ON ANALYTIC CONTINUATION OF MULTIPLE L-FUNCTIONS AND RELATED ZETA-FUNCTIONS

SHIGEKI AKIYAMA & HIDEAKI ISHIKAWA

1. Introduction

Analytic continuation of Euler-Zagier's multiple zeta function of two variables was first established by F.V.Atkinson [3] with an application to the mean value problem of the Riemann zeta function. We can find recent developments in [8],[7] and [5]. From an analytic point of view, these results suggest broad applications of multiple zeta functions. In [9] and [10], D.Zagier pointed out an interesting interplay between positive integer values and other areas of mathematics, which include knot theory and mathematical physics. Many works had been done according to his motivation but here we restrict our attention to the analytic continuation. T.Arakawa and M.Kaneko [2] showed an analytic continuation with respect to the last variable. To speak about the analytic continuation with respect to all variables, we have to refer to J. Zhao [11] and S.Akiyama, S.Egami and Y.Tanigawa [1]. In [11], an analytic continuation and the residue calculation were done by using the theory of generalized functions in the sense of I.M. Gel'fand and G.E. Shilov. In [1], they gave an analytic continuation by means of a simple application of the Euler-Maclaurin formula. The advantage of this method is that it gives the complete location of singularities. This work also includes some study on the values at non positive integers.

In this paper we consider a more general situation, which seems important for number theory, in light of the method of [1]. We shall give an analytic continuation of multiple Hurwitz zeta functions (Theorem 1) and also multiple L functions (Theorem 2) defined below. In special cases, we can completely describe the whole set of singularities, by using a property of zeros of Bernoulli polynomials (Lemma 4) and a non vanishing result on a certain character sum (Lemma 2).

We explain notations used in this paper. The rational integers is denoted by $\mathbb{Z}$, the rational numbers by $\mathbb{Q}$, the complex numbers by $\mathbb{C}$ and the positive integers by $\mathbb{N}$. We write $\mathbb{Z}_{\le \ell}$ for the integers not greater than ℓ. Let χ_i $(i = 1, 2, \dots, k)$ be Dirichlet characters of the same conductor $q \ge 2$ and β_i $(i = 1, 2, \cdots, q)$ be real numbers in the half open interval $[0, 1)$. The principal character is denoted by χ_0. Then multiple Hurwitz zeta function and multiple L function are defined respectively by:

$$(1)\quad \zeta_k(s_1, \dots, s_k \mid \beta_1, \dots, \beta_k) = \sum_{0<n_1<\cdots<n_k} \frac{1}{(n_1+\beta_1)^{s_1}(n_2+\beta_2)^{s_2}\dots(n_k+\beta_k)^{s_k}}$$

and

$$(2)\qquad L_k(s_1, \dots, s_k \mid \chi_1, \dots, \chi_k) = \sum_{0<n_1<\cdots<n_k} \frac{\chi_1(n_1)}{n_1^{s_1}}\frac{\chi_2(n_2)}{n_2^{s_2}}\dots\frac{\chi_k(n_k)}{n_k^{s_k}},$$

where $n_i \in \mathbb{N}$ $(i = 1, \dots, k)$. If $\Re(s_i) \ge 1$ $(i = 1, 2, \dots, k-1)$ and $\Re(s_k) > 1$, then these series are absolutely convergent and define holomorphic functions of k complex variables in this region. In the sequel we write them by $\zeta_k(s \mid \beta)$ and

$L_k(s \mid \chi)$, for abbreviation. The Hurwitz zeta function $\zeta(s,\alpha)$ in the usual sense for $\alpha \in (0,1)$ is written as

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} = \frac{1}{\alpha^s} + \zeta_1(s \mid \alpha),$$

by the above notation.

We shall state the first result. Note that $\beta_j - \beta_{j+1} = 1/2$ for some j implies $\beta_{j-1} - \beta_j \neq 1/2$, since $\beta_j \in [0,1)$.

Theorem 1. *The multiple Hurwitz zeta function $\zeta_k(s \,|\, \beta)$ is meromorphically continued to $\mathbb{C}^k$ and has possible singularities on:*

$$s_k = 1, \qquad \sum_{i=1}^{j} s_{k-i+1} \in \mathbb{Z}_{\leq j} \ \ (j = 2, 3, \ldots, k).$$

Let us assume furthermore that all β_i $(i = 1, \ldots, k)$ are rational. If $\beta_{k-1} - \beta_k$ is not 0 nor $1/2$, then the above set coincides with the set of whole singularities. If $\beta_{k-1} - \beta_k = 1/2$ then

$$\begin{aligned} s_k &= 1 \\ s_{k-1} + s_k &= 2, 0, -2, -4, -6, \ldots \\ \sum_{i=1}^{j} s_{k-i+1} &\in \mathbb{Z}_{\leq j} \quad \textit{for } j = 3, 4, \ldots, k \end{aligned}$$

forms the set of whole singularities. If $\beta_{k-1} - \beta_k = 0$ then

$$\begin{aligned} s_k &= 1 \\ s_{k-1} + s_k &= 2, 1, 0, -2, -4, -6, \ldots \\ \sum_{i=1}^{j} s_{k-i+1} &\in \mathbb{Z}_{\leq j} \quad \textit{for } j = 3, 4, \ldots, k \end{aligned}$$

forms the set of whole singularities.

