Automata for arithmetic Meyer sets*

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Abstract. The set $\mathbb{Z}_\beta$ of β-integers is a Meyer set when β is a Pisot number, and thus there exists a finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. We give finite automata describing the expansions of the elements of $\mathbb{Z}_\beta$ and of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$. We present a construction of a such a finite set F, and a method to minimize the size of F. We obtain in this way a finite transducer that performs the decomposition of the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ as a sum belonging to $\mathbb{Z}_\beta + F$.

1 Introduction

The so-called Meyer sets have been introduced by Meyer [11, 12] under the name of "quasicrystals" in order to formalize the quasicrystals discovered by the physicists in the eighties. A set X is a *Delaunay set* if it is uniformly discrete and relatively dense. A set X is a *Meyer set* if it is a Delaunay set and there exists a finite set F such that $X - X \subset X + F$. There exist strong relations between Meyer sets and some algebraic integers. Recall that a *Pisot number* (or a Pisot-Vijayaraghavan number) is an algebraic integer > 1 such that all its algebraic conjugates have modulus strictly less than one. A *Salem number* is an algebraic integer such that every conjugate has modulus smaller than or equal to 1, and at least one of them has modulus 1. The following result from Meyer makes the connection between Meyer sets and those algebraic integers. If $X \subset \mathbb{R}^n$ is a Meyer set and if $\beta > 1$ is a real number such that $\beta X \subset X$ then β is a Pisot or a Salem number. Conversely for each n and for each Pisot or Salem number β, there exists a Meyer set $X \subset \mathbb{R}^n$ such that $\beta X \subset X$.

Note that all the quasicrystals encountered in the real world are linked to quadratic Pisot numbers, namely $\frac{1+\sqrt{5}}{2}$, $1+\sqrt{2}$ and $2+\sqrt{3}$.

In this paper we study Meyer sets $\mathbb{Z}_\beta$ associated with β-expansions, β being a Pisot number, and give a construction of a minimal finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$.

Lagarias [8] gave a general construction of a finite set F satisfying $X - X \subset X + F$ for a Delaunay set X such that $X - X$ is also a Delaunay set. But the

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sets obtained are huge and no method of minimization of these sets is known. Minimal sets F are given in [3] for $\mathbb{Z}_\beta$ when β is a quadratic Pisot unit. When β is a quadratic Pisot number, a possible set F for $\mathbb{Z}_\beta$ is exhibited in [6].

We first give finite automata describing the formal addition and substraction of beta-integers. We characterize the cases when the formal addition gives a system of finite type when the original system $\mathbb{Z}_\beta$ is of finite type.

We then give a construction of a family of finite sets F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$, and a method to minimize the size of the sets F we built. We obtain in this way a finite transducer that performs the decomposition of the result of the formal substraction $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ into a sum belonging to $\mathbb{Z}_\beta + F$.

2 Preliminaries

Let A be a finite alphabet. A concatenation of letters of A is called a *word*. The set A^* of all finite words equipped with the empty word ε and the operation of concatenation is a free monoid. We denote by a^k the word obtained by concatenating k letters a. The length of a word $w = w_0 w_1 \cdots w_{n-1}$ is denoted by $|w| = n$. One considers also infinite words $v = v_0 v_1 v_2 \cdots$. The set of infinite words on A is denoted by $A^{\mathbb{N}}$. An infinite word v is said to be *eventually periodic* if it is of the form $v = wz^\omega$, where w and z are in A^* and $z^\omega = zzz \cdots$. A *factor* of a finite or infinite word w is a finite word v such that $w = uvz$; if $u = \varepsilon$, the word v is a *prefix* of w. A prefix of w is *strict* if it is not equal to w.

Definitions and results on numeration systems can be found in [10, Chapter 7]. Let $\beta > 1$ be a real number. Any positive real number x can be represented in base β by the following greedy algorithm [14]. Denote by $\lfloor . \rfloor$ and by $\{.\}$ the integral part and the fractional part of a number. There exists $k \in \mathbb{Z}$ such that $\beta^k \le x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. For $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$. Then $x = x_k \beta^k + x_{k-1}\beta^{k-1} + \cdots$. If $x < 1$, we get $k < 0$ and we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. The sequence $(x_i)_{k \ge i \ge -\infty}$ is called the *β-expansion* of x, and is denoted by

$$\langle x \rangle_\beta = x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots$$

most significant digit first. The part $x_{-1}x_{-2}\cdots$ after the “decimal” point is called the *β-fractional part* of x.

