PENTAGONAL DOMAIN EXCHANGE

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ABSTRACT. Self-inducing structure of pentagonal piecewise isometry is applied to show detailed description of periodic and aperiodic orbits, and further dynamical properties. A Pisot number appears as a scaling constant and plays a crucial role in the proof. Further generalization is discussed in the last section.

Adler-Kitchens-Tresser [1] and Goetz [10] initiated the study of piecewise isometries. This class of maps shows the way to possible generalizations of results on interval exchanges to higher dimensions [16, 30]. In this paper we examine the detailed properties of the map shown in Figure 1 from an algebraic point of view.
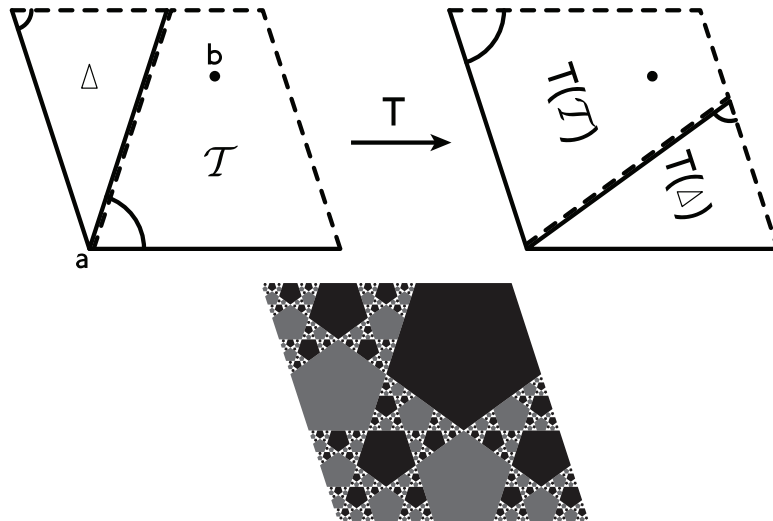

FIGURE 1. A piecewise rotation T on two pieces. The triangle is rotated $2\pi/5$ around a and the trapezium is rotated $2\pi/5$ around b. Periodic points with short periods are shown below, in two colours to illustrate that they cluster into groups, each forming a pentagon.

The goal of this paper is to see how this map is applied to show number theoretical results. First we reprove that almost all orbits in the sense of

Lebesgue measure are periodic, and in addition, there are explicit aperiodic points. Second we show that aperiodic points forms a proper dense subset of an attractor of some iterated function system and are recognized by a Büchi automaton (c.f. Figure 14). The dynamics acting on this set of aperiodic points are conjugate to the 2-adic odometer (addition of one) whose explicit construction is given (Theorem 3). As a result, we easily see that all aperiodic orbits are dense and uniformly distributed in the attractor. We finally give a characterization of points which have purely periodic multiplicative coding by constructing its natural extension (Theorem 6). In doing so we obtain an intriguing picture Figure 16 that emerges naturally from taking algebraic conjugates, whose structure is worthy of further study. We discuss possible generalizations for 7-fold and 9-fold piecewise rotations in Section 3.

A dynamical system is *self-inducing* if the first return map to some subset has the same dynamics as the full map. The most important example is the irrational rotation, presented as exchange of two intervals. An elementary example begins with Φ shown in Figure 2. For this interval exchange,
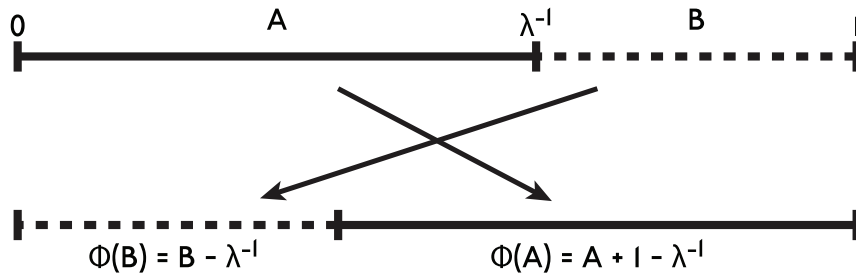

FIGURE 2. An interval exchange map Φ, where $\lambda = \frac{1+\sqrt{5}}{2}$

now consider the second interval B. As shown in Figure 3 this interval is translated to the left once, and to the right. Thus $\Phi^2(B_1)$ is back in B, the interval B_2 requires one more step, but $\Phi^3(B_2)$ also lies within B. This first return dynamics on B is therefore conjugate to the dynamics on $A \cup B$. Self-inducing subsystem of two interval exchange corresponds to purely periodic orbits of continued fraction expansion and they are efficiently captured by the continued fraction algorithm.

This gives a motivation to study the interval exchange transform (IET) of three or more pieces, trying to find higher dimensional continued fraction with good Diophantine approximation properties. The study of self-inducing structure of IET's was started by a pioneer work of Rauzy [26], now called Rauzy induction, and got extended in a great deal by many authors including Veech [31] and Zorich [33], see [32] for historical developments.

Self-inducing piecewise isometries emerged from dynamical systems as a natural generalization of IET [21, 1, 5, 11, 12, 8, 22] and the first return dynamics appears in outer billiards [28, 6]. Like IET they provide a simple
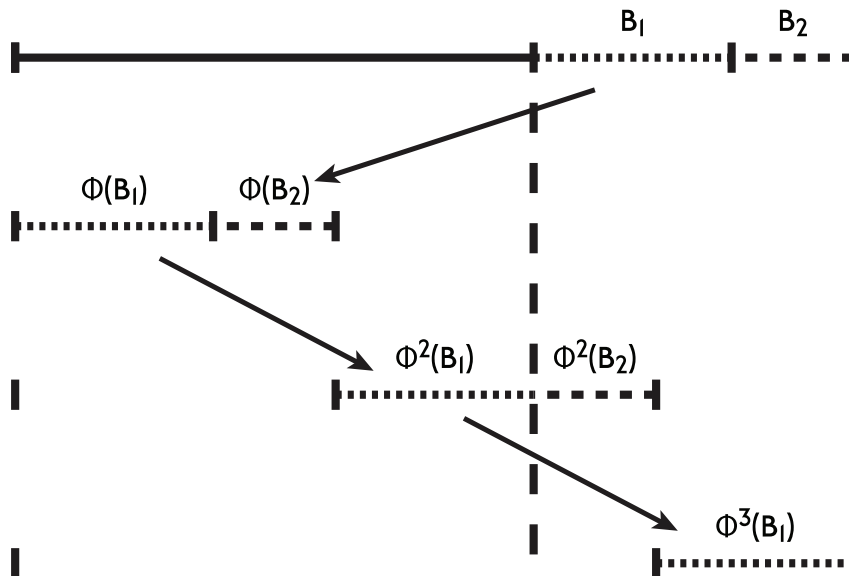
FIGURE 3. The interval exchange Φ is self-inducing. The intervals B_1 and B_2 are swapped by the first return map of Φ on the interval B.

setting to study many of the deep and perplexing behaviors that can emerge from a dynamical system.

The self-inducing structure links such dynamical systems to number theoretical algorithms, such as, digital expansions and Diophantine approximation algorithms, and allows us to study their periodic orbits by constructing their natural extensions. This idea leads to complex and beautiful fractal behavior.

Our target is the piecewise isometry in Figure 1, but to illustrate the bridge formed between the two fields let us begin with a simple conjecture from number theory:

Conjecture 1. *For any* $-2 < \lambda < 2$, *each integer sequence defined by* $0 \le a_{n+1} + \lambda a_n + a_{n-1} < 1$ *is periodic.*

Since $a_{n+2} \in \mathbb{Z}$ is uniquely determined by $(a_n, a_{n+1}) \in \mathbb{Z}^2$, we treat this recurrence as a map $(a_n, a_{n+1}) \mapsto (a_{n+1}, a_{n+2})$ acting on $\mathbb{Z}^2$. It is natural to set $\lambda = -2\cos(\theta)$ to view this map as a 'discretized rotation':

$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$$

with eigenvalues $\exp(\pm\sqrt{-1}\theta)$. As the matrix is conjugate to the planar rotation matrix of angle θ, putting $P = \begin{pmatrix} 1 & 0 \\ \cos\theta & -\sin\theta \end{pmatrix}$, we have

$$P \begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} P \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} + P \begin{pmatrix} 0 \\ \langle \lambda a_{n+1} \rangle \end{pmatrix}$$

where $\langle x \rangle$ is the fractional part of x. Therefore this gives a rotation map of angle θ acting on a lattice $P\mathbb{Z}^2$ but the image requires a bounded perturbation of modulus less than two to fit into lattice points of $P\mathbb{Z}^2$. For conjecture 1 we expect that such perturbations do not cumulate and the orbits stay bounded, equivalently, all orbits become periodic.

A nice feature of the map $(a_n, a_{n+1}) \mapsto (a_{n+1}, a_{n+2})$ is that it is clearly bijective on $\mathbb{Z}^2$ by symmetry, while under the usual round off scheme, the digital information should be more or less lost by the irrational rotation. This motivates dynamical study of global stability of this algorithm.

The conjecture is trivial when $\lambda = 0, \pm 1$. Among non-trivial cases, the second tractable case is when θ is rational and λ is quadratic over $\mathbb{Q}$. Akiyama, Brunotte, Pethő and Steiner [3] proved:

Theorem 1. *The conjecture is valid for* $\lambda = \frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}$.

It seems hard to prove Conjecture 1 for other values. The case $\lambda = \frac{1-\sqrt{5}}{2}$ was firstly shown by Lowenstein, Hatjispyros and Vivaldi [21] with heavy computer assistance. A number theoretical proof for $\frac{1+\sqrt{5}}{2}$ appeared in [2], whose proof is short but not so easy to generalize. We try to give an accessible account using self-inducing piecewise isometry in the case $\lambda = \omega = \frac{1+\sqrt{5}}{2}$, together with its further dynamical behavior. The proof in Section 1 is basically in [3]. However this version may elucidate the background idea and is directly connected to the scaling constant of self-inducing structure of piecewise isometry acting on a lozenge.

A Pisot number is an algebraic integer > 1 whose conjugates have modulus less than 1. Throughout the paper, we will see the importance of the fact that the scaling constant of self-inducing system is a Pisot number. Our all discussions heavily depend on this fact. Indeed, Pisot scaling constants often appear in self-inducing structures of several important dynamical systems, for e.g., IET and substitutive dynamical systems. We discuss this point in Section 3. It is pretty surprising that we see this phenomenon in cubic piecewise rotations as well. We hope this paper gives an easy way to access this interesting area of mathematics.

We wish to show our gratitude to P.Hubert, W.Steiner and F.Vivaldi for helpful comments and relevant literatures in the development of this manuscript.

1. Proof of the periodicity for golden mean

Setting $\zeta = \exp(2\pi i/5)$, we have $\omega = -\zeta^2 - \zeta^{-2}$ and $1/\omega = \zeta + \zeta^{-1}$. The integer ring of $\mathbb{Q}(\zeta)$ coincides with the ring $\mathbb{Z}[\zeta]$ generated by ζ in $\mathbb{Z}$, $\mathbb{Z}[\zeta]$ is a free $\mathbb{Z}$-module generated by $1, \zeta, \zeta^2, \zeta^3$. Hereafter we use a different base as a $\mathbb{Z}$-module:

Lemma 2. $\mathbb{Z}[\zeta]$ *is a free $\mathbb{Z}$-module of rank 4 generated by* $1, \omega, \zeta, \omega\zeta$.

