Salem numbers and uniform distribution modulo 1

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1 Introduction

Uniform distribution of sequences of exponential order growth is an attractive and mysterious subject. Koksma's Theorem assures that the sequence (α^n) $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1 for almost all $\alpha > 1$. See [6]. To find an example of such α has been an open problem for a long time. In [7], M. B. Levin constructed an $\alpha > 1$ with more strong distribution properties. His method gives us a way to approximate such α step by step. (See also [4, pp.118–130].) However, no 'concrete' examples of such α are known to date. For instance, it is still an open problem whether (e^n) and $((3/2)^n)$ are dense or not in $\mathbb{R}/\mathbb{Z}$. (c.f. Beukers [2])

On the other hand, one can easily construct $\alpha > 1$ that (α^n) is *not* uniformly distributed modulo 1. A Pisot number gives us such an example. We recall the definition of Pisot and Salem numbers. A *Pisot number* is a real algebraic integer greater than 1 whose conjugates other than itself have modulus less than 1. A *Salem number* is a real algebraic integer greater than 1 whose conjugates other than itself have modulus less than or equal to 1 and at least one conjugate has modulus equal to 1. It is shown that (α^n) tends to 0 in $\mathbb{R}/\mathbb{Z}$ when α is a Pisot number. If α is a Salem number, (α^n) is dense in $\mathbb{R}/\mathbb{Z}$ but not uniformly distributed modulo 1. (See [1, pp.87–89].) Moreover, Salem numbers are the only known 'concrete' numbers whose powers are dense in $\mathbb{R}/\mathbb{Z}$.

In this short note, we will consider a quantitative problem:

How far is the sequence (α^n) from the uniform distribution for a Salem number α?

Let (a_n), $n = 0, 1, \ldots$ be a real sequence and I be an interval in $[0, 1]$. Define a counting function $A_N((a_n), I)$ by the cardinality of $n \in \mathbb{Z} \cap [1, N]$ such that $\{a_n\}$, the fractional part of a_n, lie in I. We shall show

Theorem 1. *Let α be a Salem number of degree greater than or equal to 8. Then $\lim_{N\to\infty}\frac{1}{N}A_N((\alpha^n),I)$ exists and satisfies*

$$\left|\lim_{N\to\infty}\frac{1}{N}A_N((\alpha^n),I)-|I|\right|\le 2\zeta\Big(\frac{\deg\alpha-2}{4}\Big)(2\pi)^{1-\frac{\deg\alpha}{2}}|I|,$$

where $\zeta(s)$ is the Riemann zeta function, $\deg\alpha$ is the degree of α over $\mathbb{Q}$ and $|I|$ is the length of I.

Theorem 2. *Let α be a Salem number of degree 4 or 6. Then $\lim_{N\to\infty}\frac{1}{N}A_N((\alpha^n),I)$ exists and satisfies*

$$\left|\lim_{N\to\infty}\frac{1}{N}A_N((\alpha^n),I)-|I|\right|\le 4\pi^{-\frac{3}{2}}\sqrt{|I|}\quad \textit{for}\quad \deg\alpha=4,$$

and

$$\left|\lim_{N\to\infty}\frac{1}{N}A_N((\alpha^n),I)-|I|\right|\le \frac{|I|}{2\pi^2}\left(\log\frac{1}{|I|}+1+|I|\right)\quad \textit{for}\quad \deg\alpha=6.$$

These theorems show that the sequence (α^n) is quite 'near' to uniformly distributed sequences when the degree of a Salem number α is large.

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2 Proof of the Theorem 1

Let α be a Salem number of degree s. From the definition of Salem numbers, s is an even integer not less than 4, whose conjugates are

$$\alpha,\alpha^{-1},\alpha^{(1)},\dots,\alpha^{(s-2)}$$

with complex $\alpha^{(j)}$ of modulus 1 [1, p.85]. Assume that $\alpha^{(j+r)}=\overline{\alpha^{(j)}}$ for $j=1,\dots,r$ with $r=\frac{s-2}{2}$. Put

$$\alpha^{(j)}=\exp(2\pi i\theta_j)\qquad(0<\theta_j<1)\tag{1}$$

for $1\le j\le r$.

