Almost uniform distribution modulo 1 and the distribution of primes *†

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Abstract

Let $(a_n),\ n = 1, 2, \ldots$ be a sequence of real numbers which is related with number theoretic functions such as P_n, the n-th prime. We study the distribution of the fractional parts of (a_n) using the concept of "almost uniform distribution" defined in [9]. Then we can show a generalization of the results of [2] on the convex property of $\log P_n$. The method may be extended as well to other oscillation problems of number theoretical interest.

Let $(a_n),\ n = 1, 2, \ldots$ be a sequence of real numbers and $A(I, (a_n), N)$ be the *counting function*, that is, the number of $n = 1, 2, \ldots, N$ that $\{a_n\}$ is contained in a certain interval $I \subset [0, 1]$. Here we denote by $\{a_n\} = a_n - [a_n]$, the fractional part of a_n. First we recall a kind of generalization of the classical definition of uniform distribution modulo 1 (see [9], [3] and [8]).

Definition. The sequence (a_n) is said to be *almost uniformly distributed modulo* 1 if there exist a strictly increasing sequence of natural numbers $(n_j),\ j = 1, 2, \ldots$ and, for every pair of a, b with $0 \le a < b \le 1$,

$$\lim_{j \to \infty} \frac{A([a, b), (a_n), n_j)}{n_j} = b - a.$$

For example, we define (c_n) by

$$c_n = \frac{n}{2^{1+[\log_2 n]}}.$$

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Then (c_n) is almost uniformly distributed modulo 1 but not uniformly distributed modulo 1. It is obvious that if the sequence (a_n) is uniformly distributed modulo 1, then almost uniformly distributed modulo 1. On the contrary, if

$$n_{j+1} - n_j = o(n_j),$$

then almost uniformly distributed modulo 1 implies uniformly distributed modulo 1. Using the classical method of uniform distribution theory (see e.g. [6]), we can show the following

Proposition 1. The sequence $(a_n),\ n = 1, 2, \dots$ is almost uniformly distributed modulo 1 if and only if there exist a strictly increasing sequence of natural numbers $(n_j),\ j = 1, 2, \dots$, such that for every real-valued continuous function on the interval $[0, 1]$, we have

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{i=1}^{n_j} f(\{a_i\}) = \int_0^1 f(x)dx.$$

Proposition 2. (Weyl's Criterion for almost uniformly distributed modulo 1**)** The sequence $(a_n),\ n = 1, 2, \dots$ is almost uniformly distributed modulo 1 if and only if there exist a strictly increasing sequence of natural numbers $(n_j),\ j = 1, 2, \dots$, such that for every integer h, we have

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{i=1}^{n_j} \exp(2\pi h\sqrt{-1}a_i) = 0.$$

We should mention the next generalization of Fejér's Theorem.

Theorem 1. (Fejér's Theorem for almost uniformly distributed modulo 1**)** Let $(f(n)),\ n = 1, 2, \dots$ be a sequence of real numbers and $\Delta f(n) = f(n+1) - f(n)$. If the following three conditions is satisfied, then $(f(n))$ is almost uniformly distributed modulo 1:

1. There exists a natural number N that $\Delta f(n)$ is monotone when $n \geq N$ (hereafter, we say this property as *ultimately monotone*),
2. $\lim_{n\to\infty} \Delta f(n) = 0$,
3. $\limsup_{n\to\infty} n|\Delta f(n)| = \infty$.

Note that the corresponding third condition for uniformly distributed modulo 1 is:

$$\lim_{n\to\infty} n|\Delta f(n)| = \infty.$$

Moreover, it is shown in [4] (see also [5]) that $\limsup_{n\to\infty} n|\Delta f(n)| = \infty$ is a necessary condition for uniformly distributed modulo 1. Concerning this fact, in [3], it is shown that $(\log n)$ is not almost uniformly distributed modulo 1 but almost uniformly distributed modulo 1 in the "average" sense. It is an interesting problem to study this delicate difference between uniformly distributed modulo 1 and almost uniformly distributed modulo 1. We can show the following:

Corollary 1. Let $(g(n))$ be a sequence of real numbers which satisfies three conditions:

(C1) $g(n) = o(n)$,

(C2) The average $f(n) = \frac{1}{n}\sum_{k=1}^{n} g(k)$ is *not* almost uniformly distributed modulo 1,

(C3) $\limsup\limits_{n\to\infty} |f(n) - g(n+1)| = \infty$.

Then $\Delta^2 f(n)$ changes its sign infinitely many times. Here $\Delta^2 f(n) = \Delta(\Delta f(n))$.

Proof. We have

$$\begin{aligned} \Delta f(n) &= \frac{1}{n+1}\sum_{k=1}^{n+1} g(k) - \frac{1}{n}\sum_{k=1}^{n} g(k) \\ &= \frac{1}{n+1} g(n+1) - \frac{1}{n(n+1)}\sum_{k=1}^{n} g(k). \end{aligned} \tag{1}$$

This shows that $\lim\limits_{n\to\infty} \Delta f(n) = 0$. And by (1),

$$(n+1)\Delta f(n) = g(n+1) - f(n).$$

Thus

$$\limsup_{n\to\infty}\ n|\Delta f(n)| = \infty.$$

If $\Delta f(n)$ is ultimately monotone, then $f(n)$ is almost uniformly distributed modulo 1, which contradicts the assumption. $\square$

Let P_n be the n-th prime. Now we show

Theorem 2. $(\log P_n)$ is not almost uniformly distributed modulo 1.