Proof. From $\omega = -\zeta^2 - \zeta^{-2}$, we have

$$x_1 + x_2\omega + (y_1 + y_2\omega)\zeta = (x_1 + y_2) + (y_1 + y_2)\zeta + (y_2 - x_2)\zeta^2 - x_2\zeta^3.$$

On the other hand

$$a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 = (a_0 - a_2 + a_3) - a_3\omega + ((a_1 - a_2 + a_3) + (a_2 - a_3)\omega)\zeta.$$

$\square$

Taking the complex conjugate, the same statement is valid with another basis $1, \omega, \zeta^{-1}, \omega\zeta^{-1}$. Thus each element in $\mathbb{Z}[\zeta]$ has a unique expression:

$$x - \zeta^{-1}y \qquad (x, y \in \mathbb{Z}[\omega]).$$

Denote by $\langle x \rangle$ the fractional part of $x \in \mathbb{R}$. Then a small computation gives

$$0 \leq a_n + \omega a_{n+1} + a_{n+2} < 1$$

$$a_n + \omega a_{n+1} + a_{n+2} = \langle \omega a_{n+1} \rangle$$

$$\langle \omega a_n \rangle - \frac{1}{\omega}\langle \omega a_{n+1} \rangle + \langle \omega a_{n+2} \rangle \equiv 0 \pmod{\mathbb{Z}}$$

$$x_n - (\zeta + \zeta^{-1})x_{n+1} + x_{n+2} \equiv 0 \pmod{\mathbb{Z}}$$

$$(x_{n+1} - \zeta^{-1}x_{n+2}) \equiv \zeta^{-1}(x_n - \zeta^{-1}x_{n+1}) \pmod{\zeta^{-1}\mathbb{Z}}$$

and $x_n = \langle \omega a_n \rangle$. Our problem is therefore embedded into a piecewise isometry T acting on a lozenge $[0, 1) + (-\zeta^{-1})[0, 1)$:

$$T(x) = \begin{cases} x/\zeta & \text{Im}(x/\zeta) \geq 0 \\ (x-1)/\zeta & \text{Im}(x/\zeta) < 0 \end{cases}.$$

The action of T is geometrically described in Figure 1. The lozenge $L = [0, 1) + (-\zeta^{-1})[0, 1)$ is rotated by the multiplication of $-\zeta^{-1}$ and then the trapezoid $\mathcal{Z}$ which falls outside L is pulled back in by adding $-\zeta^{-1}$. In total, the isosceles triangle Δ is rotated clockwise by the angle $3\pi/5$ around the origin and the trapezoid $\mathcal{Z}$ is rotated by the same angle but around the point $\frac{1}{2} + i\frac{\sqrt{5(5+2\sqrt{5})}}{10} \simeq 0.5 + 0.6882i$ indicated by a black spot, that is the intersection of two diagonals. Our aim is to show that each point $x \in \mathbb{Z}[\zeta] \cap L$ gives a periodic T-orbit.

A. Goetz [10] gave a slightly different map. Ours is an 'inclined' modification of [16] and [3].

Clearly the map T is bijective and preserves 2-dimensional Lebesgue measure μ. However the measure dynamical system $(L, \nu, \mathbb{B}, T)$ (with the σ-algebra $\mathbb{B}$ of Lebesgue measurable sets) is far from ergodic. It turned out that orbits of T is periodic for almost all points but for an exceptional set of Lebesgue measure zero. Our goal is to prove that the set $\mathbb{Z}[\zeta]$ has no intersection with this exceptional set. This is not so obvious since $\mathbb{Z}[\zeta]$ is dense in L because $\mathbb{Z}[\omega]$ is dense in $\mathbb{R}$.

To illustrate the situation, it is instructive to describe an orbit of $1/3$. See Figure 4.
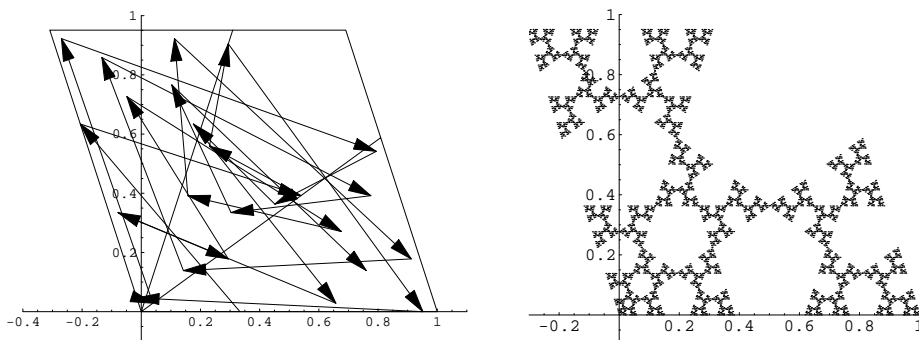
FIGURE 4. The orbit of $1/3$

Later we will show that the orbit of $1/3$ is aperiodic and forms a dense subset of the exceptional set of aperiodic points. Roughly speaking, our task is to show that $\mathbb{Z}[\zeta] \cap L$ has no intersection with the fractal set appeared Figure 4.

The key to the proof is a self-inducing structure with a scaling constant ω^2. We consider a region $L' = \omega^{-2}L$ and consider the first return map

$$\hat{T}(x) = T^{m(x)}(x)$$

for $x \in L'$ where $m(x)$ is the minimum positive integer such that $T^{m(x)}(x) \in L'$. For any $x \in L'$, the value $m(x) = 1, 3$ or 6. We can show that

$$(1) \qquad \omega^2\hat{T}(\omega^{-2}x) = T(x)$$

for $x \in L$. The proof is geometric, shown in Figure 5. The return time $m(x) = 3$ in the open pentagon $\Delta' = \omega^{-2}\Delta$ [this is marked Δ in the figure] and $m(x) = 6$ in the shaded pentagonal region D with three closed and two open edges. In the remaining isosceles triangle in $\omega^{-2}L$ (whose two equal edges are closed and the other open), the return time $m(x)$ is 1.

Note that the equation is valid for **all** $x \in L'$. This makes the later discussion very simple. Unfortunately this is not the case for other quadratic values of γ and we have to study the behavior of the boundary independently, see [3].

Let U be the 1-st hitting map to L' for $x \in L$, i.e., $U(x) = T^{m(x)}(x)$ for the minimum non-negative integer $m(x)$ such that $T^{m(x)}(x) \in L'$. Note that U is a partial function, i.e., $U(x)$ is not defined when there is no positive integer m such that $T^m(x) \in L'$. Since

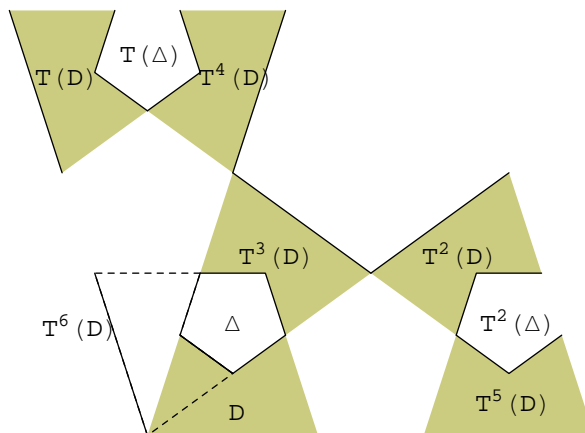$$T(x) = \begin{cases} x/\zeta & x \in \Delta \\ (x-1)/\zeta & x \in T(\mathcal{Z}) \setminus \Delta \end{cases},$$

FIGURE 5. Self Inducing structure

it is easy to make the map U explicit:

$$
U(x) = \begin{cases}
x & x \in L' \\
(x-1)/\zeta & x \in T^5(D) \\
(x-\zeta)/\zeta^2 & x \in T^4(D) \\
\left(x - \frac{\zeta}{\omega}\right)/\zeta^3 & x \in T^3(D) \\
\left(x + \frac{1}{\omega\zeta^2}\right)/\zeta^4 & x \in T^2(D) \\
x + \frac{1}{\omega\zeta} & x \in T(D) \\
\text{Not defined} & x \in P_0 \cup P_1 \cup P_2
\end{cases}
$$

where P_0 is the largest open pentagon and P_1 and $P_2 = P_1/\zeta$ are two second largest closed pentagons in Figure 6.
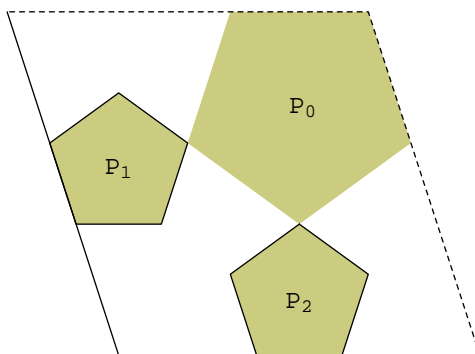

FIGURE 6. Period Pentagons

Set

$$Q = \left\{ 0, 1, \zeta, \frac{\zeta}{\omega}, -\frac{1}{\omega\zeta^2}, -\frac{1}{\omega\zeta} \right\} = \{d_0, d_1, d_2, d_3, d_4, d_5\} \subset \mathbb{Z}[\zeta]$$

to use later.

We introduce a crucial map S which is the composition of the 1-st hitting map U and expansion by ω^2, i.e. $S(x) = \omega^2 U(x)$. Denote by $\pi(x)$ the period of T-orbits of $x \in L$ and put $\pi(x) = \infty$ if x is not periodic by T. (We easily see $\pi(x) = 5$ in P_0 and $\pi(x) = 10$ in $P_1 \cup P_2$ unless x is the centroid of the pentagon.) Then if $\pi(x)$ and $\pi(S(x))$ are defined and finite, then we see that $\pi(S(x)) < \pi(x)$ which is a consequence of Equation (1). Therefore if $\pi(x)$ is finite then we have a decreasing sequence

$$\pi(x) > \pi(S(x)) > \pi(S^2(x)) > \ldots$$

of positive integers. This shows that there exists a positive integer k such that $S^k(x)$ is not defined. In this case we say that S-orbit of x in finite. We easily see that if S-orbit of $x \in L$ is finite, then clearly $\pi(x)$ is finite by Equation (1). Thus we have a clear distinction: $x \in L$ is T-periodic if and only if its S-orbit is finite. Assume that $x \in L \cap \mathbb{Z}[\zeta]$ gives an infinite S-orbit. When $U(x)$ is defined, we have $U(x) = (x - d_{m(x)})/\zeta^{m(x)}$ with $m(x) = \{0, 1, 2, 3, 4, 5\}$ and $d_i \in Q$ for all $x \in L$. Thus we have

$$S^k(x) = \omega^{2k} \frac{x}{\zeta^{\sum_{j=1}^k m_j}} - \sum_{i=1}^k \omega^{2(k-i+1)} \frac{d_{m_i}}{\zeta^{\sum_{j=i}^k m_j}}.$$

By the assumption $S^k(x)$ is defined for $k = 1, 2, \ldots$ and stays in L. Consider the conjugate map ϕ which sends $\zeta \to \zeta^2$. As $\phi(\omega) = -1/\omega$, we have

$$\phi(S^k(x)) = \frac{\phi(x)}{\omega^{2k} \zeta^{2 \sum_{j=1}^k m_j}} - \sum_{i=1}^k \frac{d'_{m_i}}{\omega^{2(k-i+1)} \zeta^{2 \sum_{j=i}^k m_j}}$$

with $d'_i = \phi(d_i) \in \phi(Q)$. Put $A = \max\{|d'_i| : d_i \in Q\}$. Then we have

$$|\phi(S^k(x))| \le |\phi(x)| + \frac{A}{\omega^2 - 1}$$

Thus we have $S^k(x), \phi(S^k(x))$ and their complex conjugates are bounded by a constant which does not depend on k. This implies that the sequence $(S^k(x))_k$ must be eventually periodic.