For the simplicity, we only concerned with special cases and determined the whole set of singularities in Theorem 1. The reader can easily handle the case when all $\beta_i - \beta_{i+1}$ $(i = 1, \ldots, k-1)$ are not necessary rational and fixed. So we have enough information on the location of singularities of multiple Hurwitz zeta functions. For the case of multiple L functions, our knowledge is rather restricted.

Theorem 2. *The multiple L-function $L_k(s \,|\, \chi)$ is meromorphically continued to $\mathbb{C}^k$ and has possible singularities on:*

$$s_k = 1, \quad \sum_{i=1}^{j} s_{k-i+1} \in \mathbb{Z}_{\leq j} \ \ (j = 2, 3, \ldots, k).$$

Especially for the case $k = 2$, we can state the location of singularities in detail as follows:

Corollary 1. *We have a meromorphic continuation of $L_2(s \mid \chi)$ to $\mathbb{C}^2$. $L_2(s \mid \chi)$ is holomorphic in*

$$(3) \qquad \begin{cases} \{(s_1, s_2) \in \mathbb{C}^2 \ \mid s_1 + s_2 \notin \mathbb{Z}_{\leq 2}, \quad s_2 \neq 1\} & \textit{if} \ \ \chi_1 = \chi_0, \quad \chi_2 = \chi_0 \\ \{(s_1, s_2) \in \mathbb{C}^2 \ \mid s_1 + s_2 \notin \mathbb{Z}_{\leq 1}, \quad s_2 \neq 1\} & \textit{if} \ \ \chi_1 \neq \chi_0, \quad \chi_2 = \chi_0 \\ \{(s_1, s_2) \in \mathbb{C}^2 \ \mid s_1 + s_2 \notin \mathbb{Z}_{\leq 1}\} & \textit{if} \ \ \chi_2 \neq \chi_0, \end{cases}$$

where the excluded sets are possible singularities. Suppose that χ_1 and χ_2 are primitive characters with $\chi_1\chi_2 \neq \chi_0$. Then $L_2(s \mid \chi)$ is a holomorphic function in

$$(4) \quad \begin{cases} \{(s_1, s_2) \in \mathbb{C}^2 \ \mid s_1 + s_2 \neq 0, -2, -4, -6, -8, \dots \} & \textit{if} \quad \chi_1\chi_2(-1) = 1, \\ \{(s_1, s_2) \in \mathbb{C}^2 \ \mid s_1 + s_2 \neq 1, -1, -3, -5, -7, \dots \} & \textit{if} \quad \chi_1\chi_2(-1) = -1, \end{cases}$$

where the excluded set forms the whole set of singularities.

Unfortunately the authors could not get the complete description of singularities of multiple L function for $k \geq 3$.

2. Preliminaries

Let $N_1, N_2 \in \mathbb{N}$ and η be a real number. Suppose that a function $f(x)$ is $l+1$ times continuously differentiable. By using Stieltjes integral expression, we see

$$\begin{aligned} \sum_{N_1+\eta < n \leq N_2} f(n) &= \int_{N_1+\eta}^{N_2} f(x) d[x] \\ &= \int_{N_1+\eta}^{N_2} f(x) dx - [f(x)\tilde{B}_1(x)]_{N_1+\eta}^{N_2} + \int_{N_1+\eta}^{N_2} f^{'}(x)\tilde{B}_1(x) dx \end{aligned}$$

where $\tilde{B}_j(x) = B_j(x - [x])$ is the j-th *periodic Bernoulli polynomial.* Here j-th Bernoulli polynomial $B_j(x)$ is defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j(x)}{j!} t^j$$

and $[x]$ is the largest integer not exceeding x. Define the Bernoulli number B_r by the value $B_r = B_r(0)$. Repeating integration by parts,

$$(5) \quad \begin{aligned} \sum_{N_1+\eta < n \leq N_2} f(n) &= \int_{N_1+\eta}^{N_2} f(x) dx + \frac{1}{2} f(N_2) + f(N_1+\eta)\tilde{B}_1(\eta) \\ &\quad + \sum_{r=1}^{l} \frac{(-1)^{r+1}}{(r+1)!} \Big(B_{r+1} f^{(r)}(N_2) - f^{(r)}(N_1+\eta)\tilde{B}_{r+1}(\eta) \Big) \\ &\quad - \frac{(-1)^{l+1}}{(l+1)!} \int_{N_1+\eta}^{N_2} f^{(l+1)}(x) \tilde{B}_{l+1}(x) dx. \end{aligned}$$

When $\eta = 0$, the formula (5) is nothing but the standard Euler-Maclaurin summation formula. This slightly modified summation formula by a parameter η works quite fine in studying our series (1) and (2).