Proof. Let t be a real number and $\pi(x)$ be the number of primes less than or equal to x. Consider the sum over primes p:

$$\sum_{p\le N} p^{\sqrt{-1}t} = \int_{3/2}^{N} x^{\sqrt{-1}t} d\pi(x).$$

Integrating by parts of the right hand side, by using the prime number theorem of the form:

$$\pi(x) = \frac{x}{\log x} + O(\frac{x}{\log^2 x}),$$

we have

$$\begin{aligned}\sum_{p\le N} p^{\sqrt{-1}t} &= \frac{N^{1+\sqrt{-1}t}}{\log N} - \sqrt{-1}t\int_{3/2}^{N} \frac{x^{\sqrt{-1}t}}{\log x}dx + O(\frac{N}{\log^2 N}) \\ &= \frac{N^{1+\sqrt{-1}t}}{(1+\sqrt{-1}t)\log N} + O(\frac{N}{\log^2 N}).\end{aligned}$$

Thus we see

$$\frac{1}{\pi(N)}\sum_{p\le N} e^{\sqrt{-1}t\log p} \sim \frac{N^{\sqrt{-1}t}}{1+\sqrt{-1}t}$$

The right hand side is not zero. By using Proposition 2, we get the result. □

Now we give a very different proof of the results of [2].

Theorem 3. $\Delta^2 \log P_n$ changes its sign infinitely many times.

Proof. Let $g(n) = n\log P_n - (n-1)\log P_{n-1}$ and $f(n) = \log P_n$ in Corollary 1. (Here we put $P_0 = 1$ for example.) By using Theorem 2, it suffice to show (C1) and (C3). By using prime number theorem, we have

$$\begin{aligned} g(n) &\le n\frac{P_n - P_{n-1}}{P_{n-1}} + \log P_{n-1}, \\ &= o(n).\end{aligned}$$

For the condition (C3),

$$\begin{aligned} g(n+1) - f(n) &= (n+1)(\log P_{n+1} - \log P_n) \\ &> \frac{(n+1)(P_{n+1} - P_n)}{P_{n+1}} \\ &\sim \frac{P_{n+1} - P_n}{\log P_n}.\end{aligned}$$

Here we write $f \sim g$ if $|f/g| \to 1$. P. Erdös [1] was the first to obtain

$$\limsup_{n\to\infty} \frac{P_{n+1}-P_n}{\log P_n} = \infty,$$

by showing

$$\limsup_{n\to\infty} \frac{(P_{n+1}-P_n)(\log\log\log P_n)^2}{\log P_n \log\log P_n \log\log\log\log P_n} > c > 0.$$

About the improvement of the constant c, see [7]. This completes the proof. $\square$

Our method to show this type of results can be generalized by a kind of "linearity" in many cases. To explain this, we notice

Theorem 4. Let l be a fixed positive integer, and C_i $(i = 1, 2, \ldots, l)$ be the real numbers with $\sum C_i \neq 0$. The sequence $(\sum_{i=0}^{l-1} C_i \log P_{n+i})$ is not almost uniformly distributed modulo 1.

Proof. First, we consider the case $(C \log P_n)$. Without loss of generality, we may assume that $C > 0$. Then we write $C \log P_n = \log_b P_n$ with a constant $b > 1$. To see the assertion, replace e with b in the proof of Theorem 2.

If $l > 1$, it suffice to note that

$$\sum_{i=0}^{l-1} C_i \log P_{n+i} - \log P_n \sum_{i=0}^{l-1} C_i = o(1).$$

This shows the assertion. $\square$

Theorem 5. Let l be a fixed positive integer, and f_i $(i = 1, 2, \ldots, l)$ be the positive real numbers. Then

$$\Delta^2 \log(P_n^{f_1} P_{n+1}^{f_2} \ldots P_{n+l-1}^{f_l})$$

changes its sign infinitely many times.

Proof. Put

$$\begin{aligned} g(n) &= n(\sum_{i=1}^{l} f_i \log P_{n+i-1}) - (n-1)(\sum_{i=1}^{l} f_i \log P_{n+i-2}) \\ f(n) &= \sum_{i=1}^{l} f_i \log P_{n+i-1}. \end{aligned}$$

By using Corollary 1 and Theorem 4, in a similar manner as in the proof of Theorem 3, we obtain the assertion. Here, we essentially used the positiveness of f_i $(i = 1, 2, \ldots l)$ in proving (C3). $\square$

We expect that the conditions $f_i > 0$ $(i = 1, 2, \dots l)$ can be dropped.

Our method is applicable to a lot of arithmetic functions $g(n)$ such that $1/n \sum_{k \le n} g(k)$ is not almost uniformly distributed modulo 1. For example, we can show similar assertions for the divisor function $d(n) = \sum_{d|n} 1$ as

$$\frac{1}{n} \sum_{k=1}^{n} d(k) = \log n + (2\gamma - 1) + O(\frac{1}{\sqrt{n}}),$$

with the Euler constant γ. The proof for this case is easier, but the results do not seem well worthy of stating here.

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