Summing up, for a point x in $\mathbb{Z}[\zeta]$, its S-orbit is finite or eventually periodic. When it is finite then its T-orbit is periodic and when its S-orbit is eventually periodic then T-orbit is aperiodic. Thus we have an algorithm for $x \in \mathbb{Z}[\beta] \cap L$ to tell whether its T-orbit is periodic or not. Since

$$|\phi(S^k(x))| \le \frac{|\phi(x)|}{\omega^{2k}} + \frac{A}{1 - \omega^{-2}},$$

for any positive ε, the right hand side is bounded by

$$\varepsilon + \frac{A}{\omega^2 - 1}$$

for a sufficiently large k. This means that under the assumption that there is an infinite S-orbit, the set

$$\left\{ x \in \mathbb{Z}[\zeta] \cap L \ : \ |\phi(x)| \leq \varepsilon + \frac{A}{\omega^2 - 1} \right\}$$

contains x with $\pi(x) = \infty$. Since this set is finite, it is equal to

$$B = \left\{ x \in \mathbb{Z}[\zeta] \cap L \ : \ |\phi(x)| \leq \frac{A}{\omega^2 - 1} \right\}$$

for a sufficiently small ε. Since there are only finitely many candidates in B, we obtain an algorithm to check whether an element $x \in \mathbb{Z}[\zeta] \cap L$ with $\pi(x) = \infty$ exists. In fact, all elements in B gives a finite S-expansion, we are done.

The same algorithm applies to $\frac{1}{M}\mathbb{Z}[\zeta]$ with a fixed positive integer M. In this way, we can also show that points in $\frac{1}{2}\mathbb{Z}[\zeta]$ are periodic. We can find aperiodic orbits in $\frac{1}{3}\mathbb{Z}[\zeta]$. For example, one can see that $1/3$ has an aperiodic T-orbit because its S-orbit:

$$\frac{1}{3}, \frac{w^2}{3}, -\frac{\zeta^{-1}}{3}, -\frac{\omega^2\zeta^{-1}}{3} - \frac{2\zeta^{-1}}{3}, -\frac{\omega^{-2}\zeta^{-1}}{3}, -\frac{\zeta^{-1}}{3}, \ldots$$

satisfies $S^2(1/3) = S^6(1/3)$.

It is crucial in the above proof that the scaling constant of the self-inducing structure is a Pisot number. Scaling constants of piecewise isometries often become Pisot numbers, moreover algebraic units. We discuss these phenomena in Section 3.

2. Coding of aperiodic T-orbits

Denote by $\mathbf{A}$ the set of all T-aperiodic points in L. By the proof of the previous section, we have

$$\mathbf{A} = \{x \in L \mid S^k(x) \text{ is defined for all } k = 1, 2, \ldots \}.$$

We also have $S(\mathbf{A}) \subset \mathbf{A}$. This means that for $x_1 \in \mathbf{A}$, there is a $m_i \in \{0, 1, 2, 3, 4, 5\}$ and $x_i \in \mathbf{A}$ such that $\omega^2\zeta^{-m_i}(x_i - d_{m_i}) = x_{i+1} \in \mathbf{A}$ for $i = 1, 2, \ldots$. We therefore have an expansion

$$(2) \qquad x_1 = d_{m_1} + \frac{\zeta^{m_1}}{\omega^2}\left(d_{m_2} + \frac{\zeta^{m_2}}{\omega^2}\left(d_{m_3} + \frac{\zeta^{m_3}}{\omega^2}\left(d_{m_4} + \frac{\zeta^{m_4}}{\omega^2} \cdots \right.\right.\right.$$

Conversely a sequence $\{m_i\}_{i=1,2,\ldots}$ defines a single point of Y. Therefore $\mathbf{A}$ must be a subset of the attractor Y of the iterated function system (IFS):

$$Y = \bigcup_{i=0}^{5} \left(\frac{\zeta^i}{\omega^2}Y + d_i \right),$$

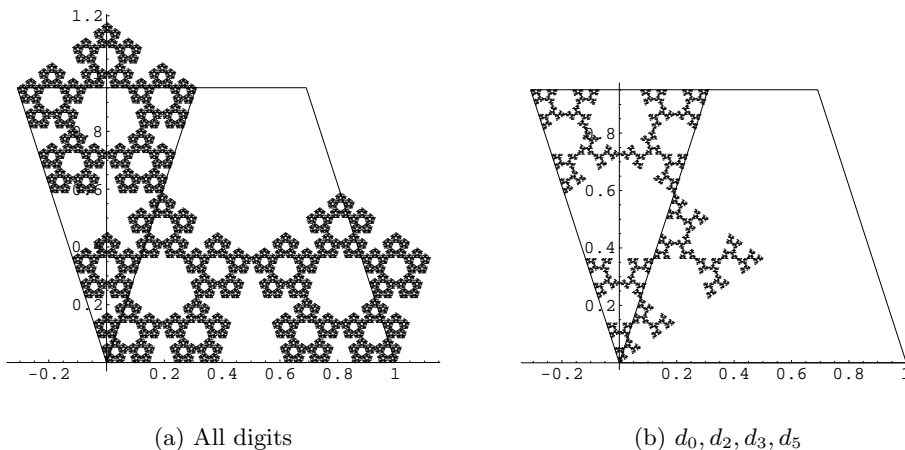an approximation of which is depicted in Figure 7(a). At this point we can

(a) All digits

(b) d_0, d_2, d_3, d_5

FIGURE 7. Attractors containing **A**

assert that 2-dimensional Lebesgue measure of aperiodic points in L must be zero, because $\omega^4 \simeq 6.854 \cdots > 6$.

We notice that the digits in Q are not arbitrarily chosen because the image of S must be in $T(\mathcal{Z})$. Thus the digits d_1 and d_4 appears only at the beginning in the expression of Equation (2). Therefore it is more suitable to study $\mathbf{A} \cap T(\mathcal{Z})$. The attractor

$$(3) \quad Y' = \left(\frac{1}{\omega^2} Y' + d_0 \right) \cup \left(\frac{\zeta^2}{\omega^2} Y' + d_2 \right) \cup \left(\frac{\zeta^3}{\omega^2} Y' + d_3 \right) \cup \left(\frac{\zeta^5}{\omega^2} Y' + d_5 \right)$$

is depicted in Figure 7(b).

This iterated function system satisfies OSC by a pentagonal shape K with whose vertices are

$$0, -\zeta^{-1}, \zeta, -\zeta\omega^{-1} - \zeta^{-1}, -\zeta^2\omega^{-1}$$

as in Figure 8. We confirm that the pieces $K_m = \frac{\zeta^m}{\omega^2} K + d_m$ do not overlap.

We consider the induced system of $(L, \mathbb{B}, \nu, T)$ to $T(\mathcal{Z})$. Denote by $\widetilde{T}$ the first return map on $T(\mathcal{Z})$. Then the induced system $(T(\mathcal{Z}), \widetilde{T})$ is the domain exchange of two isosceles triangle A and B depicted in Figure 9. The triangle A has two closed edges of equal length and one open edge, while B has one closed edge and two open edges of the same length. The open regular pentagon P_0 and the triangle B move together by $\widetilde{T}$ and can be merged into a single shape.

We see

$$(4) \qquad \widetilde{T}(x) = \begin{cases} T^2(x) & x \in \Delta \\ T(x) & x \in T(\mathcal{Z}) \setminus \Delta. \end{cases}$$
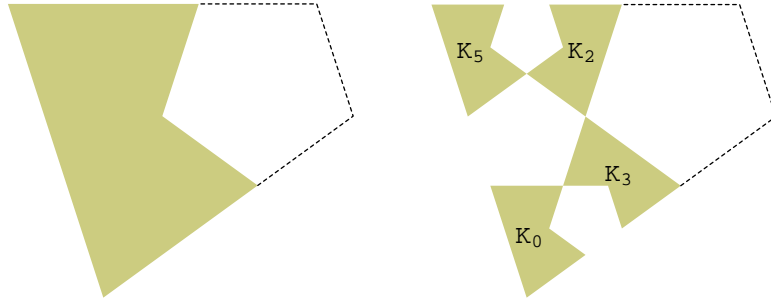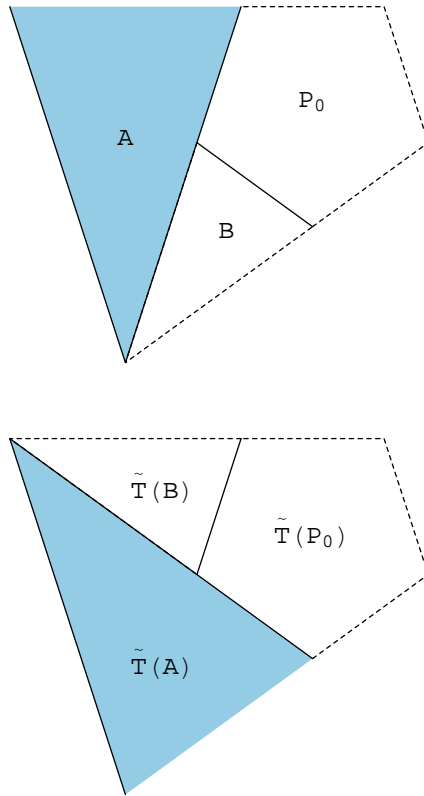
FIGURE 8. Open set condition

FIGURE 9. Induced Rotation $\widetilde{T}$ on $T(\mathcal{Z})$

Again we find self-inducing structure with the scaling constant ω^2:

$$(5) \qquad \omega^2 \widetilde{T}(\omega^{-2}x) = \widetilde{T}(x)$$

for all $x \in T(\mathcal{Z})$. This can be seen in Figure 10 with $\alpha = \omega^{-2}A$, $\beta = \omega^{-2}B$ and $R = \omega^{-2}P_0$. This induced dynamics $(T(\mathcal{Z}), \widetilde{T})$ is essential in describing
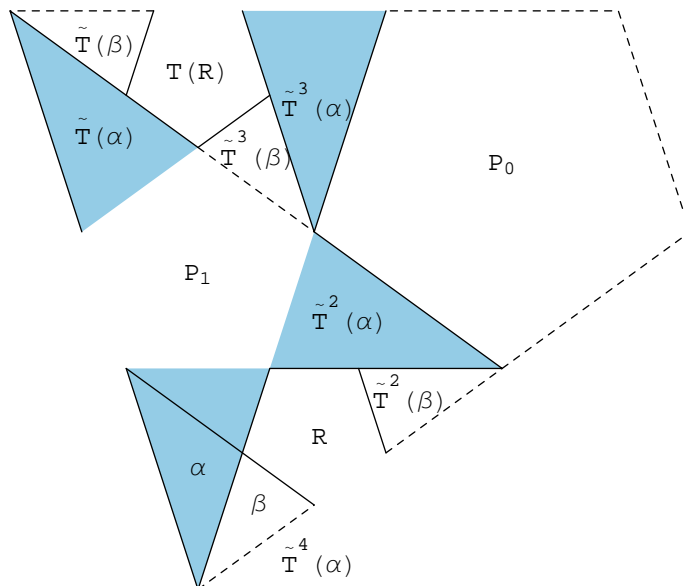

FIGURE 10. Self Inducing Structure of $(T(\mathcal{Z}), \widetilde{T})$

the set $\mathbf{A}$.

