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uniform distribution theory

MAHLER'S Z-NUMBER AND 3/2 NUMBER SYSTEMS

Shigeki Akiyama

ABSTRACT. We improve the results in [1] on the characterization of multiple points in rational based number system, in connection with Mahler's Z-number problem. As a by-product, we show that when $p > q^2$, there exists a positive x such that the fractional part of $x(p/q)^n$ $(n = 0, 1, \dots)$ stays in a Cantor set (Theorem 2.5). Hausdorff dimension of the set is positive but tends to zero as $p \to \infty$ when q is fixed.

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1. Representations in a rational base

Let us review the result in [1]. Let p, q be coprime integers with $p > q > 1$ and consider a digit set $\mathcal{A} = \{0, 1, \dots, p-1\}$. Every positive integer u has a unique representation:

$$u = u_0 \frac{1}{q} + u_1 \frac{p}{q^2} + u_2 \frac{p^2}{q^3} + \cdots + u_\ell \frac{p^\ell}{q^{1+\ell}}$$

with $u_i \in \mathcal{A}$. The digits u_i are successively determined by taking module p of both sides in the ring $\mathbb{Z}_q = \{z/q^n \mid z \in \mathbb{Z}, n \geq 0\}$, the localization of $\mathbb{Z}$ by q. Following the convention of decimal expression, we write $u = u_\ell u_{\ell-1} \dots u_1 u_0$ and identify with the word in $\mathcal{A}^*$. The set of words which represent positive integers is denoted by $L_{p/q} \subset \mathcal{A}^*$. Then the set $L_{p/q}$ is not even context free since no infinite repetition is allowed but 0^∞. However the odometer is given by an automaton. A positive real number x not greater than a given constant $\theta = \theta(p/q) > 1$ has a representation in a form:

$$x = x_{-1} \frac{1}{p} + x_{-2} \frac{q}{p^2} + \cdots + x_{-\ell} \frac{q^\ell}{p^{1+\ell}} + \cdots = .x_{-1} x_{-2} \dots$$

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with the property that $x_{-1}x_{-2}\dots x_{-m} \in L_{p/q}$ for all positive integers m. This $\theta(p/q)$ corresponds to the maximal word $W(0)$ by the notation of [1] and is explicitly written as:

$$\theta\left(\frac{p}{q}\right) = \sum_{i=0}^{\infty}\left(\frac{qG_{i+1}}{p} - G_i\right)\left(\frac{q}{p}\right)^i$$

with $G_0 = 0$ and $G_{i+1} = \lfloor (pG_i + p - 1)/q \rfloor$. For any real number x, there exists $M > 0$ that $x(q/p)^M \le \theta(p/q)$. This means that we can expand any $x > 0$ into

$$x = x_M x_{M-1}\dots x_0.x_{-1}x_{-2}\dots$$

using the decimal point '.' in a usual manner. From the property of $L_{p/q}$, there are no eventually periodic expansions. The p/q-integer part (resp. p/q-fractional part) of x is defined to be $x_M x_{M-1}\dots x_0.$ (resp. $.x_{-1}x_{-2}\dots$). We put the decimal point to distinguish them with other representations. This representation is unique but countably many exceptions.

Define $\langle x \rangle = x - \lfloor x \rfloor$, the fractional part of x. If $p \ge 2q - 1$, then there are no x which admit three different expressions and we have a good characterization of such exceptional x's having two p/q-representations:

Theorem 1.1 (Akiyama-Frougny-Sakarovitch [1]). *Let $p \ge 2q - 1$. Then a positive real number x has two p/q-representations if and only if there exists n_0 so that*

$$\left\langle \frac{x}{q}\left(\frac{p}{q}\right)^n \right\rangle \in \bigcup_{0 \le c \le q-1} \left[\frac{k_c}{p}, \frac{k_c + 1}{p}\right[\tag{1}$$

holds for all $n \ge n_0$. The number $k_c \in \mathcal{A}$ is defined by $qk_c \equiv c \pmod p$.