Lemma 1. *Letting*

$$\Phi_l(s \,|N_1+\eta, \alpha) = \frac{(s)_{l+1}}{(l+1)!} \int_{N_1+\eta}^{\infty} \frac{\tilde{B}_{l+1}(x)}{(x+\alpha)^{s+l+1}} dx$$

and

$$(s)_r = \begin{cases} s(s+1)(s+2)\dots(s+r-1) & \textit{if} \quad r \geq 1 \\ 1 & \textit{if} \quad r = 0 \\ (s-1)^{-1} & \textit{if} \quad r = -1 \end{cases},$$

it follows that

$$\sum_{N_1+\eta < n} \frac{1}{(n+\alpha)^s} = \sum_{r=-1}^{l} \frac{\tilde{B}_{r+1}(\eta)}{(r+1)!} \frac{(s)_r}{(N_1+\alpha+\eta)^{s+r}} - \Phi_l(s \mid N_1+\eta, \alpha)$$

with

$$\Phi_l(s \mid N_1+\eta,\alpha) \ll N_1^{-(\Re s+l+1)}.$$

Proof. Put $f(x)=(x+\alpha)^{-s}$. Then we have $f^{(r)}(x)=(-1)^r(s)_r(x+\alpha)^{-s-r}$. So from (5),

$$\begin{aligned}\sum_{N_1+\eta<n}^{N_2}\frac{1}{(n+\alpha)^s} &= \left[\frac{1}{1-s}\frac{1}{(x+\alpha)^{s-1}}\right]_{N_1+\eta}^{N_2}+\frac{1}{2}\frac{1}{(N_2+\alpha)^s}+\frac{\tilde{B}_1(\eta)}{(N_1+\eta+\alpha)^s}\\ &\quad -\sum_{r=1}^{l}\frac{(s)_r}{(r+1)!}\left(\frac{B_{r+1}}{(N_2+\alpha)^{s+r}}-\frac{\tilde{B}_{r+1}(\eta)}{(N_1+\eta+\alpha)^{s+r}}\right)\\ &\quad -\frac{1}{(l+1)!}\int_{N_1+\eta}^{N_2}\tilde{B}_{l+1}(x)\frac{(s)_{l+1}}{(x+\alpha)^{s+l+1}}dx.\end{aligned}$$

When $\Re s>1$, we have

$$\begin{aligned}\sum_{N_1+\eta<n}\frac{1}{(n+\alpha)^s} &= \frac{1}{s-1}\frac{1}{(N_1+\alpha+\eta)^{s-1}}+\frac{\tilde{B}_1(\eta)}{(N_1+\eta+\alpha)^s}\\ &\quad +\sum_{r=1}^{l}\frac{\tilde{B}_{r+1}(\eta)}{(r+1)!}\frac{(s)_r}{(N_1+\alpha+\eta)^{s+r}}-\Phi_l(s\mid N_1+\eta,\alpha)\end{aligned}$$

as $N_2\to\infty$. When $\Re s\le 1$, if we take a sufficiently large l, the integral in the last term $\Phi_l(s\mid N_1+\eta,\alpha)$ is absolutely convergent. Thus this formula gives an analytic continuation of the series of the left hand side. Performing integration by parts once more and comparing two expressions, it can be easily seen that $\Phi_l(s\mid N_1+\eta,\alpha)\ll N_1^{-(\Re s+l+1)}$. $\square$

Let $A_{\chi_1,\chi_2}(j)$ be a sum

$$\sum_{a_1=1}^{q-1}\sum_{a_2=1}^{q-1}\chi_1(a_1)\chi_2(a_2)\tilde{B}_j(\frac{a_1-a_2}{q}).$$

Lemma 2. *Suppose χ_1 and χ_2 are primitive characters modulo q with $\chi_1\chi_2\neq\chi_0$. Then we have: for $1\le j$*

$$A_{\chi_1,\chi_2}(j)=\begin{cases}-2\frac{j!}{(2\pi i)^j}\tau(\chi_1)\overline{\tau(\overline{\chi_2})}L(j,\overline{\chi_1\chi_2}) & \text{if } \quad (-1)^j\chi_1\chi_2(-1)=1,\\ 0 & \text{if } \quad (-1)^j\chi_1\chi_2(-1)=-1,\end{cases}$$

where $\tau(\chi)$ is the Gauss sum defined by $\tau(\chi)=\sum_{u=0}^{q-1}\chi(u)e^{2\pi iu/q}$.

Proof. Recall the Fourier expansion of Bernoulli polynomial:

$$\tilde{B}_j(y)=-j!\lim_{M\to\infty}\sum_{\substack{n=-M\\ n\neq 0}}^{M}\frac{e^{2\pi iny}}{(2\pi in)^j} \tag{6}$$

for $1\le j$, $0\le y<1$ except $(j,y)=(1,0)$. First suppose $j\ge 2$, then the right hand side of (6) is absolutely convergent. Thus it follows from (6) that

$$A_{\chi_1,\chi_2}(j)=\sum_{a_1=1}^{q-1}\sum_{a_2=1}^{q-1}\chi_1(a_1)\chi_2(a_2)\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\exp[2\pi in\frac{a_1-a_2}{q}]}{(2\pi in)^j}\right).$$