Readers may notice that we can find a self-inducing structure by smaller scaling constant ω in Figure 7(b) by taking two connected pieces. However this choice of inducing region is not suitable because the self-inducing relation (with flipping) is measure theoretically valid, but has different behavior on the boundary.

Let us introduce two codings. First is the coding of T-orbits of a point x in L in two symbols $\{0,1\}$: $\mathbf{d}(x) = (\psi(T^n(x))_n \in \{0,1\}^{\mathbb{N}}$ where

$$\psi(x) = \begin{cases} 0 & x \in \Delta \\ 1 & x \in \mathcal{Z} \end{cases}.$$

For e.g., the $\mathbf{d}(1/3) = 10110101011010101101101101\ldots$ The second coding is defined by $\widetilde{\mathbf{d}}(x) = (\widetilde{\psi}(\widetilde{T}^n(x)))_n \in \{a,b\}^{\mathbb{N}}$ for $x \in T(\mathcal{Z})$ where

$$\widetilde{\psi}(x) = \begin{cases} a & x \in \Delta \\ b & x \in T(\mathcal{Z}) \setminus \Delta. \end{cases}$$

For a point x in $T(\mathcal{Z})$ we have two codings by $\{a,b\}$ and by $\{0,1\}$. From Equation (4), the two codings are equivalent through the substitution $a \to 01$, $b \to 1$. For a given coding of T-orbit by $\{0,1\}$, there is a unique way

to retrieve the coding of $\widetilde{T}$-orbit by $\{a, b\}$, because the symbol 0 must be followed by 1. For e.g., $T(1/3) = -2\zeta^{-1}/3 \in T(\mathcal{Z})$ is coded in two ways as:

$$\widetilde{\mathbf{d}}(-2\zeta^{-1}/3) = a\,b\,a\,a\,a\,b\,a\,a\,a\,b\,a\,b\,a\,b\,a\,b\ldots$$

and

$$\mathbf{d}(-2\zeta^{-1}/3) = 01\,1\,01\,01\,01\,1\,01\,01\,01\,1\,01\,1\,01\,1\,01\,1\ldots$$

Hereafter we discuss the coding $\widetilde{\mathbf{d}}$. Observing the trajectory of the region $\omega^{-2}(\Delta)$ and $\omega^{-2}(T(\mathcal{Z}) \setminus \Delta)$ by the first return map by the iteration of $\widetilde{T}$ to the region $\omega^{-2}T(\mathcal{Z})$, it is natural to introduce a substitution σ_0:

$$a \to aaba, \quad b \to baba.$$

on $\{a, b\}^*$ and we have

$$\widetilde{\mathbf{d}}(\omega^{-2}x) = \sigma_0(\widetilde{\mathbf{d}}(x))$$

for $x \in T(\mathcal{Z})$. More generally, following the analogy of the previous section, the first hitting map to the region $\omega^{-2}(T(\mathcal{Z}))$ provide us an expansion of a point $x \in T(\mathcal{Z})$ exactly in the same form as (2) with restricted digits $\{d_0, d_2, d_3, d_5\}$. One can confirm that

$$(6) \qquad \widetilde{\mathbf{d}}\left(\frac{\zeta^m}{\omega^2}x + d_m\right) = \begin{cases} \sigma_0(\widetilde{\mathbf{d}}(x)) & m = 0 \\ a \oplus \sigma_0(\widetilde{\mathbf{d}}(\widetilde{T}(x))) & m = 2 \\ ba \oplus \sigma_0(\widetilde{\mathbf{d}}(\widetilde{T}^2(x))) & m = 3 \\ aba \oplus \sigma_0(\widetilde{\mathbf{d}}(\widetilde{T}^3(x))) & m = 5 \end{cases}$$

where $\oplus$ is the concatenation of letters. Defining conjugate substitutions by $\sigma_1 = a\sigma_0 a^{-1}$, $\sigma_2 = ba\sigma_1 a^{-1}b^{-1}$ and $\sigma_3 = aba\sigma_0 a^{-1}b^{-1}a^{-1}$, i.e.,

$$\begin{aligned} \sigma_0(a) &= aaba, & \sigma_0(b) &= baba \\ \sigma_1(a) &= aaab, & \sigma_1(b) &= abab \\ \sigma_2(a) &= baaa, & \sigma_2(b) &= baba \\ \sigma_3(a) &= abaa, & \sigma_3(b) &= abab \end{aligned}$$

one may rewrite

$$\widetilde{\mathbf{d}}\left(\frac{\zeta^m}{\omega^2}x + d_m\right) = \begin{cases} \sigma_0(\widetilde{\mathbf{d}}(x)) & m = 0 \\ \sigma_1(\widetilde{\mathbf{d}}(\widetilde{T}(x))) & m = 2 \\ \sigma_2(\widetilde{\mathbf{d}}(\widetilde{T}^2(x))) & m = 3 \\ \sigma_3(\widetilde{\mathbf{d}}(\widetilde{T}^3(x))) & m = 5. \end{cases}$$

We say that an infinite word y in $\{a, b\}^{\mathbb{N}}$ is an S-adic limit of σ_i $(i = 0, 1, 2, 3)$ if there exist $y_i \in \{a, b\}^{\mathbb{N}}$ for $i = 1, 2, \ldots$ such that

$$y = \lim_{\ell \to \infty} \sigma_{m_1} \circ \sigma_{m_2} \circ \sigma_{m_3} \circ \cdots \circ \sigma_{m_\ell}(y_\ell).$$

with $m_i \in \{0, 1, 2, 3\}$. Since each element $x \in T(\mathcal{Z}) \cap \mathbf{A}$ has an infinite expansion (2) with digits $\{d_0, d_2, d_3, d_5\}$, we find $x_i \in \omega^{-2}T(\mathcal{Z})$ such that

$$\widetilde{\mathbf{d}}(x) = \lim_{\ell \to \infty} \sigma_{m_1} \circ \sigma_{m_2} \circ \sigma_{m_3} \circ \cdots \circ \sigma_{m_\ell}(\widetilde{\mathbf{d}}(x_\ell)).$$

This shows that $\widetilde{\mathbf{d}}(x)$ is an S-adic limit of σ_i $(i = 0, 1, 2, 3)$.

Note that from the definition (7) of σ_i, for a given S-adic limit y there is an algorithm to retrieve uniquely the sequence $(\sigma_{m_i})_i$. Checking first four letters of y, we know the first letter of y_1 and to determine m_1 we need first 6 letters. We can iterate this process easily.

Summing up, we embedded the set $\mathbf{A} \cap T(\mathcal{Z})$ into the attractor Y' of an IFS (3) and succeeded in characterizing the coding of $\widetilde{T}$-orbits of points in this attractor as a set of S-adic limits on $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$. However recalling that points in closed pentagons P_1 and P_2 are T-periodic and Y' is a non-empty compact set, we see from Figure 7(b) that $\mathbf{A}$ is a proper subset of Y'.

We wish to characterize the set of aperiodic points in Y' and its coding through $\widetilde{\mathbf{d}}$. Recalling the discussion in the previous section, if $x \in T(\mathcal{Z})$ has periodic T-orbits if and only if there exists a positive integer k such that $S^k(x) \in P_0 \cup P_1 \cup P_2$. The equivalent statement in the induced system $(T(\mathcal{Z}), \widetilde{T})$ is that $x \in T(\mathcal{Z})$ is $\widetilde{T}$-periodic if and only if there exists a positive integer k such that $S^k(x) \in P_0 \cup P_1$. Note that we have:

$$\widetilde{T}(x) = \begin{cases} \zeta^{-1}(x - p) + p & x \in P_0 \\ \zeta^{-2}(x - q) + q & x \in P_1 \end{cases}$$

where $p = \frac{1}{2} + i\frac{\sqrt{5(5+2\sqrt{5})}}{10}$ (resp. $q = i\sqrt{\frac{5+\sqrt{5}}{10}}$) is the center of P_0 (resp. P_1) and consequently $\widetilde{T}^5(x) = x$ holds for $x \in P_0 \cup P_1$. If $x \in T(\mathcal{Z})$ and x is $\widetilde{T}$-periodic, then there exist $x_i \in T(\mathcal{Z})$ such that $x_\ell \in P_0 \cup P_1$ and

$$x = d_{m_1} + \frac{\zeta^{m_1}}{\omega^2}\left(d_{m_2} + \frac{\zeta^{m_2}}{\omega^2}\left(d_{m_3} + \frac{\zeta^{m_3}}{\omega^2} \ldots \left(d_{m_\ell} + \frac{\zeta^{m_\ell}}{\omega^{2\ell}}x_\ell\right)\ldots\right)\right),$$

with $m_i \in \{0, 2, 3, 5\}$. Thus the set of $\widetilde{T}$-periodic points in $T(\mathcal{Z})$ consists of all the pentagons of the form

$$(7) \quad d_{m_1} + \frac{\zeta^{m_1}}{\omega^2}\left(d_{m_2} + \frac{\zeta^{m_2}}{\omega^2}\left(d_{m_3} + \frac{\zeta^{m_3}}{\omega^2} \ldots \left(d_{m_\ell} + \frac{\zeta^{m_\ell}}{\omega^{2\ell}}P_j\right)\ldots\right)\right)$$

with $j = 0, 1$ and $m_j \in \{0, 2, 3, 5\}$. From the self-inducing structure (5), it is easy to see that if two points x, x' are in the same pentagon of above shape and none of them is the center, then they have exactly the same periods. Moreover two $\widetilde{T}$-orbits keeps constant distance, i.e., $\widetilde{T}^n(x) - \widetilde{T}^n(x') = \zeta^s(x - x')$ for some integer s. The period is completely determined in [3]. We have all the periodic orbits in $T(\mathcal{Z})$ and therefore have a geometric description of aperiodic points:

$$\mathbf{A} \cap T(\mathcal{Z}) = T(\mathcal{Z}) \setminus \{\text{All pentagons of the form (7)}\}.$$