Proof. We only review an easy part, the necessity of the condition (1) for the purpose of this note. It is shown in [1] that the digit-wise difference of eventually maximal word and eventually minimal word is formally: $0^*(-q)(p-q)^\infty$ by the special feature of our representation. Therefore x has double representations if and only if x has a suffix in $\{0, \dots, q-1\}^{\mathbb{N}}$, since $p-1-(p-q) = q-1$. Thus there exists n_0 that for $n \ge n_0$ we can expand $(p/q)^n x = c_M c_{M-1}\dots c_0.c_{-1}c_{-2}\dots$ with $M = M(n)$ and $c_{-j} \in \{0, \dots, q-1\}$ for $j = 1, 2, \dots$. We have an estimate

$$.c_{-1}c_{-2}\dots < \frac{q-1}{p-q} \le 1.$$

Since p/q-integer parts have integer values, this inequality means that the p/q-fractional part (resp. p/q-integer part) of x coincides with the *usual* fractional part (resp. integer part) of x. Let us consider a function $f(x) = q(\lfloor xp/q \rfloor - (p/q)\lfloor x \rfloor)$. By the above claim, if x admits double expressions, then we have

$$f((p/q)^n x) = q(c_M c_{M-1}\dots c_0 c_{-1}. - c_M c_{M-1}\dots c_0 0.) = q \times c_{-1}. = c_{-1}$$

Thus for large n, $f((p/q)^n x)$ takes values only in $\{0, 1, \ldots, q-1\}$. Note that f is a periodic function of period q and the value of $f((p/q)^n x)$ is determined by $x \pmod q \in \mathbb{R}/q\mathbb{Z}$. Now using $p \geq 2q-1$, it is easy to show

$$f^{-1}(\{0, 1, \ldots, q-1\}) = \bigcup_{0 \leq c \leq q-1} \left[\frac{qk_c}{p}, \frac{q(k_c+1)}{p} \right[$$

which shows the necessity. □

The same idea allows us to show

Theorem 1.2. *For an integer k with $p-1 \leq k(p-q)$, if a real number x has k different p/q-representations then there exists n_0 so that*

$$\left\langle \frac{x}{q} \left(\frac{p}{q} \right)^n \right\rangle \in \bigcup_{0 \leq c \leq (k-1)q-(k-2)p-1} \left[\frac{k_c}{p}, \frac{k_c+1}{p} \right[\tag{2}$$

holds for all $n \geq n_0$.

Proof. We proceed in a similar manner as the above proof of Theorem 1.1. Only thing to note is that x has k different representations if and only if x has a suffix in $\{0, \ldots, (k-1)q-(k-2)p-1\}^*$, since $p-1-(k-1)(p-q) = (k-1)q-(k-2)p-1$ and $(p/q)^n x = c_M c_{M-1} \ldots c_0 . c_{-1} c_{-2} \ldots$ for large n satisfies

$$.c_{-1}c_{-2} \cdots < \frac{(k-1)q-(k-2)p-1}{p-q} \leq 1.$$

□

Corollary 1.3. *A real number has at most $1 + \left\lfloor \frac{p-2}{p-q} \right\rfloor$ different p/q-representations.*

Proof. An inequality $1 \leq (k-1)q-(k-2)p-1$ is necessary to have an aperiodic expansion of $x > 0$. □

As far as we computed, there seems no triple points for any p/q-representations. Perhaps it is reasonable to pose a

Conjecture 1.4. *There are no positive real x so that*

$$\left\langle x \left(\frac{p}{q} \right)^n \right\rangle \in \bigcup_{0 \leq c \leq 2q-p-1} \left[\frac{k_c}{p}, \frac{k_c+1}{p} \right[$$

holds for all n,

which implies that there are no x with triple expressions when $p-1 \le 3(p-q)$. For e.g., if $p=4$ and $q=3$ then the conjecture asserts that there are no positive x such that

$$\left\langle x\left(\frac{4}{3}\right)^n\right\rangle \in [0,1/4)\ \cup\ [3/4,1)$$

holds for all $n \ge 0$. This is also equivalent to the statement that there are no real x such that $||x(4/3)^n|| < 1/4$ for all n, where $||y||$ is the distance of y from the nearest integer. Here the left endpoint of $[3/4,1)$ can be neglected. In fact, $\langle x(4/3)^n\rangle = 3/4$ occurs only when x is rational and at most once for such a x by seeing the denominator of x. However we may substitute x by $x(4/3)^{n+1}$ in such a case. The end points usually do no harm by this trick.