The digits x_i are elements of the *canonical* alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$ and $A_\beta = \{0, \ldots, \beta - 1\}$ otherwise. When a β-expansion ends in infinitely many zeroes, it is said to be *finite*, and the 0's are omitted.

A finite or infinite word w on A_β which is the β-expansion of some number x is said to be *admissible*. Leading 0's are allowed.

The set $\mathbb{Z}_\beta$ of *β-integers* is the set of real numbers x such that the β-fractional part of $|x|$ is equal to 0,

$$\mathbb{Z}_\beta = \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} = \mathbb{Z}_\beta^+ \cup \mathbb{Z}_\beta^-$$

where $\mathbb{Z}_\beta^+$ is the set of non-negative beta-integers, and $\mathbb{Z}_\beta^- = -\mathbb{Z}_\beta^+$.

Denote by D_β the set of β-expansions of numbers of $[0,1)$ and the shift by σ. Then D_β is shift-invariant. Let S_β be its closure in $A_\beta^{\mathbb{N}}$. The set S_β is a symbolic dynamical system, called the *β-shift*. The set $\mathbb{Z}_\beta^+$ is equal to the set of finite factors of S_β.

There is a peculiar representation of the number 1 which plays an important role in the theory. It is denoted by $d_\beta(1)$, and computed by the following process [14]. Let the *β-transform* be defined on $[0,1]$ by $T_\beta(x) = \beta x \bmod 1$. Then $d_\beta(1) = (t_i)_{i \geq 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$. Note that $\lfloor \beta \rfloor = t_1$. We recall a result of Parry [13]: a sequence s of natural integers is an element of D_β if and only if for every $p \geq 1$, $\sigma^p(s)$ is strictly less in the lexicographic order than $d_\beta(1)$ if $d_\beta(1)$ is infinite, or less than $d_\beta^*(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$ if $d_\beta(1) = t_1 \cdots t_m$ is finite.

A word $w_1 \cdots w_n$ of A_β^* is said to be a *minimal forbidden* word for S_β if it is not a factor of S_β and if $w_1 \cdots w_{n-1}$ and $w_2 \cdots w_n$ are factors of S_β. Recall that a symbolic dynamical system is said to be *of finite type* if the set of its minimal forbidden words is finite. More generally it is said to be *sofic* if the set of its finite factors is recognized by a finite automaton. The β-shift is sofic if and only if $d_\beta(1)$ is eventually periodic, and it is of finite type if and only if $d_\beta(1)$ is finite. By abuse we say that the set $\mathbb{Z}_\beta$ of β-integers is of finite type (resp. sofic) if $d_\beta(1)$ is finite (resp. infinite eventually periodic). Recall that if β is a Pisot number, then $d_\beta(1)$ is finite or eventually periodic [2, 15].

A set $X \subset \mathbb{R}^n$ is *uniformly discrete* if there exists a positive real r such that for any $x \in \mathbb{R}^n$, the open ball of center x and radius r contains at most one point of X. If $Y \subset X$ and X is uniformly discrete, then Y is uniformly discrete. A set $X \subset \mathbb{R}^n$ is *relatively dense* if there exists a positive real R such that for any $x \in \mathbb{R}^n$, the open ball of center x and radius R contains at least one point of X. If $X \subset Y$ and X is relatively dense, then Y is relatively dense. A set X is a *Delaunay set* if it is uniformly discrete and relatively dense. A set X is a *Meyer set* if it is a Delaunay set and there exists a finite set F such that $X - X \subset X + F$. Lagarias proved [8] that a set X is a Meyer set if and only if both X and $X - X$ are Delaunay sets. Note that when X is a Delaunay set, then $X - X$ is relatively dense, but not necessarily uniformly discrete. For example $X = \{n + \frac{1}{|n|+2}\}$ is a Delaunay set and $X - X$ has 1 as point of accumulation.