Subtraction of these pentagons from $T(\mathcal{Z})$ is described by an algorithm. The initial set is $D_0 = T(\mathcal{Z}) \setminus P_0$ with two open and three closed edges as in the left Figure 11. The interior $\text{Inn}(D_0)$ gives another feasible open set

to assure the open set condition of the IFS of (3). Inductively we define the decreasing sequence of sets

$$D_{i+1} = \bigcup_{m \in \{0,2,3,5\}} \left(\frac{\zeta^m}{\omega^2} D_i + d_m \right)$$

for $i = 0, 2, \ldots$. Then D_i consists of 4^i pieces congruent to $\omega^{-2i} D_0$ without overlapping. Note that since

$$D_1 = D_0 \setminus (P_1 \cup \omega^{-2} P_0 \cup \frac{\omega^{-2} P_0 - 1}{\zeta}),$$

D_1 is obtained by subtracting from D_0 one closed and two open regular pentagons as in Figure 11. To generate D_{i+1}, each 4^i pieces in D_i are
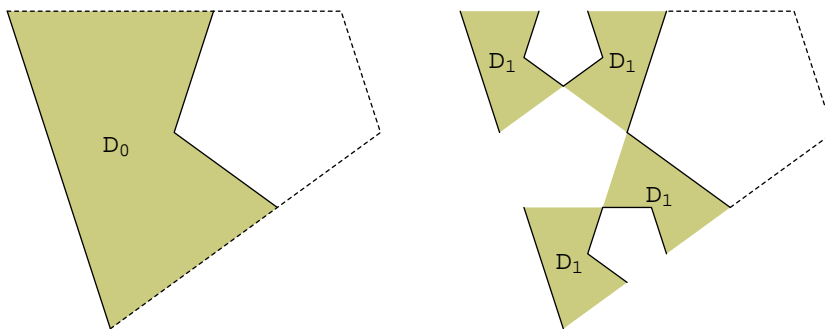

FIGURE 11. Pentagon Removal Algorithm

subdivided into 4 sub-pieces by subtracting three small regular pentagons. Clearly all regular pentagons of the shape (7) are subtracted by this iteration and we obtain

$$\mathbf{A} \cap T(\mathcal{Z}) = \bigcap_{i=0}^{\infty} D_i.$$

This observation allows us to symbolically characterize aperiodic points in Y'. First, every point x of Y' has an address $d_{m_1} d_{m_2} \cdots \in \{d_0, d_2, d_3, d_5\}^{\mathbb{N}}$ by the expansion (2). The address is unique but for countable exceptions. The exceptional points forms the set of cut points of Y' having the eventually periodic expansion:

$$\begin{aligned} d_0 d_2 (d_0)^{\infty} &\simeq d_3 d_3 (d_5)^{\infty} \\ d_3 (d_0)^{\infty} &\simeq d_2 (d_5)^{\infty} \\ d_2 d_2 (d_0)^{\infty} &\simeq d_5 d_3 (d_5)^{\infty} \end{aligned}$$

in the suffix of its address, which is understood by Figure 12 where $K_{mn} = \frac{\zeta^m}{\omega^2} (\frac{\zeta^n}{\omega^2} K + d_n) + d_m$.
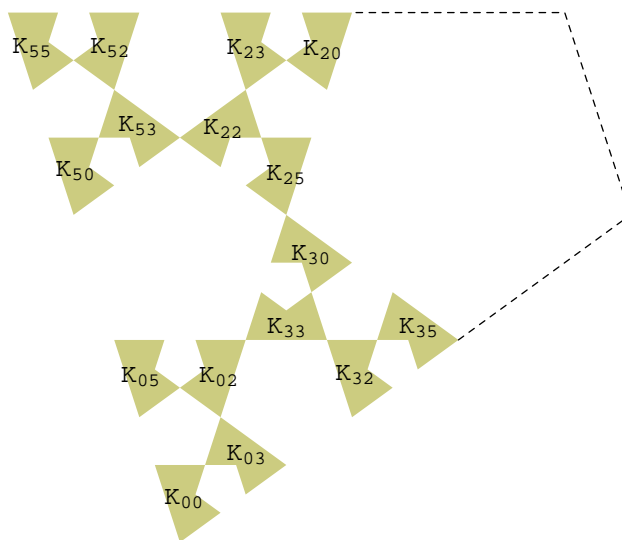
FIGURE 12. Subdivision procedure

Note that if a point x in $T(\mathcal{Z})$ is periodic, then there exists a non-negative integer k such that $S^k(x) \in P_0 \cup P_1$. Moreover, if $x \in Y' \cap T(\mathcal{Z})$, then there exists a non-negative integer k such that $S^k(x) \in \partial(P_1)$, because it can not be an inner point of P_0 or P_1. In other words, such x must be located in the open edge of one of 4^k pieces of D_k. From Figure 11, one can construct the following Figure 13 which recognize points of two open edges in $\partial(D_0)$. For construction, we introduce a new symbol set $\{R, L\}$ (right and left) to distinguish which open edge of D_k is into focus.

To read the graph and obtain the previous sequences, ignore $\{R, L\}$ and substitute $\{0, 2, 3, 5\}$ with $\{d_0, d_2, d_3, d_5\}$. A point $x \in Y'$ is periodic (or in the open edge of D_0) if and only if a suffix of the address $d_{m_1} d_{m_2} \cdots \in \{d_0, d_2, d_3, d_5\}^{\mathbb{N}}$ is in Figure 13. Note that the points with double addresses are on the open edge of some D_i and consequently their suffixes are read in Figure 13. Figure 12 helps this construction. For e.g., the right open edge of K_5 consists of the left open edge of K_{53} and the right open edge of K_{50}, therefore we draw outgoing edges from $5R$ to $3L$ and $0R$.

As a result, the set of addresses of the points in $\mathbf{A} \cap T(\mathcal{Z})$ are recognized by a Büchi automaton which is the complement of the Büchi automaton of Figure 14. Here the double bordered states in Figure 14 are final states. Each infinite word produced by the edge labels $\{d_0, d_2, d_3, d_5\}$ on this directed graph is accepted, because it visits infinitely many times the final states. We do not give here the exact shape of its complement. It is known that complementation of a Büchi automaton is much harder than the one of a
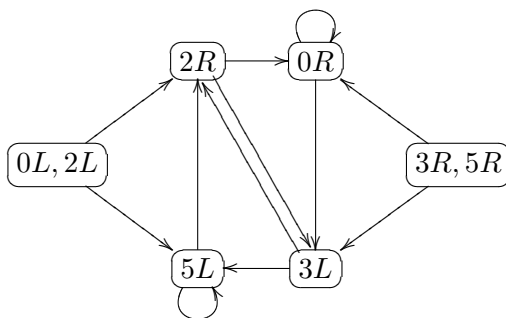
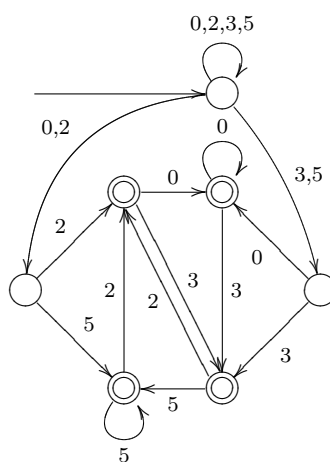FIGURE 13. $\widetilde{T}$-periodic expansions

FIGURE 14. Büchi automaton for periodic points in Y'

finite automaton, because the subset construction does not work (c.f. [29, 23]).

Now consider the topology of $\{a, b\}^{\mathbb{N}}$ induced from the metric defined by $2^{-\max_{x_i \neq y_i} i}$ for $x = x_1 x_2 \ldots, y = y_1 y_2 \cdots \in \{a, b\}^{\mathbb{N}}$. Take a fixed point $w = (w_i)_{i=0,1,2,\ldots} \in \{a, b\}^{\mathbb{N}}$ with $\sigma_0(w) = w$. This is computed for e.g., by $\lim_n \sigma_0^n(a)$. The shift map V is a continuous map from $\{a, b\}^{\mathbb{N}}$ to itself defined by $V((w_i)) = (w_{i+1})$. Letting X_{σ_0} be the closure of the set $\{V^n(w) \mid n = 0, 1, \ldots\}$, we can define the substitutive dynamical system (X_{σ_0}, V) associated with σ_0. Since σ_0 is primitive the set X_{σ_0} does not depend on the choice of the fixed point and (X_{σ_0}, V) is minimal and uniquely ergodic (see [9]). Let τ be the invariant measure of (X_{σ_0}, V). On the other hand, for the attractor Y' there is the self-similar measure ν, i.e., a unique probability measure (c.f. Hutchinson [13]) satisfying

$$\nu(X) = \frac{1}{4} \sum_{m \in \{0,2,3,5\}} \nu\left(\frac{\omega^2}{\zeta^m}(X - d_m)\right)$$

for ν-measurable sets $\mathbb{B}_{Y'}$ in Y'.

Theorem 3. *The restriction of $\widetilde{T}$ to Y' is measure preserving and $(Y', \mathbb{B}_{Y'}, \nu, \widetilde{T})$ is isomorphic to the 2-adic odometer $(\mathbb{Z}_2, x \mapsto x + 1)$ as measure dynamical systems:*

$$
\begin{array}{ccc}
\mathbb{Z}_2 & \xrightarrow{+1} & \mathbb{Z}_2 \\
\phi \downarrow & & \phi \downarrow \\
Y' & \xrightarrow{\widetilde{T}} & Y'
\end{array}
$$
(8)

where $\phi : \mathbb{Z}_2 \to Y'$ is almost one to one and measure preserving, which will be made explicit in the proof. Moreover the map

$$ \rho : x \mapsto \frac{x - (x \bmod 4)}{4} $$

from $\mathbb{Z}_2$ to itself gives a commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Z}_2 & \xrightarrow{\rho} & \mathbb{Z}_2 \\
\phi \downarrow & & \phi \downarrow \\
Y' & \xrightarrow{S} & Y'.
\end{array}
$$
(9)

The above theorem may be read that $(Y', \mathbb{B}_{Y'}, \nu, \widetilde{T})$ gives a one-sided variant of numeration system in the sense of Kamae [15].

Proof. First we confirm that $\widetilde{T}$ is measure preserving. Denote by $[d_{m_1}, d_{m_2}, \ldots, d_{m_\ell}]$ the cylinder set:

$$
(10) \qquad d_{m_1} + \frac{\zeta^{m_1}}{\omega^2} \left(d_{m_2} + \frac{\zeta^{m_2}}{\omega^2} \left(d_{m_3} + \frac{\zeta^{m_3}}{\omega^2} \ldots \left(d_{m_\ell} + \frac{\zeta^{m_\ell}}{\omega^{2\ell}} Y' \right) \ldots \right) \right)
$$

By the OSC, we have $\nu([d_{m_1}, d_{m_2}, \ldots, d_{m_\ell}]) = 4^{-\ell}$. From Figure 10, we see that $\widetilde{T}^{-1}([d_3]) = [d_5]$, $\widetilde{T}^{-1}([d_2]) = [d_3]$, $\widetilde{T}^{-1}([d_0]) = [d_2]$ but $\widetilde{T}^{-1}([d_5])$ intersects both A and B. Hence if $m_1 = 0, 2, 3$, then $\nu(\widetilde{T}^{-1}([d_{m_1}, d_{m_2}, \ldots, d_{m_\ell}])) = 4^{-\ell}$. By using the self-inducing structure in Figure 10, we also have $\widetilde{T}^{-1}([d_5 d_3]) = [d_0 d_5]$, $\widetilde{T}^{-1}([d_5 d_2]) = [d_0 d_3]$ and $\widetilde{T}^{-1}([d_5 d_0]) = [d_0 d_2]$. Thus if $m_2 = 0, 2, 3$, then $\nu(\widetilde{T}^{-1}([d_5, d_{m_2}, \ldots, d_{m_\ell}])) = 4^{-\ell}$. Repeating this, we can show that

$$ \nu(\widetilde{T}^{-1}([d_{m_1}, d_{m_2}, \ldots, d_{m_\ell}])) = 4^{-\ell} $$

holds for all $m_i \in \{0, 2, 3, 5\}$ but a single exception $m_1 = m_2 = \cdots = m_\ell = 5$. Since ℓ is arbitrary chosen, a simple approximation argument shows that $\widetilde{T}$ is measure preserving and $(Y', \mathbb{B}_{Y'}, \nu, \widetilde{T})$ forms a measure dynamical system.