Lemma 1. *Let θ_j be the numbers defined by (1). Then $1,\theta_1,\dots,\theta_r$ are linearly independent over $\mathbb{Q}$.*

Proof. . See for example [1, pp.88–89].

From this lemma, $\{(m\theta_1, m\theta_2, \dots, m\theta_r)\}_{m=1}^{\infty}$ is uniformly distributed mod $\mathbb{Z}^r$. Hence for any Riemannian integrable function $f(x)$ on $(\mathbb{R}/\mathbb{Z})^r$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} f(m\theta_1, \dots, m\theta_r)$$

exists and is equal to

$$\int_{(\mathbb{R}/\mathbb{Z})^r} f(x_1, \dots, x_r) dx_1 \cdots x_r.$$

Let $I = [a, b]$ be an interval in $[0, 1]$ and χ_I the characteristic function of I. We extend χ_I as a periodic function on $\mathbb{R}$ by a period 1. Since $A_N((\alpha^n), I) = \sum_{m=1}^{N} \chi_I(\alpha^m)$ and

$$\alpha^m + \alpha^{-m} + 2\sum_{j=1}^{r} \cos(2\pi m\theta_j) \in \mathbb{Z},$$

we study the limit of

$$S_N(\alpha, I) := \frac{1}{N} \sum_{m=1}^{N} \chi_I\Big(-\alpha^{-m} - 2\sum_{j=1}^{r} \cos(2\pi m\theta_j)\Big) \tag{2}$$

as $N \to \infty$.

For that purpose, we recall the Selberg polynomial which approximates the characteristic function of an interval. Let $\Delta_K(x)$ be the Fejér's kernel defined by

$$\Delta_K(x) = 1 + \sum_{\substack{|k|\le K\\ k\neq 0}} \Big(1 - \frac{|k|}{K}\Big) e^{2\pi i k x},$$

and $V_K(x)$ be the Vaaler's polynomial:

$$V_K(x) = \frac{1}{K+1} \sum_{k=1}^{K} f\Big(\frac{k}{K+1}\Big) \sin(2\pi k x)$$

where $f(u) = -(1-u)\cot(\pi u) - \frac{1}{\pi}$. It is clear that for any η $(0 < \eta \le 1/2)$,

$$|f(u)| \le \begin{cases} \dfrac{\pi\eta}{\sin \pi\eta} \dfrac{1}{\pi u} + \dfrac{1}{\pi} & \text{for } 0 < u \le \eta \\ \dfrac{1-\eta}{\sin \pi(1-\eta)} + \dfrac{1}{\pi} & \text{for } \eta < u < 1. \end{cases} \tag{3}$$

Furthermore let $B_K(x)$ denote the Beurling polynomial:

$$B_K(x) = V_K(x) + \frac{1}{2(K+1)}\Delta_{K+1}(x). \tag{4}$$

Take an interval $J = [a,b]$ in $[0,1]$. Then Selberg polynomials for the interval J are

$$S_K^+(x) = b - a + B_K(x-b) + B_K(a-x) \tag{5}$$

and

$$S_K^-(x) = b - a - B_K(b-x) - B_K(x-a). \tag{6}$$

These functions $S_K^{\pm}(x)$ are trigonometric polynomials of degree at most K and satisfy

$$S_K^-(x) \le \chi_{_J}(x) \le S_K^+(x). \tag{7}$$

See [8] for further properties of Selberg polynomials.

Lemma 2. *Let k be a positive integer. Then we have*

$$|J_0(2\pi k)| \le \frac{1}{\pi\sqrt{2k}}. \tag{8}$$

Proof. Let $H_\nu^{(j)}(z)\,(j=1,2)$ be the Hankel functions. Asymptotic expansions of $H_\nu^{(j)}(z)$ are given by

