anyhow the conjectures 1 and 2 are well worth consideration. We shall discuss the conjecture 1 in the forthcoming paper [2].

Table (I)

n	Q_n
0	2^{3}
1	2^{10}
2	$2^{7} \cdot 3^{3}$
3	2^{23}
4	$2^{16} \cdot 3^{3} \cdot 5^{3}$
5	$2^{33} \cdot 3^{6}$
6	$2^{29} \cdot 3^{2} \cdot 5^{3} \cdot 7^{3}$
7	$2^{48} \cdot 3^{6} \cdot 5^{3}$
8	$2^{37} \cdot 3^{8} \cdot 5^{3} \cdot 7^{3}$
9	$2^{58} \cdot 3^{4} \cdot 5^{6} \cdot 7^{3}$
10	$2^{54} \cdot 3^{8} \cdot 5^{2} \cdot 7^{3} \cdot 11^{3}$
11	$2^{71} \cdot 3^{11} \cdot 5^{6} \cdot 7^{3}$
12	$2^{62} \cdot 3^{9} \cdot 5^{6} \cdot 7^{3} \cdot 11^{3} \cdot 13^{3}$
13	$2^{81} \cdot 3^{12} \cdot 5^{6} \cdot 7^{6} \cdot 11^{3}$
14	$2^{76} \cdot 3^{15} \cdot 5^{9} \cdot 7^{2} \cdot 11^{3} \cdot 13^{3}$
15	$2^{96} \cdot 3^{11} \cdot 5^{4} \cdot 7^{6} \cdot 11^{3} \cdot 13^{3}$

Table (II)

n	i	$R_n(i)$
0	0	2^{2}
	1	3

1	0	$-2^{4} \cdot 3$
	1	-2^{3}
	2	$3 \cdot 23$
2	0	2^{6}
	1	2^{4}
	2	$-2^{2} \cdot 29$
	3	3^{3}
3	0	$-2^{8} \cdot 3 \cdot 101$
	1	$-2^{8} \cdot 101$
	2	$2^{5} \cdot 5 \cdot 1039$
	3	$-2^{4} \cdot 3821$
	4	$3 \cdot 1867$
4	0	$2^{10} \cdot 3 \cdot 373$
	1	$2^{8} \cdot 5 \cdot 373$
	2	$-2^{7} \cdot 113 \cdot 193$
	3	$2^{5} \cdot 5 \cdot 19 \cdot 419$
	4	$-2^{2} \cdot 3 \cdot 13 \cdot 1259$
	5	$3^{4} \cdot 5^{3}$
5	0	$-2^{12} \cdot 19 \cdot 251 \cdot 997$
	1	$-2^{11} \cdot 19 \cdot 251 \cdot 997$
	2	$2^{8} \cdot 3 \cdot 68420351$
	3	$-2^{8} \cdot 2593 \cdot 40939$
	4	$2^{4} \cdot 13 \cdot 23 \cdot 67 \cdot 16901$
	5	$-2^{3} \cdot 3^{3} \cdot 2254159$
	6	$3^{6} \cdot 23003$

6	0	2^14 · 3 · 8241137
	1	2^12 · 7 · 8241137
	2	-2^10 · 5 · 7^2 · 1489 · 3181
	3	2^8 · 3^2 · 401 · 719723
	4	-2^6 · 7 · 11 · 31406659
	5	2^4 · 7^2 · 127 · 185483
	6	-2^2 · 3 · 443 · 213947
	7	3^3 · 5^5 · 7^3
7	0	-2^16 · 3 · 37 · 4480225531
	1	-2^17 · 37 · 4480225531
	2	2^14 · 281 · 22269851869
	3	-2^13 · 3 · 5 · 37 · 97 · 136878251
	4	2^9 · 7 · 11 · 17 · 31 · 748699387
	5	-2^9 · 7 · 23 · 359 · 1621 · 46171
	6	2^6 · 3 · 7 · 79 · 1741850399
	7	-2^5 · 3^4 · 3307958239
	8	3^7 · 5^3 · 337 · 1861
8	0	2^18 · 15031 · 8306071
	1	2^16 · 3 · 15031 · 8306071
	2	-2^16 · 3^2 · 11 · 23 · 2477 · 295517
	3	2^14 · 3 · 1483 · 911092657
	4	-2^11 · 1660103 · 5394731
	5	2^9 · 3 · 1918266195107
	6	-2^8 · 1168380534109
	7	2^6 · 83 · 3596927419
	8	-2^2 · 3^2 · 19594805623
	9	3^8 · 5^3 · 7^3 · 41