Let us define a map η from X_{σ_0} to Y'. Take an element $z = x_1 x_2 \cdots \in X_{\sigma_0}$. Then each prefix $x_1 x_2 \ldots x_\ell$ with $\ell > 3$ is a subword of the fix point v of σ_0 starting with a. Therefore there is a word $y \in \{\lambda, a, ba, aba\}$ and $z_1 \in X_{\sigma_0}$ such that $x = y_1 \sigma_0(z_1)$. It is easy to see from (6) that this y and z_1 are unique.
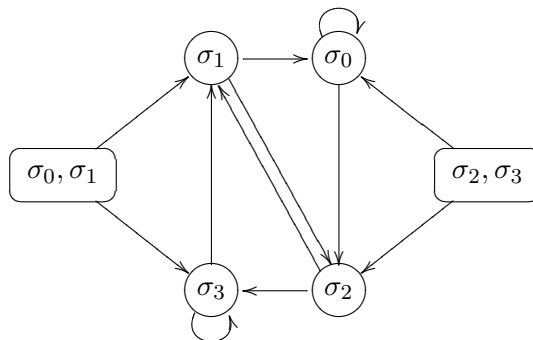
FIGURE 15. Forbidden suffix of $\mathbf{A'}$

Iterating this we have $z_i = y_{i+1}\sigma(z_{i+1})$ with $y_i \in \{\lambda, a, ba, aba\}$, $z_{i+1} \in X_{\sigma_0}$ and $z_0 = z$. Thus we have for any ℓ,

$$\begin{aligned}
z &= y_1\sigma_0(y_2\sigma_0(y_3\sigma_0 \ldots y_\ell(\sigma_0(z_\ell)))) \\
&= y_1\sigma_0(y_2)\sigma_0^2(y_3)\ldots\sigma_0^{\ell-1}(y_\ell)\sigma_0^\ell(z_\ell).
\end{aligned}$$

Define a map from $\{\lambda, a, ba, aba\}$ to $\mathbb{Z}$ by

$$\kappa(\lambda) = 0, \kappa(a) = 1, \kappa(ba) = 2, \kappa(aba) = 3.$$

Then $z_i = y_{i+1}\sigma(z_{i+1})$ is equivalent to $z_i = \sigma_{\kappa(y_{i+1})}(z_{i+1})$ and z is represented as an S-adic limit:

$$z = \lim_{\ell \to \infty} \sigma_{\kappa(y_1)} \circ \sigma_{\kappa(y_2)} \circ \cdots \circ \sigma_{\kappa(y_\ell)}(z_\ell).$$

for $\ell = 1, 2, \ldots$. This gives a **multiplicative coding $\mathbf{d'}$** $: X_{\sigma_0} \to \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}^{\mathbb{N}}$. Let $\mathbf{A'}$ be the points of X_{σ_0} whose multiplicative coding does not end up in an infinite word produced by reading the vertex labels of Figure 15. Let us associate to z a 2-adic integer $\iota(z) = -\sum_{i=0} \kappa(y_i)2^{2i} \in \mathbb{Z}_2$. The map ι is clearly bijective bi-continuous and the value $\iota(z)$ is also called the multiplicative coding of z. We write down first several iterates of V on the fix point of σ_0, to illustrate the situation:

$$\begin{aligned}
\sigma_0\sigma_0\sigma_0\sigma_0\ldots &\xrightarrow{\iota} -0000\ldots \\
\sigma_3\sigma_3\sigma_3\sigma_3\ldots &\xrightarrow{\iota} -3333\ldots \\
\sigma_2\sigma_3\sigma_3\sigma_3\ldots &\xrightarrow{\iota} -2333\ldots \\
\sigma_1\sigma_3\sigma_3\sigma_3\ldots &\xrightarrow{\iota} -1333\ldots \\
\sigma_0\sigma_3\sigma_3\sigma_3\ldots &\xrightarrow{\iota} -0333\ldots \\
\sigma_3\sigma_2\sigma_3\sigma_3\ldots &\xrightarrow{\iota} -3233\ldots
\end{aligned}$$

One can see that the following commutative diagram (11) holds.

(11)
$$\begin{array}{ccc}
X_{\sigma_0} & \xrightarrow{V} & X_{\sigma_0} \\
\iota \downarrow & & \iota \downarrow \\
\mathbb{Z}_2 & \xrightarrow{+1} & \mathbb{Z}_2
\end{array}$$

Therefore (X_{σ_0}, V) is topologically conjugate to the 2-adic odometer $(\mathbb{Z}_2, x \mapsto x + 1)$. Here the consecutive digits $\{0, 1\}$ in $\mathbb{Z}_2$ are glued together to give $\{0, 1, 2, 3\} = \{0, 1\} + 2\{0, 1\}$. Indeed, σ_0 satisfies the coincidence condition of height one in the sense of Dekking [25, 9] and above conjugacy is a consequence of this. $(\mathbb{Z}_2, x \mapsto x + 1)$ is a translation of a compact group $\mathbb{Z}_2$ which is minimal and uniquely ergodic with the Haar measure of $\mathbb{Z}_2$. Moreover one can confirm that ι preserves the measure and (X_{σ_0}, V) and $(\mathbb{Z}_2, x \mapsto x + 1)$ are isomorphic through ι as measure dynamical systems. In view of (6), we define

$$\xi(i) = \begin{cases} 0 & i = \lambda \\ 2 & i = a \\ 3 & i = ba \\ 5 & i = aba \end{cases}$$

and the map $\eta : X_{\sigma_0} \to Y'$ by

$$(12) \qquad \eta(x) = d_{\xi(y_1)} + \frac{\zeta^{\xi(y_1)}}{\omega^2} \left(d_{\xi(y_2)} + \frac{\zeta^{\xi(y_2)}}{\omega^2} \left(d_{\xi(y_3)} + \frac{\zeta^{\xi(y_3)}}{\omega^2} \dots \right. \right.$$

Then η is clearly surjective, continuous, and measurable because both τ and ν are Borel probability measures. Since the set of points with double addresses is on the open edge, the map η is bijective from $\mathbf{A}'$ to $\mathbf{A} \cap T(\mathcal{Z})$. Since $\widetilde{\mathbf{d}}(T(x)) = V(\widetilde{\mathbf{d}}(x))$, we have a commutative diagram:

$$(13) \qquad \begin{array}{ccc} \mathbf{A}' & \xrightarrow{V} & \mathbf{A}' \\ \eta \downarrow & & \eta \downarrow \\ \mathbf{A} \cap T(\mathcal{Z}) & \xrightarrow{\widetilde{T}} & \mathbf{A} \cap T(\mathcal{Z}). \end{array}$$

From Figure 14, it is easy to see that the set $\mathcal{P}$ of $\widetilde{T}$-periodic points in Y' is measure zero by ν, i.e., $\nu(\mathbf{A} \cap T(\mathcal{Z})) = \nu(Y' \cap T(\mathcal{Z})) = 1$, because the number of words of length n in Figure 14 is $O(2^n)$. Similarly as the Perron-Frobenius root of the substitution σ_0 is 4 and the number of words of lengths n in Figure 15 are $O(2^n)$, we see that $\tau(\mathbf{A}') = \tau(X_{\sigma_0}) = 1$. From (13) the pull back measure $\nu \circ \eta^{-1}$ of X_{σ_0} is invariant by V, we have $\tau = \nu \circ \eta^{-1}$ by unique ergodicity. Therefore by taking $\phi = \eta \circ \iota$, we have the commutative diagram (8) with measure zero exceptions. Let V' be a map from X_{σ_0} to itself which acts as the shift operator on the multiplicative coding $\mathbf{d}'$, i.e., $(\mathbf{d}'(V'(z)) = \sigma_{n_2}\sigma_{n_3} \dots$ for $\mathbf{d}'(z) = \sigma_{n_1}\sigma_{n_2} \dots$. Then we see that

$$(14) \qquad \begin{array}{ccc} \mathbf{A}' & \xrightarrow{V'} & \mathbf{A}' \\ \eta \downarrow & & \eta \downarrow \\ \mathbf{A} \cap T(\mathcal{Z}) & \xrightarrow{S} & \mathbf{A} \cap T(\mathcal{Z}). \end{array}$$

and the commutative diagram (9) is valid but for measure zero exceptions.

$\square$

Corollary 4. *Each aperiodic point $x \in \mathbf{A} \cap T(\mathcal{Z})$, the $\widetilde{T}$-orbit of x is uniformly distributed in Y' with respect to the self similar measure ν.*

Proof. In the proof of Theorem 3 the map η is bijective form $\mathbf{A}'$ to $\mathbf{A} \cap T(\mathcal{Z})$. Therefore if $x \in \mathbf{A} \cap T(\mathcal{Z})$, then there exists a unique element in $z \in X_{\sigma_0}$ with $\eta(z) = x$. Therefore there exist an element $z_0 \in \mathbb{Z}_2$ such that $\phi(z_0) = x$. The Haar measure μ_2 on Z_2 is given by the values on the semi-algebra:

$$\mu_2([c_0, c_1, \ldots, c_{\ell-1}]) = 4^{-\ell}$$

for each cylinder set $[c_0, c_1, \ldots, c_{\ell-1}] = \{y \in \mathbb{Z}_2 \mid y \equiv \sum_{i=0}^{\ell-1} c_i 4^i \pmod{4^\ell}\}$. Since $(\mathbb{Z}_2, x \mapsto x + 1)$ is uniquely ergodic, the assertion follows immediately from the commutative diagram (8). $\square$

Not all points in Y' gives a dense orbit as we already mentioned that $\mathbf{A} \cap Y'$ is a proper dense subset of Y'. There are many periodic points in Y' as well. This gives a good contrast to usual minimal topological dynamics given by a continuous map acting on a compact metrizable space.