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})}\left\{\sum_{m=0}^{p-1}\frac{(-1)^m(\nu,m)}{(2iz)^m} + R_p^{(1)}(z)\right\}$$

and

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})}\left\{\sum_{m=0}^{p-1}\frac{(\nu,m)}{(2iz)^m} + R_p^{(2)}(z)\right\},$$

where $(\nu,m) = \frac{(4\nu^2-1)(4\nu^2-3^2)\cdots(4\nu^2-(2m-1)^2)}{2^{2m}m!}$, $(\nu,0)=1$ and $R_p^{(j)}(z)\,(j=1,2)$ are remainder terms ([9, pp. 197–198]). Taking $\nu=0$, $p=2$, we get

$$\begin{aligned}
J_\nu(z) &= \frac{1}{2}\left(H_\nu^{(1)}(2\pi k) + H_\nu^{(2)}(2\pi k)\right)\\
&= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left\{\cos\left(z-\frac{\pi}{4}\right) + \frac{1}{8z}\sin\left(z-\frac{\pi}{4}\right) + \frac{1}{2}\left(R_2^{(1)}(z) + R_2^{(2)}(z)\right)\right\}.
\end{aligned}$$

It is easily seen that for $j=1,2$

$$|R_2^{(j)}(z)| \le \frac{9}{128z^2} \quad \text{for } z>0$$

(see the integral representation of $R_p^{(j)}(z)$ in [9, p.197]). Hence

$$J_0(2\pi k) = \frac{1}{\pi\sqrt{k}}\left(\frac{1}{\sqrt{2}} - \frac{1}{16\sqrt{2}\pi k} + R\right).$$

Since

$$\begin{aligned}|R| &\le \frac{1}{2}\left(|R_2^{(1)}(2\pi k)| + |R_2^{(2)}(2\pi k)|\right) \le \frac{9}{512\pi^2 k^2}\\ &\le \frac{1}{16\sqrt{2}\pi k},\end{aligned}$$

we get the assertion of the lemma.

Lemma 3. *Take a and b in $[0,1]$ with $a < b$ and let $J = (a,b), [a,b], (a,b]$ or $[a,b)$. Let r be an integer not less than 3. Then we have*

$$\left|\int_{(\mathbb{R}/\mathbb{Z})^r} \chi_J\left(-2\sum_{j=1}^r \cos(2\pi x_j)\right) dx_1\cdots dx_r - |J|\right| \le 2\zeta\left(\frac{r}{2}\right)(2\pi)^{-r}|J|. \qquad (9)$$

Proof. Hereafter we write $z = 2\sum_{j=1}^r \cos(2\pi x_j)$ and $W = (\mathbb{R}/\mathbb{Z})^r$ for simplicity. By (7), we evaluate the integrals:

$$\int_W \Big\{B_K(\mp(z+b)) + B_K(\pm(z+a))\Big\} dx_1\cdots dx_r. \qquad (10)$$

Substituting by the definition of $B_K(x)$ (4) and using the integral formula

$$\begin{aligned}\int_W e^{\pm 2\pi i k(z+a)} dx_1\cdots dx_r &= e^{\pm 2\pi i k a}\left(\int_0^1 e^{4\pi i k\cos 2\pi x} dx\right)^r\\ &= e^{\pm 2\pi i k a} J_0(4\pi k)^r,\end{aligned}$$

(see [5, p.81]), we have

$$\begin{aligned}\int_W B_K(z+a) dx_1\cdots dx_r &= \int_W \left\{V_K(z+a) + \frac{\Delta_{K+1}(z+a)}{2(K+1)}\right\} dx_1\cdots dx_r\\ &= \frac{1}{K+1}\sum_{k=1}^K f\left(\frac{k}{K+1}\right)\sin(2\pi k a) J_0(4\pi k)^r\\ &\quad + \frac{1}{2(K+1)}\left\{1 + \sum_{\substack{|k|\le K+1\\ k\ne 0}}\left(1 - \frac{|k|}{K+1}\right) e^{2\pi i k a} J_0(4\pi k)^r\right\}.\end{aligned} \qquad (11)$$

From (8) the absolute value of the last term on the right hand side of (11) is estimated as

$$
\begin{aligned}
&\leq \frac{1}{2(K+1)}\Big\{1 + 2(2\pi)^{-r}\sum_{k=1}^{K+1}\Big(1-\frac{k}{K+1}\Big)k^{-r/2}\Big\} \\
&\leq \frac{1}{2(K+1)}\Big\{1 + 2(2\pi)^{-r}\zeta\Big(\frac{r}{2}\Big)\Big\} \leq \frac{1}{K}.
\end{aligned}
$$