Proposition 1. [3] *If β is a Pisot number, then the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set.*

3 Automata for formal addition and substraction

In this section we construct automata that symbolically describe the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ when β is a Pisot number. Note that

$$\mathbb{Z}_\beta - \mathbb{Z}_\beta = (\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+) \cup (\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+) \cup -(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+). \tag{1}$$

The reader is referred to [4] and [16] for definitions and results in automata theory. We introduce some notations. Denote by $L_\beta^+ \subset A_\beta^*$ the set of β-expansions of elements of $\mathbb{Z}_\beta^+$ with possible leading 0's. Set $\bar{k} = -k$, where k is an integer, and let $\overline{A_\beta} = \{\overline{\lfloor\beta\rfloor}, \dots, \bar{1}, 0\}$. We denote by $L_\beta^- \subset \overline{A_\beta}^*$ the set $\{\overline{w} = \overline{w_N} \cdots \overline{w_0} \mid w = w_N \cdots w_0 = \langle -x \rangle_\beta,\ x \in \mathbb{Z}_\beta^-\}$.

When $d_\beta(1)$ is finite or eventually periodic, the set L_β^+ is recognizable by a finite automaton [5], of which we recall the construction. If $d_\beta(1) = t_1 \cdots t_m$ is finite, the automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ recognizing L_β^+ has m states $q_1, \dots, q_m$. For each $1 \leq i \leq m-1$ there is an edge between q_i and q_{i+1} labelled by t_i. For each $1 \leq i \leq m$ there are t_i edges between q_i and q_1 labelled by $0, \dots, t_i - 1$. The initial state is q_1; every state is terminal.

If $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$ is infinite eventually periodic, the automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ recognizing L_β^+ has $m+p$ states $q_1, \dots, q_{m+p}$. For each $1 \leq i \leq m+p-1$ there is an edge between q_i and q_{i+1} labelled by t_i. For each $1 \leq i \leq m+p$ there are t_i edges between q_i and q_1 labelled by $0, \dots, t_i - 1$. There is an edge from q_{m+p} to q_{m+1} labelled by t_{m+p}. The initial state is q_1; every state is terminal.

Clearly the set L_β^- is recognizable by the same automaton as L_β^+, but with negative labels on edges. Then the automaton for $\mathbb{Z}_\beta$ is $\mathcal{A}_{\mathbb{Z}_\beta} = \mathcal{A}_{\mathbb{Z}_\beta^+} \cup \mathcal{A}_{\mathbb{Z}_\beta^-}$.

By a general construction one can compute the "sum" of two automata. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite automata with labels in an alphabet of integers. One constructs a finite automaton $\mathcal{S}$ as follows :

- the set of states of $\mathcal{S}$ is the cartesian product $Q_\mathcal{S} = Q_\mathcal{A} \times Q_\mathcal{B}$
- there is an edge in $\mathcal{S}$ from (p, q) to (p', q') labelled by $a + b$ if and only if there is an edge from p to p' labelled by a in $\mathcal{A}$ and an edge from q to q' labelled by b in $\mathcal{B}$.
- the set of initial (resp. terminal) states is the cartesian product of the sets of initial (resp. terminal) states of $\mathcal{A}$ and $\mathcal{B}$.

Clearly the automaton $\mathcal{S}$ recognizes the set $\{s_N \cdots s_0 \mid N \geq 0,\ s_i = a_i + b_i,\ 0 \leq i \leq N,\ a_N \cdots a_0$ is recognized by $\mathcal{A}$ and $b_N \cdots b_0$ is recognized by $\mathcal{B}\}$.

The *formal addition* of elements of $\mathbb{Z}_\beta^+$ consists in adding elements without carry. More precisely,

$$L_\beta^+ + L_\beta^+ = \{(a_N + b_N) \cdots (a_0 + b_0) \mid a_N \cdots a_0,\ b_N \cdots b_0 \in \mathbb{Z}_\beta^+\} \subset \{0, \cdots, 2\lfloor\beta\rfloor\}^*.$$

Similarly the *formal subtraction* of elements of $\mathbb{Z}_\beta^+$ is defined by