Corollary 5. *Each aperiodic point $x \in \mathbf{A}$, the T-orbit of x is dense in the set X.*

Proof. It is clear from the fact that $(T(\mathcal{Z}), \widetilde{T})$ is the induced system of (L, T). $\square$

One can construct a dual expansion of the non-invertible dynamics (Y', S) by the conjugate map $\phi : \zeta \to \zeta^2$ in $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ and then make a natural extension: an invertible dynamics which contains (Y', S). The idea comes from symbolic dynamics. We wish to construct the reverse expansion of (12) to the other direction. To this matter, we compute in the following way:

$$\frac{\omega^2(\eta(x) - d_{\xi(y_1)})}{\zeta^{\xi(y_1)}} = d_{\xi(y_2)} + \frac{\zeta^{\xi(y_2)}}{\omega^2}\left(d_{\xi(y_3)} + \frac{\zeta^{\xi(y_3)}}{\omega^2}\left(d_{\xi(y_4)} + \frac{\zeta^{\xi(y_4)}}{\omega^2}(\ldots\right.\right.$$

and

$$\frac{\omega^2}{\zeta^{\xi(y_2)}}\left(\frac{\omega^2(\eta(x) - d_{\xi(y_1)})}{\zeta^{\xi(y_1)}} - d_{\xi(y_2)}\right) = d_{\xi(y_3)} + \frac{\zeta^{\xi(y_3)}}{\omega^2}\left(d_{\xi(y_4)} + \frac{\zeta^{\xi(y_4)}}{\omega^2}(\ldots\right.$$
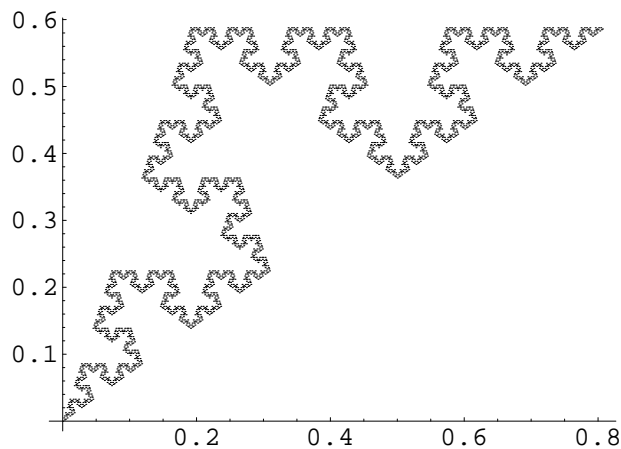
Therefore it is natural to introduce a left 'expansion':

$$\frac{\omega^2}{\zeta^{i_1}}\left(\frac{\omega^2}{\zeta^{i_2}}\left(\frac{\omega^2}{\zeta^{i_3}}((\ldots) - d_{i_3}) - d_{i_2}\right) - d_{i_1}\right)$$

with $i_k \in \{0, 2, 3, 5\}$. As this expression does not converge, we take the image of ϕ because $\phi(\omega) = -1/\omega$. Let us denote by $u_{i_k} = \phi(d_{i_k})$. Then the expansion

$$\frac{\zeta^{-2i_1}}{\omega^2}\left(\frac{\zeta^{-2i_2}}{\omega^2}\left(\frac{\zeta^{-2i_3}}{\omega^2}((\ldots) - u_{i_3})\right) - u_{i_2}\right) - u_{i_1}$$

converges and the closure of the set of such expansions gives a compact set $\mathcal{Y}$ depicted in figure 16.

FIGURE 16. The dual attractor $\mathcal{Y}$

Of course the set is an attractor of the IFS:

$$\mathcal{Y} = \frac{1}{\omega^2}(\mathcal{Y} - u_0) \cup \frac{\zeta}{\omega^2}(\mathcal{Y} - u_2) \cup \frac{\zeta^{-1}}{\omega^2}(\mathcal{Y} - u_3) \cup \frac{1}{\omega^2}(\mathcal{Y} - u_5).$$

Combining $\mathcal{Y}$ we can construct a natural extension of (Y', S) as:

$$Y' \times \mathcal{Y} \ni (\eta, \theta) \overset{\hat{S}}{\mapsto} \left(\frac{(\eta - d_i)\omega^2}{\zeta^i}, \frac{\zeta^{-2i}(\theta - \phi(d_i))}{\omega^2} \right) \in Y' \times \mathcal{Y}$$

On the other hand $(\mathbb{Z}_2, \rho)$ have a natural extension:

$$\mathbb{Z}_2 \times [0, 1) \ni (x, y) \overset{\hat{\rho}}{\mapsto} \left(\frac{x - (x \bmod 4)}{4}, \frac{y + (x \bmod 4)}{4} \right) \in \mathbb{Z}_2 \times [0, 1)$$

and two systems are isomorphic both as topological and measure theoretical dynamics:

(15)
$$\begin{array}{ccc} \mathbb{Z}_2 \times [0, 1) & \overset{\hat{\rho}}{\longrightarrow} & \mathbb{Z}_2 \times [0, 1) \\ {\scriptstyle \phi \times \phi'} \downarrow & & {\scriptstyle \phi \times \phi'} \downarrow \\ Y' \times \mathcal{Y} & \overset{\hat{S}}{\longrightarrow} & Y' \times \mathcal{Y}. \end{array}$$

where ϕ' is given as:

$$\sum_{i=1}^{\infty} x_i 4^{-i} \mapsto g_{x_1}(g_{x_2}(g_{x_3}(\dots)))$$

where $g_0(x) = (x - u_0)/\omega^2, g_1(x) = (x - u_2)\zeta/\omega^2, g_2(x) = (x - u_3)\zeta^{-1}/\omega^2$ and $g_3(x) = (x - u_5)/\omega^2$.

From this 'algebraic' natural extension construction, we can characterize purely S-periodic points in $Y' \cap \mathbb{Q}(\zeta)$.

Theorem 6. *A point y in $Y' \cap \mathbb{Q}(\zeta)$ has purely periodic multiplicative coding with four digits $\sigma_0, \sigma_2, \sigma_3, \sigma_5$ if and only if $(y, \phi(y)) \in Y' \times \mathcal{Y}$.*

This is an analogy of the results [14] for β-expansion. The proof below is on the same line.

Proof. As ω is an algebraic unit and $d_i \in \mathbb{Z}[\zeta]$, the denominator of $g_i(y)$ is the same as that of y for $i = 0, 1, 2, 3$. Therefore the module $y \in \mathcal{M} = \frac{1}{M}\mathbb{Z}[\zeta]$ is stable by g_i for some positive integer M. Note that points $y \in \mathcal{M}$ with $(y, \phi(y)) \in Y' \times \mathcal{Y}$ is finite, because $y, \phi(y)$ and their complex conjugates are bounded in $\mathbb{C}$. One can confirm that the map $\hat{S}$ becomes surjective from $\mathcal{M}$ to itself. For a finite set, surjectivity implies bijectivity. Therefore a point $y \in \mathcal{M}$ with $(y, \phi(y)) \in Y' \times \mathcal{Y}$ produces a purely periodic orbit. On the other hand if x has purely periodic multiplicative coding, it is easy to see $(y, \phi(y)) \in Y' \times \mathcal{Y}$. $\square$

3. Other self-similar systems

Pisot scaling constants appear in several important dynamics. For irrational rotations (2IET), it is well known that scaling constants of self-inducing systems must be quadratic Pisot units. A typical example Figure 2 was shown in the introduction. They are computed by the continued fraction algorithm as fundamental units of quadratic number fields. Poggiaspalla-Lowenstein-Vivald [24] showed that the scaling constant must be an algebraic unit for self-inducing uniquely ergodic IET. When the scaling constant of self-inducing IET is a cubic Pisot unit, we have further nice properties [4, 19, 20].

A necessary condition that 1-dimensional substitutive point sets give point diffraction is that the scaling constant is a Pisot number [7]. Suspension tiling dynamics of such substitution is conjectured to have pure discrete spectrum if the characteristic polynomial of its substitution matrix is irreducible. For higher dimensional tiling dynamics the Pisot (or Pisot family) property is essential to have relatively dense point spectra, see for e.g. [27, 17].

Pisot scaling properties seem to extend to the case of piecewise isometries. To conclude we present some examples, though we do not make a systematic study.

It is already observed in [16, 3] that Pisot scaling constants appear in our problem if θ is the n-th root of unity for $n = 4, 6, 8, 10, 12$ in the same way as we did in $n = 5$ but in a more involved manner. In each case they are quadratic Pisot units. What about if $\lambda = -2\cos(\theta)$ is cubic? In this case, the dynamics of Conjecture 1 are embedded into the piecewise affine mapping acting on $(\mathbb{R}/\mathbb{Z})^4$ which is harder to visualize. Instead let us consider formal analogies of piecewise isometries generated by cubic n-th fold rotation in the plane. At the expense of losing connection to Conjecture 1, we find many Pisot unit scaling constants! Being an algebraic unit is natural and may

be explained from invertibility of dynamics. However we have no idea why the Pisot numbers turn up or even how to formulate these phenomena as a suitable conjecture.

3.1. **Seven-fold.** We start with 7-fold case. Both pieces are rotated clockwise by $4\pi/7$ as in Figure 17. The triangle is rotated around A and the trapezium around B. The first return map to a region and a smaller region with the same first return map (up to scaling) are described. Unlike the five fold case, returning to the subregion does not cover the full region. A simple consequence is that there are infinitely many possible orbit closures for non-periodic orbits in the system. The scaling constant $\alpha \approx 5.04892$ is a Pisot number whose minimal polynomial is $x^3 - 6x^2 + 5x - 1$. Figure 18 shows how this remaining space can be filled in. As this region is already a little small we will zoom in and now consider just this induced sub-system in Figure 19. The smaller substitutions are easier to see as there are two scalings giving the same dynamics (A and B). The scaling constant $\beta \approx 16.3937$ for these subregions is the Pisot number associated to $x^3 - 17x^2 + 10x - 1$. The proof that the remaining substitutions work is shown in Figure 20. The first return map to the two lower triangles is shown. The same dynamics occur on a smaller region. The orbit of the smaller region covers all the regions left out of Figure 19 and so the substitution rule from that figure is now complete. The scaling constant for this triangle is α. This gives an example of recursive tiling structure by Lowenstein-Kouptsov-Vivaldi [18]. Knowing that every aperiodic orbits are in one of the above self-inducing structures, we can show that

Theorem 7. *Almost all points of this 7-fold lozenge have periodic orbits.*

The argument is similar to that given around Figure 7(a). We easily find decreasing series X_n of union of polygons satisfying $\mu(\alpha X_{n+1}) < \alpha^2 \mu(X_n)$ (or $\mu(\beta X_{n+1}) < \beta^2 \mu(X_n)$) which cover all self-inducing structures.

The fundamental units of the maximal real subfield $\mathbb{Q}(\cos(4\pi/7))$ of the cyclotomic field $\mathbb{Q}(\zeta_7)$ are given by b and $b-1$ where $b = 1/(2\cos(3\pi/7)) \approx 2.24698$. Here b is the Pisot number satisfying $x^3 - 2x^2 - x + 1$. We see that $\alpha = b^2$ and $\beta = b^4/(b-1)^2$ and thus α and β generates a subgroup of fundamental units of $\mathbb{Q}(\cos(4\pi/7))$. Note that both $\sqrt{\alpha} = b$ and $\sqrt{\beta} = b^2/(b-1)$ are Pisot numbers but $b-1$ is not. Our piecewise isometry somehow selects Pisot units out of the unit group!