Hence the integral of $B_K(z+a)$ is given by

$$
\int_W B_K(z+a)dx_1\cdots dx_r = \frac{1}{K+1}\sum_{k=1}^{K} f\Big(\frac{k}{K+1}\Big)\sin(2\pi ka)J_0(4\pi k)^r + G_1(a)
$$

with the bound $|G_1(a)| \leq \frac{1}{K}$. The integral of $B_K(-z-b)$ is given in the same way,

$$
\int_W B_K(-z-b)dx_1\cdots dx_r = -\frac{1}{K+1}\sum_{k=1}^{K} f\Big(\frac{k}{K+1}\Big)\sin(2\pi kb)J_0(4\pi k)^r + G_2(b)
$$

with the same upper bound $|G_2(b)| \leq \frac{1}{K}$. Adding the above expressions we have

$$
\begin{aligned}
&\Big|\int_W \Big\{B_K(-z-b) + B_K(z+a)\Big\}dx_1\cdots dx_r\Big| \\
&\quad\leq \Big|\frac{1}{K+1}\sum_{k=1}^{K} f\Big(\frac{k}{K+1}\Big)(\sin 2\pi ka - \sin 2\pi kb)J_0(4\pi k)^r\Big| + \frac{2}{K} \\
&\quad\leq \frac{2}{K+1}\sum_{k=1}^{K}\Big|f\Big(\frac{k}{K+1}\Big)\Big||\sin \pi k(a-b)|(2\pi)^{-r}k^{-\frac{r}{2}} + \frac{2}{K} \\
&\quad\leq \frac{(2\pi)^{1-r}}{K+1}(b-a)\sum_{k=1}^{K}\Big|f\Big(\frac{k}{K+1}\Big)\Big|k^{1-\frac{r}{2}} + \frac{2}{K}.
\end{aligned}
$$

Now we estimate the sum in the above equation. Let ε be a small positive number, and take $\eta < \frac{1}{2}$ to be a small positive number which satisfies $\frac{\pi\eta}{\sin \pi\eta} <$

$1+\varepsilon$. Dividing the sum into two parts at $\big[\eta(K+1)\big]$ and using (3), we have

$$\frac{1}{K+1}\sum_{k=1}^{K}\Big|f\Big(\frac{k}{K+1}\Big)\Big|k^{1-\frac{r}{2}} \le \frac{1}{K+1}\sum_{k=1}^{[\eta(K+1)]}\left(\frac{\pi\eta}{\sin\pi\eta}\frac{K+1}{\pi k}+\frac{1}{\pi}\right)k^{1-\frac{r}{2}}$$

$$+\frac{1}{K+1}\left(\frac{1-\eta}{\sin\pi(1-\eta)}+\frac{1}{\pi}\right)\sum_{k=[\eta(K+1)]+1}^{K}k^{1-\frac{r}{2}}$$

$$\le \frac{1}{\pi}(1+\varepsilon)\zeta\left(\frac{r}{2}\right)+O\left(\frac{1}{\sqrt{K}}\right).$$

where the implied constant in the last equation does not depend on K. Therefore

$$\left|\int_W\Big\{B_K(-z-b)+B_K(z+a)\Big\}dx_1\cdots dx_r\right|$$

$$\le 2(2\pi)^{-r}(b-a)(1+\varepsilon)\zeta\Big(\frac{r}{2}\Big)+O\Big(\frac{1}{\sqrt{K}}\Big).$$

In the same manner we have

$$\left|\int_W\Big\{B_K(z+b)+B_K(-z-a)\Big\}dx_1\cdots dx_r\right|$$

$$\le 2(2\pi)^{-r}(b-a)(1+\varepsilon)\zeta\Big(\frac{r}{2}\Big)+O\Big(\frac{1}{\sqrt{K}}\Big).$$

Thus from (5), (6) and (7) we get the upper bound of the left hand side of (9):

$$\left|\int_W\chi_J\Big(-2\sum_{j=1}^{r}\cos(2\pi x_j)\Big)dx_1\cdots dx_r-|J|\right|$$

$$\le 2(1+\varepsilon)\zeta\Big(\frac{r}{2}\Big)(2\pi)^{-r}|J|+O\Big(\frac{1}{\sqrt{K}}\Big).$$

Now we let $K\to\infty$, as ε is arbitrary, we get the assertion of the lemma.