Since

$$\sum_{u=0}^{q-1} \chi(u) e^{2\pi i n u/q} = \bar{\chi}(n)\tau(\chi)$$

for a primitive character χ, we have

$$A_{\chi_1,\chi_2}(j) = -j! \sum_{\substack{n \neq 0 \\ n=-\infty}}^{\infty} \frac{\bar{\chi_1}(n)\bar{\chi_2}(-n)\tau(\chi_1)\tau(\chi_2)}{(2\pi i n)^j},$$

from which the assertion follows immediately by the relation $\overline{\tau(\overline{\chi})} = \chi(-1)\tau(\chi)$. Next assume that $j = 1$. Dividing $A_{\chi_1,\chi_2}(1)$ into

$$A_{\chi_1,\chi_2}(1) = \sum{}' \chi_1(a_1)\chi_2(a_2)\tilde{B}_1\left(\frac{a_1 - a_2}{q}\right) + \sum_{a_1=1}^{q-1} \chi_1\chi_2(a_1)B_1 \tag{7}$$

where $\sum'$ taken over all the terms $1 \leq a_1, a_2 \leq q-1$ with $a_1 \neq a_2$. The second sum in (7) is equal to 0 by the assumption. By using (6), the first sum is

$$\begin{aligned}
&\sum{}' \chi_1(a_1)\chi_2(a_2)\tilde{B}_1\left(\frac{a_1 - a_2}{q}\right) \\
&= \sum{}' \chi_1(a_1)\chi_2(a_2) \lim_{M\to\infty}\left(-\sum_{\substack{n=-M \\ n\neq 0}}^{M} \frac{\exp[2\pi i n(a_1 - a_2)/q]}{2\pi i n}\right) \\
&= -\lim_{M\to\infty} \sum_{\substack{n=-M \\ n\neq 0}}^{M} \frac{\sum_{a_1=1}^{q-1}\sum_{a_2=1}^{q-1} \chi_1(a_1)\chi_2(a_2)\exp[2\pi i n\frac{a_1-a_2}{q}] - \sum_{a_1=1}^{q-1}\chi_1\chi_2(a_1)}{2\pi i n}
\end{aligned}$$

Using $\sum_{a_1=1}^{q-1} \chi_1\chi_2(a_1) = 0$ again, we have

$$\sum{}' \chi_1(a_1)\chi_2(a_2)\tilde{B}_1\left(\frac{a_1 - a_2}{q}\right) = -\frac{1}{2\pi i} \lim_{M\to\infty} \sum_{\substack{n=-M \\ n\neq 0}}^{M} \frac{\bar{\chi}_1(n)\bar{\chi}_2(-n)\tau(\chi_1)\tau(\chi_2)}{n}.$$

Hence we get the result. □

We recall the classical theorem of von Staudt & Clausen.

Lemma 3.

$$B_{2n} + \sum_{p-1 \,|\, 2n} \frac{1}{p} \qquad \textit{is an integer.}$$

Here the summation is taken over all prime p such that $p-1$ divides $2n$.

Extending the former results of D.H.Lehmer and K.Inkeri, the distribution of zeros of Bernoulli polynomials is extensively studied in [4], where one can find a lot of references. On rational zeros, we quote here the result of [6].

Lemma 4. *Rational zeros of Bernoulli polynomial $B_n(x)$ must be $0, 1/2$ or 1. These zeros occur when and only when in the following cases:*

$$\begin{cases} B_n(0) = B_n(1) = 0 & n \textit{ is odd} \quad n \geq 3 \\ B_n(1/2) = 0 & n \textit{ is odd} \quad n \geq 1. \end{cases} \tag{8}$$

We shall give its proof, for the convenience of the reader.

Proof. First we shall show that if $B_n(\gamma) = 0$ with $\gamma \in \mathbb{Q}$ then $2\gamma \in \mathbb{Z}$. The Bernoulli polynomial is explicitly written as

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Let $\gamma = P/Q$ with $P, Q \in \mathbb{Z}$ and P, Q are coprime. Then we have

$$-\frac{P^n}{Q} = nB_1 P^{n-1} + \sum_{k=2}^{n} \binom{n}{k} B_k P^{n-k} Q^{k-1}.$$

Assume that there exist a prime factor $q \geq 3$ of Q. Then the right hand side is q-integral. Indeed, we see that $B_1 = -1/2$ and qB_k is q-integral since the denominator of B_k is always square free, which is an easy consequence of Lemma 3. But the left hand side is not q-integral, we get a contradiction. This shows that Q must be a power of 2. Let $Q = 2^m$ with a non negative integer m. Then we have

$$-\frac{P^n}{2^{m-1}} = -nP^{n-1} + \sum_{k=2}^{n} \binom{n}{k} B_k P^{n-k} 2^{m(k-1)+1}.$$