3.2. **Nine-fold.** The next example is 9-fold case in Figure 21. Both pieces are rotated anti-clockwise by $4\pi/9$, the triangle around A and the trapezium around B. The first return map ($\triangle$) to the triangle is also shown. In addition the same dynamics are found on a smaller piece of the map. Like the 7-fold shown in Figure 17 this does give a full description of the dynamics, but it is $\triangle^2$ not $\triangle$. The scaling constant $\gamma \approx 8.29086$ is a Pisot unit defined by $x^3 - 9x^2 + 6x - 1$. Unfortunately in this case we were not able to find a complete description of the scaling structure.
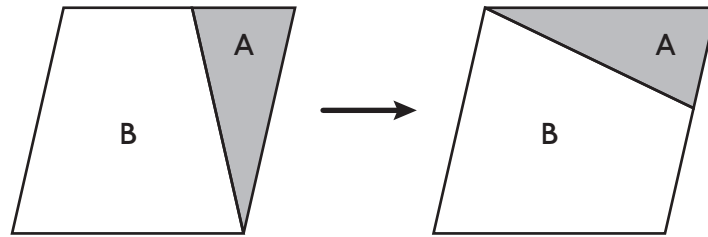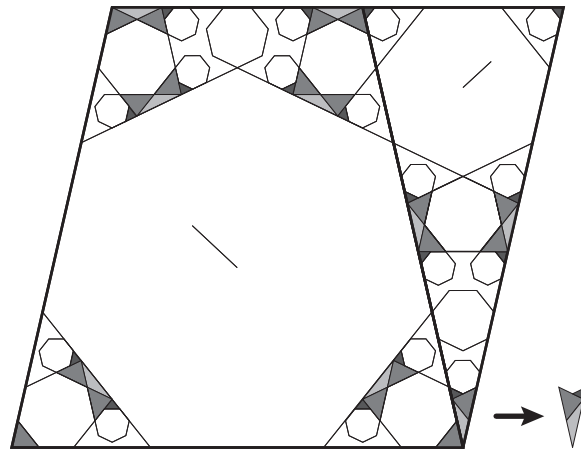
First return to region
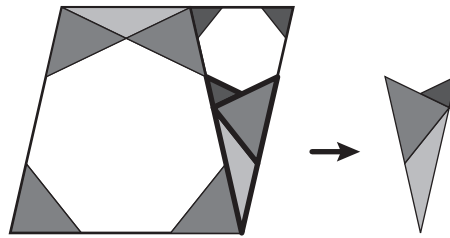
Self-inducing map

FIGURE 17. A seven-fold piecewise isometry.

The fundamental units of $\mathbb{Q}(\cos(4\pi/9))$ are b and $b^2 - 2b - 1$ where $b = 1/(2\cos(4\pi/9)) \approx 2.87939$ is a Pisot number given by $x^3 - 3x^2 + 1$. We have $\gamma = b^2$ and are expecting to find another Pisot unit $b^2/(b^2 - 2b - 1) \approx 5.41147$ (or its square) as a scaling constant in this dynamics, which would give an analogy to the seven-fold case.

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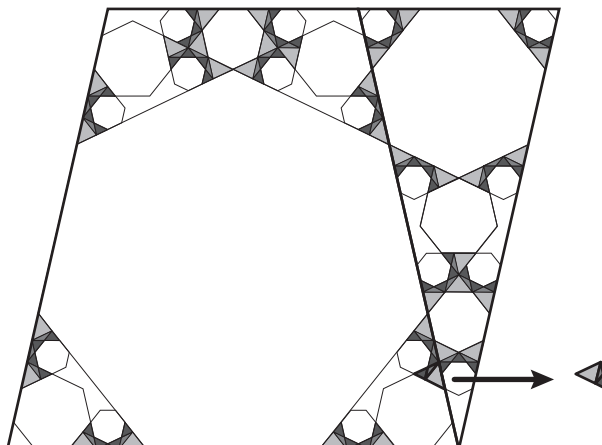
FIGURE 18. The regions remaining from the self-similarity shown in Figure 17

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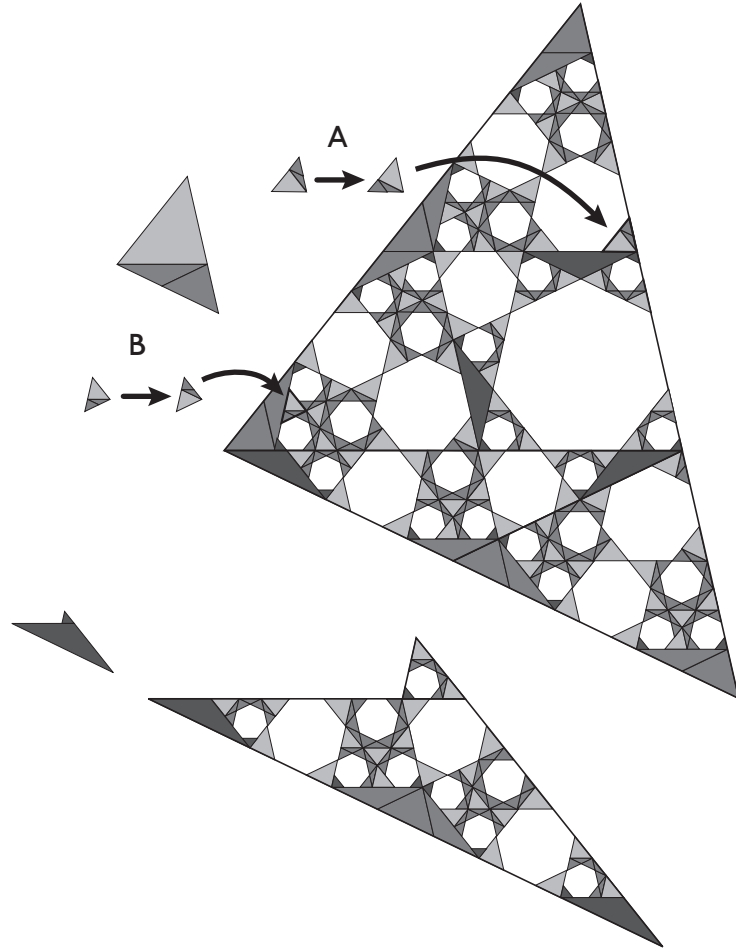
FIGURE 19. The substitution rule of the induced subsystem shown in Figure 18.

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First return map to lower two triangles
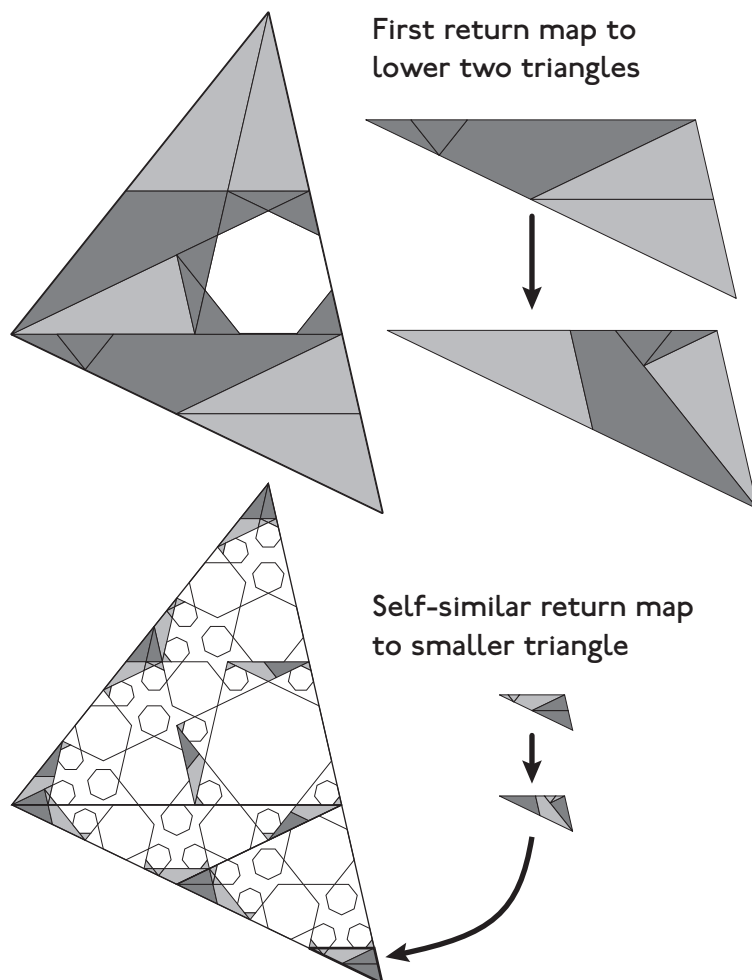
Self-similar return map to smaller triangle

FIGURE 20. The final pieces of the structure of the piecewise isometry found in Figure 17.

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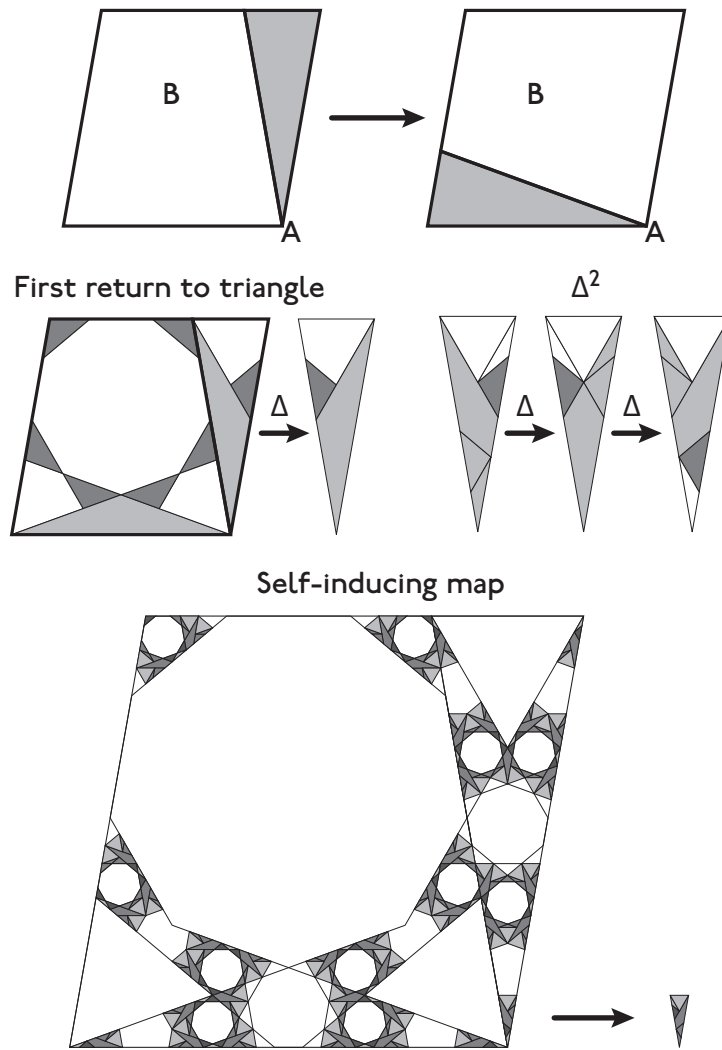
FIGURE 21. A nine fold piecewise isometry.

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