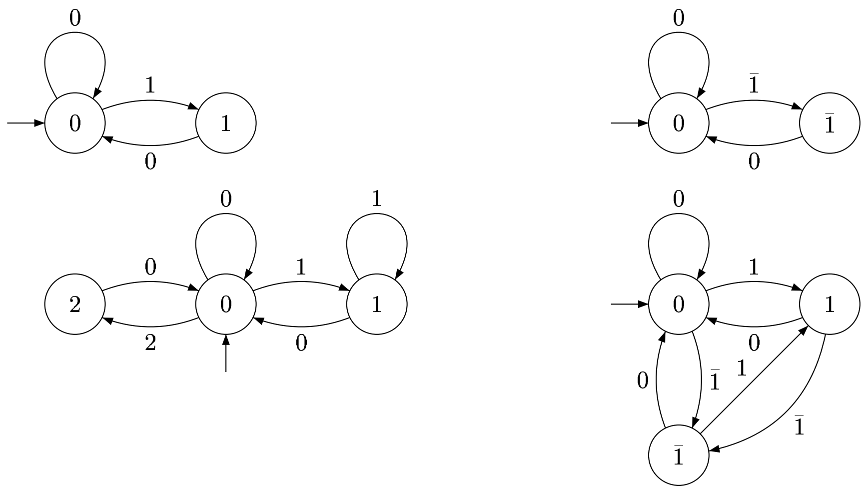
$$L_\beta^+ - L_\beta^+ = \{(a_N - b_N) \cdots (a_0 - b_0) \mid a_N \cdots a_0,\ b_N \cdots b_0 \in \mathbb{Z}_\beta^+\} \subset \{-\lfloor\beta\rfloor, \cdots, \lfloor\beta\rfloor\}^*.$$

From the construction of the sum automaton follows

Proposition 2. *If $d_\beta(1)$ is finite or eventually periodic, the set $L_\beta^+ + L_\beta^+$ corresponding to the formal addition $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+$ and the set $L_\beta^+ - L_\beta^+$ corresponding to the formal subtraction $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+$ are recognizable by a finite automaton.*

By Equation (1), $\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} = \mathcal{A}_{\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+} \cup \mathcal{A}_{\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+} \cup \mathcal{A}_{-(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+)}$. The automata given by this construction are generally not minimal.

Example 1. In the case where $\beta = \frac{1+\sqrt{5}}{2}$, $d_\beta(1) = 11$ and $d_\beta^*(1) = (10)^\omega$. We give below the minimal automata $\mathcal{A}_{\mathbb{Z}_\beta^+}$, $\mathcal{A}_{\mathbb{Z}_\beta^-}$, $\mathcal{A}_{\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+}$, and $\mathcal{A}_{\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+}$. Initial states are indicated by an incoming arrow, and every state is terminal.



It is an interesting question to see what is the result of formal addition or subtraction when the system $\mathbb{Z}_\beta$ is of finite type. First recall that, from the result of Parry cited in Sect. 2, if $d_\beta(1) = t_1 \cdots t_m$, the set of minimal forbidden words for $\mathbb{Z}_\beta^+$ is the set $I_\beta = \{t_1 \cdots t_m\} \cup \{t_1 t_2 \cdots t_{p-1} x_p \mid t_p < x_p \le t_1,\ 2 \le p \le m,\ x_p \in A_\beta\}$.

Proposition 3. *If $d_\beta(1) = t_1 \cdots t_m$ is finite, the formal subtraction $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+$ defines a system of finite type.*

Proof. Recall that if a word is admissible, any word with smaller nonnegative digits is admissible as well. Thus the set of forbidden words for the formal subtraction $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+$ is equal to $\{w, \overline{w} \mid w \in I_\beta\}$, which is finite. □

The result for formal addition is quite different.

Proposition 4. *If $d_\beta(1) = t_1 \cdots t_m$ is finite, the formal addition $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+$ defines a system of finite type if and only if $t_m = t_1$ and, for each $2 \le i \le m-1$, $t_i = t_1$ or $t_i = 0$.*

Corollary 1. *If $\beta < 2$ and $d_\beta(1)$ is finite then the formal addition $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+$ defines a system of finite type.*

The proof of Proposition 4 follows from several technical results.