9	0	-2^20 · 3 · 167 . 136952705369261
	1	-2^19 · 5 · 167 · 136952705369261
	2	2^16 · 3 · 8116307 · 158278346579
	3	-2^17 · 3^2 · 5 · 130211 · 203701880699
	4	2^13 · 5530906269839607607
	5	-2^12 · 3^2 · 5 · 1361 · 31641295099097
	6	2^9 · 3 · 4147229 · 145192913161
	7	-2^9 · 5 · 23 · 43 · 238001 · 120902413
	8	2^4 · 3 · 769 · 101535052634657
	9	-2^3 · 3^3 · 5^3 · 73037 · 57924653
	10	3^5 · 5^7 · 7^3 · 241303
10	0	2^22 · 3^2 · 9343 · 22343 · 67906849
	1	2^20 · 3 · 11 · 9343 · 22343 · 67906849
	2	2^18 · 3^4 · 11 · 19 · 71 · 383 · 16257195281
	3	2^16 · 3 · 11 · 59 · 373 · 116539 · 220926467
	4	-2^15 · 11 · 53 · 109 · 175938429894113
	5	2^13 · 17 · 29 · 31 · 48271 · 11323415153
	6	2^11 · 3 · 11 · 8431 · 46439 · 334080913
	7	2^9 · 7 · 11 · 23 · 337 · 57529 · 46048973
	8	-2^6 · 11 · 22397 · 3288078398821
	9	2^4 · 3 · 11 · 17 · 247304254372333
	10	-2^2 · 3^3 · 53 · 22091 · 453573881
	11	3^10 · 5^2 · 7^3 · 11^3 · 19 · 53
11	0	-2^24 · 2267 · 456061 · 3581958061713 1
	1	-2^24 · 2267 · 456061 · 35819580617131
	2	2^21 · 7013 · 16079170169649964603 9
	3	-2^20 3^3 · 11 · 31 · 193 · 313 · 1721891 · 1470947789
	4	2^16 · 32141 · 105211 · 2032754699837819
	5	-2^17 · 41 · 14957 · 1098815141180455393

	6	2^14 · 3 · 11· 450776321 · 50725424887157
	7	-2^13 · 31 · 71· 661·2087801 · 51021573721
	8	2^8 · 7 · 61· 673 ·13879 · 93337331882521
	9	-2^8 · 67 · 298273000066671199291
	10	2^5 · 3^2 · 1109· 4091 · 397697 · 359760029
	11	-2^4 · 3^5 · 5^3 · 37 · 1291· 259783 · 693323
	12	3^11 · 5^8 · 7^3 ; 53 · 173 ·199
12	0	2^26 · 3 · 17^2 · 4785344572266687874843
	1	2^24 · 13 · 17^2 · 4785344572266687874843
	2	-2^23 · 13 · 8389 · 1923221 · 622063199095097
	3	2^21 · 11 · 13 · 19 · 28277· 4214030374061379281
	4	-2^18·7·11 ·13 · 113 · 4649 · 1517841413421382909
	5	2^16 · 13 · 109937 · 454489792508539519913
	6	-2^16 · 636181213 · 152434984877767139
	7	2^14 · 13 · 101 · 103 · 322457957345984172893
	8	-2^10·11·13·29 ·41· 877·8514133·46586606779
	9	2^8 · 11 · 13 · 3567761 · 29360899633057831
	10	-2^7 · 3 · 13 · 17^2 · 19 · 145511 · 779617 · 56845799
	11	2^5 · 3^3 · 13 · 67 · 149 · 50150275797504937
	12	-2^2 · 3^5 · 5^3 · 84559 · 10770664702627
	13	3^10 · 5^6 · 7^4 · 11^3 · 13^3 · 157
13	0	-2^28· 3 · 89 ·19771234829 · 309404696909685413
	1	-2^27· 7 · 89 ·19771234829 · 309404696909685413
	2	2^24 · 7^4 · 97 · 401 · 1130921089434671378871583
	3	-2^24 · 149 · 505157 ·425185711 · 2021118254878477
	4	2^20 ·7·11 · 73 · 5647 · 10074235937090992833337687
	5	-2^19·7^2 ·19 · 3863 · 27539 · 1351912351099538330831
	6	2^16 · 23 · 137·1019 · 19785874237·2654751313367743