If $m \geq 2$ then we get a similar contradiction. Thus Q must divide 2, we see $2\gamma \in \mathbb{Z}$. Now our task is to study that values of Bernoulli polynomials at half integers. Since $B_0(x) = 1$ and $B_1(x) = x - 1/2$, the assertion is obvious if $n < 2$. Assume that $n \geq 2$ and even. Then by Lemma 3, the denominator of B_n is divisible by 3. Recalling the relation

$$B_n(x+1) - B_n(x) = nx^{n-1}, \tag{9}$$

we have for any integer m

$$B_n(m) \equiv B_n \pmod{\mathbb{Z}}.$$

This shows $B_n(m) \neq 0$. The relation

$$B_n(1/2) = (2^{1-n} - 1)B_n \tag{10}$$

implies that

$$B_n(1/2) \neq 0 \tag{11}$$

for $n \geq 2$ and even. We see that $B_n(1/2)$ is not 3 integral from Lemma 3 and the relation (10). Combining (9), (11), we have for any integer m and any even integer $n \geq 2$

$$B_n(1/2 + m) \neq 0.$$

It is easy to show the assertion for the remaining case when $n \geq 2$ is odd, by using (9) and (10). □

3. Analytic continuation of multiple Hurwitz zeta functions

This section is devoted to the proof of Theorem 1. First we treat the double Hurwitz zeta function. By Lemma 1, we see

$$\sum_{n_1+(\beta_1-\beta_2)<n_2} \frac{1}{(n_2+\beta_2)^{s_2}} =$$

$$\sum_{r=-1}^{l} \frac{\tilde{B}_{r+1}(\beta_1-\beta_2)}{(r+1)!} \frac{(s_2)_r}{(n_1+\beta_1)^{s_2+r}} - \Phi_l(s_2 \mid n_1+\beta_1-\beta_2, \beta_2).. \tag{12}$$

Suppose first that $\beta_1 \geq \beta_2$. Then the sum $\sum_{n_1+\beta_1-\beta_2<n_2}$ means $\sum_{n_1<n_2}$, so it follows from (12) that

$$\begin{aligned}
\zeta_2(s \mid \beta) &= \sum_{n_1=1}^{\infty} \frac{1}{(n_1+\beta_1)^{s_1}} \sum_{n_1<n_2} \frac{1}{(n_2+\beta_2)^{s_2}} \\
&= \sum_{n_1=1}^{\infty} \frac{1}{(n_1+\beta_1)^{s_1}} \sum_{n_1+(\beta_1-\beta_2)<n_2} \frac{1}{(n_2+\beta_2)^{s_2}} \\
&= \sum_{n_1=1}^{\infty} \frac{1}{(n_1+\beta_1)^{s_1}} \\
&\quad \times \left\{ \sum_{r=-1}^{l} \frac{\tilde{B}_{r+1}(\beta_1-\beta_2)}{(r+1)!} \frac{(s_2)_r}{(n_1+\beta_1)^{s_2+r}} - \Phi_l(s_2 \mid n_1+\beta_1-\beta_2, \beta_2) \right\}.
\end{aligned}$$

Suppose that $\beta_1 < \beta_2$. We consider

$$\zeta_2(s \mid \beta) = \sum_{n_1=1}^{\infty} \frac{1}{(n_1+\beta_1)^{s_1}} \left\{ \frac{-1}{(n_1+\beta_2)^{s_2}} + \sum_{n_1 \leq n_2} \right\}. \tag{13}$$

Noting that the sum $\sum_{n_1+\beta_1-\beta_2<n_2}$ means $\sum_{n_1\leq n_2}$, we apply (12) to the second term in the braces. For the first term in the braces, we use the binomial expansion:

$$\frac{1}{(n_1+\beta_2)^{s_2}} = \frac{1}{(n_1+\beta_1)^{s_2}} \left(\sum_{m=0}^{v} \frac{(-1)^m (s_2)_m}{m!} \left(\frac{\beta_2-\beta_1}{n_1+\beta_1} \right)^m + R_{v+1} \right) \tag{14}$$

with $R_{v+1} \ll n_1^{v+1}$. By applying (12) and (14) to (13), we have

$$\begin{aligned}
\zeta_2(s \mid \beta) &= \sum_{n_1=1}^{\infty} \frac{1}{(n_1+\beta_1)^{s_1}} \left\{ \frac{1}{s_2-1} \frac{1}{(n_1+\beta_1)^{s_2-1}} \right. \\
&\quad + \sum_{r=0}^{l} \left(\frac{\tilde{B}_{r+1}(\beta_1-\beta_2)}{(r+1)!} - \frac{(\beta_1-\beta_2)^r}{r!} \right) \frac{(s_2)_r}{(n_1+\beta_1)^{s_2+r}} \\
&\quad \left. - \Phi_l(s_2 \mid n_1+\beta_1-\beta_2, \beta_2) - \frac{R_{l+1}}{(n_1+\beta_1)^{s_2}} \right\}.
\end{aligned}$$

Recalling the relation (9) and combining the cases $\beta_1 \leq \beta_2$ and $\beta_1 > \beta_2$, we have

$$\begin{aligned}
\zeta_2(s \mid \beta) &= \frac{1}{s_2-1} \zeta_1(s_1+s_2-1 \mid \beta_1) \\
&\quad + \sum_{r=0}^{l} \frac{B_{r+1}(\beta_1-\beta_2)}{(r+1)!} (s_2)_r \zeta_1(s_1+s_2+r \mid \beta_1) \\
&\quad - \sum_{n_1=1}^{\infty} \frac{\Phi_l^*(s_2|n_1+\beta_1-\beta_2,\beta_2)}{(n_1+\beta_1)^{s_1}}
\end{aligned} \tag{15}$$

where

$$\Phi_l^*(s_2|n_1+\beta_1-\beta_2,\beta_2) = \begin{cases} \Phi_l(s_2|n_1+\beta_1-\beta_2,\beta_2) & \text{if } \beta_1 \geq \beta_2 \\ \Phi_l(s_2|n_1+\beta_1-\beta_2,\beta_2) + \frac{R_{l+1}}{(n_1+\beta_1)^{s_2}} & \text{if } \beta_1 < \beta_2 \end{cases}$$