Proof of Theorem 1.

Now we study $\lim_{N\to\infty}S_N(\alpha,I)$ of (2). Let (x_n) and (y_n) be real sequences with $y_n\to 0$. Then it is easily seen from [6], Chapter 1, Theorem 7.3 that if (x_n) has a continuous asymptotic density function, then (x_n+y_n) also does and their density functions are the same. Thus it is able to ignore the term α^{-m} in (2).

Our task is to consider the integral:

$$\int_W \chi_I\Big(-2\sum_{j=1}^{r}\cos(2\pi x_j)\Big)\,dx_1\cdots dx_r.$$

Applying (9) to the interval I, we get the assertion of Theorem 1.

3 Proof of Theorem 2

Let us follow the proof of Theorem 1 with $r=1,2$. In this case, we have

$$\begin{aligned} Y &:= \left|\int_W \Big\{B_K(-z-b)+B_K(z+a)\Big\}dx_1\cdots dx_r\right| \\ &= \frac{2(2\pi)^{-r}}{K+1}\sum_{k=1}^{K}\Big|f\Big(\frac{k}{K+1}\Big)\Big||\sin\pi k(a-b)|k^{-r/2}+O(K^{-1/2}). \quad (12)\end{aligned}$$

Let ε be a small positive number and take a small positive η such that $\pi\eta/(\sin\pi\eta)<1+\varepsilon$ and a large integer K such that $1/(b-a)<\eta(K+1)<K$. We also introduce another parameter $0<v<1$ which is chosen later. Divide the summation in (12) into three parts

$$\frac{2(2\pi)^{-r}}{K+1}\left\{\sum_{k\le\frac{v}{b-a}}+\sum_{\frac{v}{b-a}<k\le\eta(K+1)}+\sum_{\eta(K+1)<k\le K}\right\}=:S_1+S_2+S_3.$$

If $b-a\le v$, using $|\sin\pi k(b-a)|\le\pi k(b-a)$ and (3), we get

$$S_1\le\begin{cases}\dfrac{(1+\varepsilon)(b-a)}{\pi}\left(2\sqrt{\dfrac{v}{b-a}}-1\right)+O\left(\dfrac{1}{K}\right) & r=1,\\[2ex] \dfrac{(1+\varepsilon)(b-a)}{2\pi^2}\left(\log\dfrac{v}{b-a}+1\right)+O\left(\dfrac{1}{K}\right) & r=2,\end{cases}$$

while if $b-a>v$, S_1 is trivially zero. If $b-a\le v$, the trivial bound $|\sin\pi k(b-a)|\le1$ implies, for $r=1,2$,

$$S_2\le\frac{4(1+\varepsilon)}{(2\pi)^{r+1}}\left(\frac{b-a}{v}\right)^{\frac{r}{2}}\left(\frac{2}{r}+\frac{b-a}{v}\right)+O(K^{-1/2}),$$

while if $b-a>v$,

$$S_2\le\frac{4(1+\varepsilon)}{(2\pi)^{r+1}}\zeta\left(1+\frac{r}{2}\right)+O(K^{-1/2}).$$

Finally we have $S_3 = O(K^{-1/2})$ for $r = 1, 2$. The implied constants do not depend on K. Now we let $K \to \infty$. In the case $r = 1$ we get

$$Y \le \begin{cases} \dfrac{(1+\varepsilon)\sqrt{b-a}}{\pi}\left\{2\left(\sqrt{v} + \dfrac{1}{\pi\sqrt{v}}\right) - \sqrt{b-a} + \dfrac{b-a}{\pi v^{\frac{3}{2}}}\right\} & b-a \le v, \\ \dfrac{1+\varepsilon}{\pi^2}\zeta\left(\dfrac{3}{2}\right) & b-a > v. \end{cases}$$