Lemma 1. *Suppose that $d_\beta(1) = t_1 \cdots t_m$, and that there exists $2 \le j \le m$ with $0 < t_j < t_1$ (so $t_1 \ge 2$), and $t_i = 0$ or $t_i = t_1$ for $2 \le i \le j-1$. Then the set of minimal forbidden words in the formal addition is infinite.*

Proof. For any $k \geq 1$ consider the word $u^{(k)} = [(t_1+t_2)(t_2+t_3)\cdots(t_{j-2}+t_{j-1})(t_{j-1}+t_j-1)(t_j-1+t_1)]^k(t_1+t_2)(t_2+t_3)\cdots(t_{m-1}+t_m)$.
Let $w^{(k)} = (2t_1-1)u^{(k)}$. First we show that $w^{(k)}$ is forbidden in the formal addition system. This comes from the fact that $w^{(k)}$ is necessarily the digit-sum of the two words $x^{(k)} = (t_1-1)[t_1\cdots t_{j-1}(t_j-1)]^k t_1\cdots t_{m-1}$ and $y^{(k)} = t_1[t_2\cdots t_{j-1}(t_j-1)t_1]^k t_2\cdots t_m$. Clearly $y^{(k)}$ is not admissible for $\mathbb{Z}_\beta^+$ because it ends in the forbidden word $t_1\cdots t_m$, and $x^{(k)}$ is admissible for $\mathbb{Z}_\beta^+$ and maximal in the sense that adding 1 to one of its digits makes the word not admissible.

Note that all strict prefixes of $y^{(k)}$ are admissible for $\mathbb{Z}_\beta^+$, so all strict prefixes of $w^{(k)}$ are also admissible.

Now we show that the word $u^{(k)}$ is admissible in the formal addition system. By hypothesis the digits (t_i+t_{i+1}) for $1 \leq i \leq j-2$ are equal to $2t_1$, t_1 or 0. So $u^{(k)}$ can be obtained as the digit-sum of $v^{(k)}$ and $z^{(k)}$ with the following method: a digit $2t_1$ of $u^{(k)}$ gives a digit t_1 in $v^{(k)}$ and a digit t_1 in $z^{(k)}$; a digit t_1 of $u^{(k)}$ gives a digit t_1-1 in $v^{(k)}$ and a digit 1 in $z^{(k)}$; a digit 0 of $u^{(k)}$ gives a digit 0 in $v^{(k)}$ and in $z^{(k)}$. Since $0 < t_j < t_1$, the digits $t_{j-1}+t_j-1$ and t_j-1+t_1 are $\leq 2t_1-2$, which is the sum of t_1-1 and t_1-1. The suffix $(t_j-1+t_1)(t_1+t_2)(t_2+t_3)\cdots(t_{m-1}+t_m)$ of $u^{(k)}$ is thus the digit-sum of $at_1t_2\cdots t_{m-1}$, with $a \leq t_1-1$, and of $bt_2t_3\cdots t_m$, with $b \leq t_1-1$. Hence $u^{(k)}$ is the digit-sum of $v^{(k)}$ and $z^{(k)}$, which are both admissible for $\mathbb{Z}_\beta^+$. □

Lemma 2. *If $d_\beta(1) = t_1\cdots t_m$ is finite and if $t_m = t_1$ and, for each $2 \leq i \leq m-1$, $t_i = t_1$ or $t_i = 0$ then the formal addition is a system of finite type.*

Proof. As in Lemma 1 we consider the word $u^{(k)}$, with $t_j = t_1$ for a fixed j, $2 \leq j \leq m$. The difference with Lemma 1 is that now the suffix $s = (t_j-1+t_1)(t_1+t_2)(t_2+t_3)\cdots(t_{m-1}+t_m)$ is not admissible. Since $t_j = t_1$, s can be the the digit-sum of $(t_1-1)t_1\cdots t_{m-1}$ and $t_1t_2\cdots t_m$, or of $(t_1-1)t_1\cdots(t_{\ell-1})(t_\ell+1)t_{\ell+1}\cdots t_{m-1}$ and $t_1t_2\cdots t_{\ell-1}(t_\ell-1)t_{\ell+1}\cdots t_m$ if $t_\ell \neq 0$, for $2 \leq \ell \leq m-1$. But none of the factors $t_1\cdots t_{\ell-1}(t_\ell+1)$ is admissible for $\mathbb{Z}_\beta^+$. By considering all the positions $2 \leq j \leq m$ in $u^{(k)}$, we see that it is not possible to construct an infinite family of minimal forbidden words of type $w^{(k)}$. □

4 A family of finite sets F

When β is a Pisot number, the set of beta-integers $\mathbb{Z}_\beta$ is a Meyer set so there exists a finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Our goal is to construct sets F as small as possible for $\mathbb{Z}_\beta$.