The right hand side in (15) has meromorphic continuation except the last term. The last summation is absolutely convergent, and hence holomorphic, in $\Re(s_1+s_2+l) > 0$. Thus we now have a meromorphic continuation to $\Re(s_1+s_2+l) > 0$. Since we can choose arbitrary large l, we get a meromorphic continuation of $\zeta_2(s \mid \beta)$ to $\mathbb{C}^2$, holomorphic in

$$\left\{(s_1,s_2)\in\mathbb{C}^2 \mid s_2\neq 1,\quad s_1+s_2\neq 2,1,0,-1,-2,-3,\dots\right\}.$$

The exceptions in this set are the possible singularities occurring in $(s_2-1)^{-1}$ and

$$\frac{B_{r+1}(\beta_1-\beta_2)}{(r+1)!}(s_2)_r\zeta_1(s_1+s_2+r \mid \beta_1). \tag{16}$$

Whether they are 'real singularity' or not depends on the choice of parameters $\beta_i \quad (i=1,2)$. For the case of multiple Hurwitz zeta functions with k variables,

$$\begin{aligned}
&\zeta_k(s_1,\dots,s_k \mid \beta_1,\dots,\beta_k) = \\
&\quad \sum_{n_1=1}^{\infty}\frac{1}{(n_1+\beta_1)^{s_1}}\sum_{n_1<n_2}\frac{1}{(n_2+\beta_2)^{s_2}}\cdots \\
&\quad \cdots\sum_{n_{k-2}<n_{k-1}}\frac{1}{(n_{k-1}+\beta_{k-1})^{s_{k-1}}}\left\{\frac{1}{s_k-1}\frac{1}{(n_{k-1}+\beta_{k-1})^{s_k-1}}\right. \\
&\quad \left.+\sum_{r=0}^{l}\frac{B_{r+1}(\beta_{k-1}-\beta_k)}{(r+1)!}\frac{(s_k)_r}{(n_{k-1}+\beta_{k-1})^{s_k+r}}-\Phi_l^*(s_k \mid n_{k-1}+\beta_{k-1}-\beta_k,\beta_k)\right\} \\
= &\quad \frac{1}{s_k-1}\zeta_{k-1}(s_1,\dots,s_{k-2},s_{k-1}+s_k-1 \mid \beta_1,\dots,\beta_{k-1}) \\
&\quad +\sum_{r=0}^{l}\frac{B_{r+1}(\beta_{k-1}-\beta_k)}{(r+1)!}(s_k)_r\zeta_{k-1}(s_1,\dots,s_{k-2},s_{k-1}+s_k+r \mid \beta_1,\dots,\beta_{k-1}) \\
&\quad -\sum_{0<n_1<n_2<\dots<n_{k-1}}^{\infty}\frac{\Phi_l^*(s_k \mid n_{k-1}+\beta_{k-1}-\beta_k,\beta_k)}{(n_1+\beta_1)^{s_1}\dots(n_{k-1}+\beta_{k-1})^{s_{k-1}}}.
\end{aligned}$$

Since

$$\sum_{0<n_1<\dots<n_{k-1}}\frac{\Phi_l^*(s_k \mid n_{k-1}+\beta_{k-1}-\beta_k,\beta_k)}{(n_1+\beta_1)^{s_1}(n_2+\beta_2)^{s_2}\dots(n_{k-1}+\beta_{k-1})^{s_{k-1}}}\ll\sum_{n_{k-1}}\frac{n_{k-1}^{-l-\Re(s_k)+k-3}}{n_{k-1}^{L}}$$

with $L=\Re(s_{k-1})+\sum_{1\le j\le k-2,\Re(s_i)\le 0}\Re(s_i)$, the last summation is convergent absolutely in

$$l-k+2+\Re(s_{k-1})+\Re(s_k)+\sum_{\substack{1\le j\le k-2\\ \Re(s_i)\le 0}}\Re(s_i)>0.$$

Since l can be taken arbitrarily large, we get an analytic continuation of $\zeta_k(s \mid \beta)$ to $\mathbb{C}^k$. Now we study the set of singularities more precisely. The 'singular part' of $\zeta_2(s \mid \beta)$ is

$$\frac{\zeta_1(s_1+s_2-1 \mid \beta_1)}{s_2-1}+\sum_{r=0}^{\infty}\frac{(s_2)_r}{s_1+s_2+r-1}\frac{B_{r+1}(\beta_1-\beta_2)}{(r+1)!}..$$