	7	-2^17 · 7 · 22623818587 · 64423644409823902337
	8	2^12 · 7^2 · 3079·246063176717 · 820886445889957
	9	-2^11 · 11 · 70501 · 1936511 · 1579921939 · 1847175203
	10	2^8 · 3 · 7 · 91291081506745419301044411953
	11	-2^8 · 3^2 · 7^2 · 175347137586806799032227603
	12	2^4 · 3^4 · 238331 · 98747591 · 18249163824323
	13	-2^3 · 3^7 · 5^4 · 7^3 · 23 · 30603283 · 3697153391
	14	3^13 · 5^6 · 7^6 · 11^4 · 1875943
14	0	2^30 · 778788561641 · 36332943234792554213
	1	2^28 · 5 · 778788561641 · 36332943234792554213
	2	-2^26·3^2 ·23·173 ·6646276099 · 7872433642798479427
	3	2^24 · 3^2 ·5^2·13·677· 1321 · 98459 · 17497173963854460787
	4	-2^22 ·3·5 ·13 · 31 · 449 · 354726890521 · 5762383298522131
	5	2^20 ·3·5^2·13 ·17 ·23 · 46666157 · 395236939 · 675852838403
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	7	2^16 · 3^4 ·5^2·609452445725699 · 1304269610113751
	8	-2^14 · 3·5 · 59 ·102878519 · 7138415751880603148947
	9	2^12· 5^2 · 13 · 17 · 37460462789592889061087418103
	10	-2^10 · 13 · 18329 · 216906490142150308029751549
	11	2^8 ·3 · 5 · 13 ·1373 · 19381 · 1915131135667132507081
	12	-2^6 · 3^3 · 7 · 839 · 8973815706713482308106447
	13	2^4 · 3^6 · 5^4 · 103 · 359 · 3637 · 333331 · 7009084913
	14	-2^2 · 3^9 · 5^6 · 29403233605398528247
	15	3^16 · 5^9 · 7^2 · 11^4 · 13^3 · 2039
15	0	-2^32 · 3 · 107 · 201432839 · 786552965266940596440439
	1	-2^34 · 107 · 201432839 · 786552965266940596440439
	2	2^31 · 5^4 · 139 ·905900635639 · 5457986474481182813
	3	2^30 · 3 ·109 ·5705671 · 270449753110919443802784089

	4	2^26·3·13·167·641 ·853 · 2293·1857391 · 162263948907363727
	5	−2^26 ·3·5·11^2 · 13 · 2143 · 10614731909726835277536692159
	6	2^23 ·3^2 ·5 ·53·283 ·10799 ·21450712441 ·4701491998045259
	7	−2^22 · 3 · 5 ·59 · 1699·5381 ·24572133092202695713223843
	8	2^17 · 3 · 5 ·73·557 ·571·23227 ·2574601 · 32897561223520199
	9	−2^8 · 3·5 ·23 ·107· 499 · 1609565899039977270658826449
	10	2^15 · 13 · 103 · 12305985003076473009105689220893
	11	−2^14 · 13 · 37 · 16418668766417 · 230002714328186339
	12	2^10 ·3 · 5 · 23 ·24781 ·10191631 · 3073566823 · 4646733293
	13	−2^10 · 3^3 ·23 ·89 ·109 · 719 · 2644144267 ·3491302984189
	14	2^7 · 3^5 · 8237 · 23693807 ·153278356767248849
	15	−2^6 · 3^7 · 5^2 · 7^4 · 22613 · 1013627 · 69327308263
	16	3^13 · 5^4 · 7^6 · 11^3 · 13^3 · 15913 · 16691

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