Taking $v = 1/\pi$, it follows that

$$Y \le 4\pi^{-\frac{3}{2}}(1+\varepsilon)\sqrt{b-a}.$$

For $r = 2$, we have

$$Y \le \begin{cases} \dfrac{(1+\varepsilon)(b-a)}{2\pi^2}\left(\log\dfrac{1}{b-a} + 1 + \dfrac{1}{\pi v} + \log v + \dfrac{b-a}{\pi v^2}\right) & b-a \le v, \\ \dfrac{1+\varepsilon}{2\pi^3}\zeta(2) & b-a > v. \end{cases}$$

Now taking $v = 1/\sqrt{\pi}$, we get

$$Y \le \frac{(1+\varepsilon)(b-a)}{2\pi^2}\left(\log\frac{1}{b-a} + 1 + (b-a)\right).$$

The same estimates are valid for

$$\int_W \Big\{B_K(z+b) + B_K(-z-a)\Big\} dx_1 \cdots dx_r$$

with $r = 1, 2$. Since ε is chosen arbitrarily, we obtain Theorem 2.

4 Examples

To illustrate the result, we give examples of distributions for Salem numbers of degree 4, 6 and 8. The interval $[0, 1]$ is divided into 100 pieces. We computed the fractional part of α^n for $1 \le n \le 200000$, and counted the number of n so that the fractional part of α^n falls into each subintervals. The vertical axis indicates the number of such n.

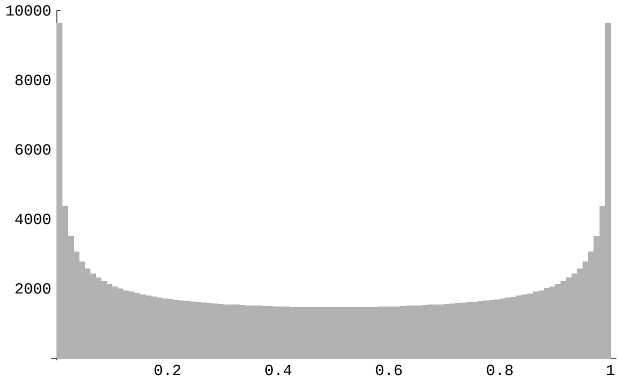

Figure 1: Salem number for $x^4 - x^3 - x^2 - x + 1 = 0$

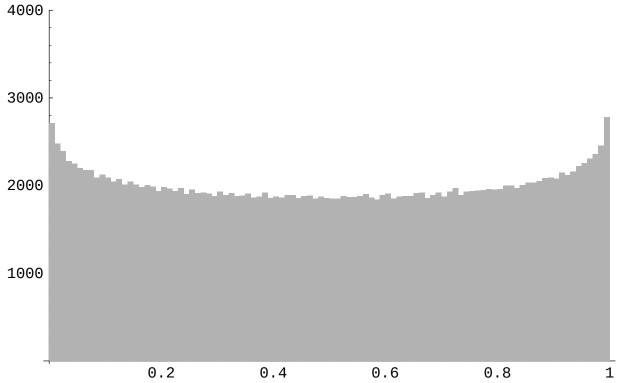

Figure 2: Salem number for $x^6 - x^5 - x^4 + x^3 - x^2 - x + 1 = 0$

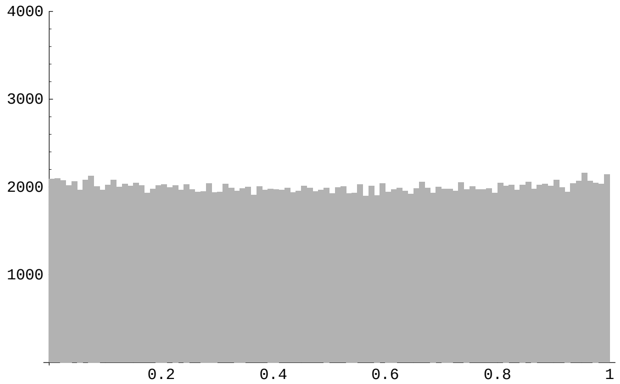

Figure 3: Salem number for $x^8 - 2x^7 + x^6 - x^4 + x^2 - 2x + 1 = 0$

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