2. A generalization of Mahler's Z-number

One can show stronger results than the ones in the previous section. Before stating the result, we begin with some terminologies. Let F be a finite union of half open subintervals $[a,b)$ of $[0,1)$ and $\mu(F)$ be the 1-dimensional Lebesgue measure of F. We study two sets $Z^+_{p/q}(F) = \{0 < x \in \mathbb{R} \mid \langle x(p/q)^n\rangle \in F\}$ and $Z_{p/q}(F) = \{x \in \mathbb{R} \mid \langle x(p/q)^n\rangle \in F\}$. In fact, our framework is much suitable for the study of $Z^+_{p/q}(F)$ but occasionally we can deduce results on $Z_{p/q}(F)$ as well. The notorious problem in this context is due to Mahler [4] whether $Z^+_{p/q}([0,1/2))$ is empty or not. Our question is to find a small $\mu(F)$ such that $Z^+_{p/q}(F) \ne \emptyset$. For developments on the distribution of limit points of $\langle x(p/q)^n\rangle$, the reader should consult series of papers by Dubickas for e.g. [2, 3]. He also derived a large $\mu(F)$ with $Z_{p/q}(F) = \emptyset$.

Theorem 1.1 implies that if $p \ge 2q-1$ then there exists some F with $\mu(F) = q/p$ that $Z^+_{p/q}(F)$ is countably infinite. For e.g., using Theorem 1.1 with $p=3$ and $q=2$, we see $Z^+_{3/2}([0,1/3)\ \cup\ [2/3,1))$ is countably infinite. Thus there exists a real x's such that $||x(3/2)^n|| < 1/3$ for all n. As a refinement of Theorem 1.1, we have

Theorem 2.1. *Let $p > q > 1$ with $p \ge 2q-1$. Then a positive real number x has two p/q-representations if and only if there exists n_0 so that*

$$\left\langle \frac{x}{q}\left(\frac{p}{q}\right)^n\right\rangle \in \bigcup_{0\le c\le q-1} \left]\frac{k_c}{p}, \frac{k_c}{p} + \frac{q-1}{p(p-q)}\right[\tag{3}$$

holds for all $n \ge n_0$. The number $k_c \in \mathcal{A}$ is defined by $qk_c \equiv c \pmod p$.

This Theorem implies that for $p \geq 2q-1$, two conditions (1) and (3) are equivalent, in fact. Further, this implies that there exists a finite union of intervals F with $\mu(F) = \frac{q(q-1)}{p(p-q)}$ such that $Z^+_{p/q}(F)$ is countably infinite.

P r o o f. As the right hand side of (3) is narrower than (1), the sufficiency of the condition (3) is obvious. We only prove the necessity. Firstly we shall show a weaker statement. The open intervals of (3) are substituted by *closed* ones. Here the idea is to generalize the function $f(x)$ to

$$f_m(x) = q^m \left\lfloor x\frac{p^m}{q^m} \right\rfloor - p^m \lfloor x \rfloor$$

with a large integer $m(\geq 2)$ in the proof of Theorem 1.1. Using the same idea, this function f_m has period q^m and if x is a double point then

$$\begin{aligned} f_m((p/q)^n x) &= q^m(c_M c_{M-1}\dots c_0 c_{-1}\dots c_{-m}. - c_M c_{M-1}\dots c_0 0^m.) \\ &= q^m \times c_{-1}\dots c_{-m}. = \sum_{j=1}^{m} p^{m-j} q^{j-1} c_{-j} \end{aligned}$$

holds for a large n. Our task is to construct concretely the inverse image of f_m. Take $k^* = k^*(c_{-1}, c_{-2}, \dots, c_{-m}) \in \{0, 1, \dots, p^m - 1\}$ which satisfies

$$\sum_{j=1}^{m} p^{m-j} q^{j-1} c_{-j} \equiv q^m k^* \pmod{p^m}. \tag{4}$$

By using the same proof of Theorem 1.1, we have

$$\left\langle \frac{x}{q^m}\left(\frac{p^m}{q^m}\right)^n \right\rangle \in \bigcup_{(c_{-1},\dots,c_{-m})\in\{0,\dots,q-1\}^m} \left[\frac{k^*}{p^m}, \frac{k^*+1}{p^m}\right[$$

for $mn \geq n_0$ where n_0 is the same as in the proof of Theorem 1.1. Multiplying q^{m-1}, we have

$$\frac{x}{q}\left(\frac{p^m}{q^m}\right)^n \pmod{q^{m-1}} \in \bigcup_{(c_{-1},\dots,c_{-m})} \left[\frac{q^{m-1}k^*}{p^m}, \frac{q^{m-1}(k^*+1)}{p^m}\right[. \tag{5}$$

From (4), one see

$$p^{m-1} k_{c_{-1}} + \sum_{j=2}^{m} p^{m-j} q^{j-2} c_{-j} \equiv q^{m-1} k^* \pmod{p^m}. \tag{6}$$