Note that this sum is by no means convergent and just indicates local singularities. From this expression we see

$$s_2=1,\ s_1+s_2\in\{2,1,0,-1,-2,-3,-4,\dots\}$$

are possible singularities and the second assertion of the Theorem 1 for $k=2$ is now clear with the help of Lemma 4. We wish to determine the whole singularities when all β_i $(i=1,\dots,k)$ are rational numbers by an induction on k. Let us consider the case of k variables,

$$\begin{aligned}
&\zeta_k(s \mid \beta) = \\
&\sum_{r=-1}^{l} \frac{B_{r+1}(\beta_{k-1}-\beta_k)}{(r+1)!}(s_k)_r \zeta_{k-1}(s_1,\dots,s_{k-2},s_{k-1}+s_k+r \mid \beta_1,\dots\beta_{k-1}) \\
&-\sum_{0<n_1<\cdots<n_{k-1}} \frac{\Phi_l^*(s_k \mid n_{k-1}+\beta_{k-1}-\beta_k,\beta_k)}{(n_1+\beta_1)^{s_1}\dots(n_{k-1}+\beta_{k-1})^{s_{k-1}}}.
\end{aligned}$$

We shall only prove the case when $\beta_{k-1}-\beta_k=0$. Other cases are left to the reader. By the induction hypothesis and Lemma 4 the singularities lie on, at least, for $r=-1,0,1,3,5,7,\dots,$

$$s_k=1, \quad s_{k-1}+s_k+r=1,$$

$$s_{k-2}+s_{k-1}+s_k+r=2,0,-2,-4,-6,\dots$$

and

$$s_{k-j}+s_{k-j+1}+\cdots+s_k+r \in \mathbb{Z}_{\le j}, \quad \text{for } j\ge 3,$$

for any three cases; $\beta_{k-2}-\beta_{k-1}=0,1/2$, and otherwise.

Thus

$$s_k=1, \quad s_{k-1}+s_k=2,1,0,-2,-4,-6,\dots$$

and

$$s_{k-j+1}+s_{k-j+2}+\cdots+s_k \in \mathbb{Z}_{\le j}, \quad \text{for } j\ge 3$$

are the possible singularities, as desired. Note that the singularities of the form

$$s_{k-2}+s_{k-1}+s_k+r=1,-1,-3,-5,\dots$$

may appear. However, these singularities don't affect our description. Next we will show that they are the 'real' singularities. For example, the singularities of the form $s_{k-2}+s_{k-1}+s_k=\eta$ occurs in several ways for a fixed η. So our task is to show that no singularities defined by one of above equations will identically vanish in the summation process. This can be shown by a small trick of replacing variables:

$$s_1=u_1,\dots,s_{k-2}=u_{k-2}, \quad s_{k-1}+s_k=u_{k-1}, \quad s_k=u_k$$

In fact, we see that the singularities of $\zeta_k(u_1,\dots,u_{k-2},u_{k-1}-u_k,u_k \mid \beta_1,\dots,\beta_k)$ appears in

$$\sum_{r=-1}^{l} \frac{B_r(\beta_{k-1}-\beta_k)}{(r+1)!}(u_k)_r \zeta_{k-1}(u_1,\dots,u_{k-2},u_{k-1}+r \mid \beta_1,\dots\beta_{k-1}).$$

By this expression we see that the singularities of $\zeta_{k-1}(u_1,\dots,u_{k-1}+r|\beta_1,\dots\beta_{k-1})$ are summed with functions of u_k of *different* degree. Thus these singularities, as weighted sum by another variable u_k, will not vanish identically. This argument seems to be an advantage of [1], which clarify the exact location of singularities. The Theorem is proved by the induction. □

4. Analytic continuation of multiple L-functions

Proof of Theorem 2. When $\Re s_i>1$ for $i=1,2,\dots,k$, the series is absolutely convergent. Rearranging the terms,

$$\sum_{n_1<\cdots<n_k} \frac{\chi_1(n_1)}{n_1^{s_1}} \cdots \frac{\chi_k(n_k)}{n_k^{s_k}} =$$

$$\frac{1}{q^{s_1+\cdots+s_k}} \sum_{a_1=1}^{q-1} \sum_{a_2=1}^{q-1} \cdots \sum_{a_k=1}^{q-1} \chi_1(a_1)\chi_2(a_2)\ldots\chi_k(a_k)$$

$$\times \Bigg[\sum_{m_1=0}^{\infty} \frac{1}{(m_1+\frac{a_1}{q})^{s_1}} \sum_{m_1+\frac{a_1-a_2}{q}<m_2} \frac{1}{(m_2+\frac{a_2}{q})^{s_2}} \cdots$$

$$\cdots \sum_{m_{k-2}+\frac{a_{k-2}-a_{k-1}}{q}<m_{k-1}} \frac{1}{(m_{k-1}+\frac{a_{k-1}}{q})^{s_{k-1}}} \sum_{m_{k-1}+\frac{a_{k-1}-a_k}{q}<m_k} \frac{1}{(m_k+\frac{a_k}{q})^{s_k}} \Bigg].$$

By this expression, it suffices to show that the series in the last brace has the desirable property. When $a_i - a_{i+1} \geq 0$ holds for $i = 1, \ldots, k-1$, this is clear form Theorem 1, since this series is just a multiple Hurwitz zeta function. Proceeding along the same line with the proof of Theorem 1, other cases are also easily deduced by recursive applications of Lemma 1. Since there are no need to use binomial expansions, this case is easier than before.