Without loss of generality, we may assume that $c_{-2}\dots c_{-m} \neq (q-1)^{m-1}$. Therefore we have an estimate

$$\sum_{j=2}^{m} p^{m-j}q^{j-2}c_{-j} < p^{m-2}\frac{q-1}{1-q/p}\left(1-\left(\frac{q}{p}\right)^{m-1}\right) \leq p^{m-1}-q^{m-1}.$$

Thus the left hand side of (6) belongs to $[0, p^m-1]\cap\mathbb{Z}$. Taking modulo 1 of (5), we have

$$\left\langle \frac{x}{q}\left(\frac{p^m}{q^m}\right)^n \right\rangle \in$$

$$\bigcup_{c_{-1}}\bigcup_{c_{-2}}\cdots\bigcup_{c_{-m}}\left[\frac{k_{c_{-1}}}{p}+\sum_{j=2}^{m}c_{-j}\frac{q^{j-2}}{p^j},\ \frac{k_{c_{-1}}}{p}+\sum_{j=2}^{m}c_{-j}\frac{q^{j-2}}{p^j}+\frac{q^{m-1}}{p^m}\right[. \quad (7)$$

Note that

$$\bigcup_{c_{-2}}\cdots\bigcup_{c_{-m}}\left[\frac{k_c}{p}+\sum_{j=2}^{m}c_{-j}\frac{q^{j-2}}{p^j},\ \frac{k_c}{p}+\sum_{j=2}^{m}c_{-j}\frac{q^{j-2}}{p^j}+\frac{q^{m-1}}{p^m}\right[$$

is contained in the interval

$$\left[\frac{k_c}{p},\ \frac{k_c}{p}+\sum_{j=2}^{m}\frac{(q-1)q^{j-2}}{p^j}+\frac{q^{m-1}}{p^m}\right[.$$

Thus we have

$$\left\langle \frac{x}{q}\left(\frac{p^m}{q^m}\right)^n \right\rangle \in \bigcup_{c}\left[\frac{k_c}{p},\ \frac{k_c}{p}+\frac{q-1}{p(p-q)}+\frac{q^{m-1}}{p^m}\frac{p-1}{p-q}\right[.$$

This implies that there exists n_1 so that for any positive ε,

$$\left\langle \frac{x}{q}\left(\frac{p}{q}\right)^n \right\rangle \in \bigcup_{c}\left[\frac{k_c}{p},\ \frac{k_c}{p}+\frac{q-1}{p(p-q)}+\varepsilon\right[$$

holds for $n \geq n_1$. This shows the weaker statement for closed intervals. Consider end points of the intervals of (3). As we may assume that $c_{-2}\dots c_{-m} \neq 0^{m-1}$ or $(q-1)^{m-1}$, we easily see that such end points can not be attained in the above proof. □

Remark 2.2. In the above proof, if $q^2 \geq p \geq 2q-1$, then

$$\bigcup_{c_{-2}}\cdots\bigcup_{c_{-m}}\left[\frac{k_c}{p}+\sum_{j=2}^{m}c_{-j}\frac{q^{j-2}}{p^j},\ \frac{k_c}{p}+\sum_{j=2}^{m}c_{-j}\frac{q^{j-2}}{p^j}+\frac{q^{m-1}}{p^m}\right[$$

is exactly equal to

$$\left[\frac{k_c}{p},\ \frac{k_c}{p} + \sum_{j=2}^{m} \frac{(q-1)q^{j-2}}{p^j} + \frac{q^{m-1}}{p^m}\right[.$$

To see this, we note

$$\frac{q^{m-k-1}}{p^{m-k+1}} \le \sum_{j=2}^{\infty} \frac{(q-1)q^{m-j}}{p^{m-j+2}} \le \sum_{j=2}^{k} \frac{(q-1)q^{m-j}}{p^{m-j+2}} + \frac{q^{m-1}}{p^m}.$$

The left inequality follows from $q^2 \ge p$ and the right from $p \ge 2q-1$.

Following the same proof, we have

Theorem 2.3. *For an integer k with $p-1 \le k(p-q)$, if a real number x has k different p/q-representations then there exists n_0 so that*

$$\left\langle \frac{x}{q}\left(\frac{p}{q}\right)^n \right\rangle \in \bigcup_{0\le c\le (k-1)q-(k-2)p-1} \left]\frac{k_c}{p}, \frac{k_c}{p} + \frac{(k-1)q-(k-2)p-1}{p(p-q)}\right[\tag{8}$$

holds for all $n \ge n_0$.

It is remarkable that if $p > q^2$, the result becomes better.

Theorem 2.4. *Let $p > q > 1$ with $p > q^2$. Then for any positive ε, there exists a finite union of intervals F with $\mu(F) < \varepsilon$ and a positive real number x has two p/q-representations if and only if there exists n_0 so that*

$$\left\langle \frac{x}{q}\left(\frac{p}{q}\right)^n \right\rangle \in F \tag{9}$$

holds for all $n \ge n_0$.

Thus for $p > q^2$ there exists a finite union of intervals F of an arbitrary small size $\mu(F)$ such that $Z^+_{p/q}(F)$ is countably infinite. In the following, we shall prove a stronger result.

A set $X = X(p/q)$ is given as a non empty compact set in $\mathbb{R}$ satisfying an iterated function system:

$$X = \bigcup_{j=0}^{q-1} \frac{qX+j}{p}.$$

It is approximated by a decreasing sequence of sets defined by $X_0 = [0, (q-1)/(p-q)]$ and $X_{k+1} = \bigcup_{j=0}^{q-1}(qX_k + j)/p$ for $k = 0, 1, \dots$. We see $X = \cap_k X_k$ and all end points of X_k are in X. As $p > q^2$, $\mu(X) = 0$ follows from the definition. The pieces $(qX+j)/p$ do not overlap, this system gives a

Cantor set in $[0,(q-1)/(p-q)]$ of Hausdorff dimension $\log q/\log(p/q) < 1$ which is positive but tends to zero as $p \to \infty$ and q is fixed.

Theorem 2.5. *Let $p > q > 1$ with $p > q^2$. Then a positive x has two p/q-representations if and only if there exists n_0 that*

$$\left\langle \frac{x}{q}\left(\frac{p}{q}\right)^n \right\rangle \in \bigcup_{c=0}^{q-1} \frac{X(p/q)+k_c}{p} \tag{10}$$

for $n \geq n_0$.

As X_k $(k = 0, 1, \dots)$ are finite unions of intervals, Theorem 2.4 follows immediately from Theorem 2.5.

P r o o f. Since $X(p/q) \subset [0,(q-1)/(p-q)]$, the sufficiency of (10) follows from Theorem 2.1. We show the necessity. It is easily seen that

$$X(p/q) = \left\{ \sum_{i=0}^{\infty} c_{-i} \frac{q^i}{p^{i+1}} \;\middle|\; c_{-i} \in [0, q-1] \cap \mathbb{Z} \right\}.$$

We proceed in the same manner as the proof of Theorem 2.1. If x admits two p/q-representations, then there exists n_0 such that (7) holds for $mn \geq n_0$. Each element u of

$$\left[\frac{k_c}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j},\ \frac{k_c}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j} + \frac{q^{m-1}}{p^m} \right[$$

has distance at most q^{m-1}/p^m from the compact set $(X(p/q)+k_c)/p$. As we can choose m large, the distance of the point $\langle x/q(p/q)^n \rangle$ and the compact set $\bigcup_{c=0}^{q-1}(X(p/q)+k_c)/p$ is zero, which proves the theorem. □

Denominators of end points of X_k are divisors of $(p(p-q))^{k+1}$ which are coprime to q. Thus, as n increases, $\langle x/q(p/q)^n \rangle$ can visit the end points at most once only when x is rational.

Note that if x is a double point, then there exists n_0 such that $\langle x(p/q)^n \rangle = .c_{-1}c_{-2}\dots$ with $c_{-i} \in [0, q-1] \cap \mathbb{Z}$ for $n \geq n_0$. This already implies that $\langle x(p/q)^n \rangle \in X(p/q)$. We observe that (10) is stronger than this inclusion. Indeed, (10) implies

$$x\left(\frac{p}{q}\right)^n \pmod q \in \bigcup_c \left(\frac{qX}{p} + \frac{qk_c}{p}\right)$$

and taking modulo 1, we get $\langle x(p/q)^n \rangle \in X(p/q)$ again.

At any rate, it is unexpected that when $p > q^2$ there exists $x > 0$ that the closure of $\langle x(p/q)^n \rangle$ $(n = 0, 1, \dots)$ is contained in the Cantor set $X(p/q)$, a compact set of measure zero. We do not know whether the closure of $K = \{\langle x(p/q)^n \rangle |\ n = 0, 1, \dots\}$ could be of Hausdorff dimension 0. In the other direction, Vijayaraghavan [5] showed that the number of accumulation points of K is infinite but it is not known whether the closure of K could be countable.

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Shigeki Akiyama
Department of Mathematics
Faculty of Science Niigaty University
Ikarashi - 28050
Niigata 950-2181
JAPAN
E-mail: akiyama@math.sc.niigata-u.ac.jp