□

Proof of Corollary 1. Considering the case $k = 2$ in Theorem 2, we see

$$L_2(s \mid \chi) = \frac{1}{q} \frac{\sum_{a_2=1}^{q-1} \chi_2(a_2)}{s_2 - 1} L(s_1+s_2-1, \chi_1) + \frac{1}{q^{s_1+s_2}} \sum_{a_1=1}^{q-1} \sum_{a_2=1}^{q-1} \chi_1(a_1)\chi_2(a_2)$$

$$\times \left\{ \sum_{r=0}^{l} \frac{\tilde{B}_{r+1}(\frac{a_1-a_2}{q})}{(r+1)!} (s_2)_r \zeta(s_1+s_2+r, \frac{a_1}{q}) - \sum_{m_1=0}^{\infty} \frac{\Phi_l(s_2 \mid m_1 + \frac{a_1-a_2}{q}, \frac{a_2}{q})}{(m_1+\frac{a_1}{q})^{s_1}} \right\}.$$

We have a meromorphic continuation of $L_2(s \mid \chi)$ to $\mathbb{C}^2$, which is holomorphic in the domain (3). Note that the singularities occur in

$$\frac{\sum_{a_2=1}^{q-1} \chi_2(a_2)}{s_2 - 1} L(s_1+s_2-1, \chi_1)$$

and

$$\sum_{a_1=1}^{q} \sum_{a_2=1}^{q} \chi_1(a_1)\chi_2(a_2) \frac{\tilde{B}_{r+1}(\frac{a_1-a_2}{q})}{(r+1)!} (s_2)_r \zeta(s_1+s_2+r, \frac{a_1}{q}).$$

If χ_2 is not principal then the first term vanishes and we see the 'singular part' is

$$\sum_{r=0}^{\infty} \frac{A_{\chi_1,\chi_2}(r+1)}{(r+1)!} \frac{(s_2)_r}{s_1+s_2+r-1}.$$

Thus we get the result by using Lemma 2 and the fact:

$$L(n, \chi) \neq 0$$

for $n \geq 1$ and a non principal character χ.

□

As we stated in the introduction, we do not have a satisfactory answer to the problem of describing whole sigularities of multiple L functions in the case $k \geq 3$, at present. For example when $k = 3$, what we have to show is the non vanishing of the sum:

$$\sum_{a_1=1}^{q-1} \sum_{a_2=1}^{q-1} \sum_{a_3=1}^{q-1} \chi_1(a_1)\chi_2(a_2)\chi_3(a_3)\tilde{B}_{r_1+1}\left(\frac{a_1-a_2}{q}\right)\tilde{B}_{r_2+1}\left(\frac{a_2-a_3}{q}\right),$$

apart from trivial cases.

References

[1] S. Akiyama, S. Egami, and Y. Tanigawa , An analytic continuation of multiple zeta functions and their values at non-positive integers, to appear in Acta Arith.

[2] T.Arakawa and M.Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, *Nagoya Math. J.* **153** (1999), 189-209.

[3] F.V. Atkinson, The mean value of the Riemann zeta-function, *Acta Math.*, **81** (1949), 353-376.

[4] K. Dilcher, Zero of Bernoulli, generalized Bernoulli and Euler polynomials, Memoirs of American Mathematical Society, number 386.

[5] S. Egami, *Introduction to multiple zeta function*, Lecture Note at Niigata University (in Japanese).

[6] K. Inkeri, The real roots of Bernoulli polynomials, *Ann. Univ. Turku. Ser. A I* **37** (1959), 20pp.

[7] M. Katsurada and K. Matsumoto, Asymptotic expansions of the mean values of Dirichlet L-functions. *Math. Z.*, **208** (1991), 23–39.

[8] Y. Motohashi, A note on the mean value of the zeta and L-functions. I, *Proc. Japan Acad.*, Ser. A Math. Sci. **61** (1985), 222–224.

[9] D. Zagier, Values of zeta functions and their applications, *First European Congress of Mathematics*, Vol. II, Birkhäuser, (1994) 210–220

[10] D. Zagier, Periods of modular forms, traces of Hecke operators, and multiple zeta values, *Research into automorphic forms and L functions (in Japanese) (Kyoto, 1992), Sūrikaisekikenkyūsho Kōkyūroku*, **843** (1993), 162–170.

[11] J. Zhao, Analytic continuation of multiple zeta functions. Proc. Amer. Math. Soc. 128 (2000), no. 5, 1275–1283.

Shigeki AKIYAMA

Department of Mathematics, Faculty of Science, Niigata University,
Ikarashi 2-8050, Niigata 950-2181, Japan
e-mail: akiyama@math.sc.niigata-u.ac.jp

Hideaki ISHIKAWA

Graduate school of Natural Science, Niigata University,
Ikarashi 2-8050, Niigata 950-2181, Japan
e-mail: isikawah@qed.sc.niigata-u.ac.jp