Remark 1. Note that there exist several sets F with minimal cardinality. For example when $\beta = (1+\sqrt{5})/2$ then $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$, with $F = \{0, \beta-1, -\beta+1\}$, or $F = \{0, \beta-2, -\beta+2\}$ or $F = \{0, \beta-1, -\beta+2\}$.

We first define finite sets from which can be extracted the finite sets F.

Lemma 3. *Let β be a Pisot number of degree d, let $I \subset \mathbb{R}$ be an interval of length 1 and let U be the following set*

$$U = \left\{ x \in \mathbb{Z}[\beta] \mid x \in I \text{ and } \forall\, 2 \le j \le d,\ |x^{(j)}| < \frac{3\lfloor\beta\rfloor}{1 - |\beta^{(j)}|} \right\},$$

where $x^{(2)}, \dots, x^{(d)}$ are the algebraic conjugates of x. Then U is finite, and there exists a subset F of U such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$.

Proof. As the maximal distance between two consecutive points of $\mathbb{Z}_\beta$ is equal to 1, one can find a set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ in any interval I of length 1.

Fix an interval I of length 1 and $F \subset I$ as small as possible such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Let $x \in F$, then $x \in (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$ and can be written as

$$x = \sum_{i=0}^{N} (a_i - b_i)\beta^i - \sum_{i=0}^{N} c_i \beta^i \quad \text{with } |a_i|, |b_i|, |c_i| \le \lfloor\beta\rfloor.$$

so

$$\forall\, 2 \le j \le d \quad x^{(j)} = \sum_{i=0}^{N} (a_i - b_i - c_i)(\beta^{(j)})^i \quad \text{with } |a_i - b_i - c_i| \le 3\lfloor\beta\rfloor.$$

As β is a Pisot number, for all $j \ge 2$, $|\beta^{(j)}| < 1$ and $|\sum_{i=0}^{N} (\beta^{(j)})^i| < (1 - |\beta^{(j)}|)^{-1}$. We obtain in this way the announced bound on the moduli of the conjugates of x and $x \in U$. So F is a subset of U.

As it contains only points of $\mathbb{Z}[\beta]$ with bounded modulus and whose all conjugates have bounded modulus, the set U is finite. Thus F is a finite set. $\square$

The choice of any interval $I \subset]-1, 1[$ of length 1 allows us to reduce the cardinality of the set containing a set F.

Lemma 4. *Let β be a Pisot number of degree d, let $I \subset]-1, 1[$ be an interval of length 1 and let U' be the following finite set*

$$U' = \left\{ x \in \mathbb{Z}[\beta] \mid x \in I \text{ and } \forall\, 2 \le j \le d,\ |x^{(j)}| < \frac{2\lfloor\beta\rfloor}{1 - |\beta^{(j)}|} \right\}.$$

Then there exists a subset F of U' such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$.

Proof. We choose here $I \subset]-1, 1[$ of length 1 and improve the bound on the moduli of the conjugates of x given in Lemma 3 by considering the decomposition

$$\mathbb{Z}_\beta - \mathbb{Z}_\beta = (\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+) \cup (\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+) \cup -(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+).$$

More precisely let $x \in F \subset I$, then $x \in (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$ and can be written as

$$x = \sum_{i=0}^{N} (a_i - b_i)\beta^i - \sum_{i=0}^{N} c_i \beta^i.$$

We study $|a_i - b_i - c_i|$ according to the signs of a_i, b_i and c_i. In $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+$, the coefficients satisfy $|a_i - b_i| \leq \lfloor\beta\rfloor$. Moreover when $F \subset]-1,1[$, $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + F$ and $-\left(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+\right) \subset \mathbb{Z}_\beta^- + F$, then we have $|a_i - c_i| \leq \lfloor\beta\rfloor$. So when $F \subset]-1,1[$, we get in all cases $|a_i - b_i - c_i| \leq 2\lfloor\beta\rfloor$. Thus

$$\forall\, 2 \leq j \leq d \quad x^{(j)} = \sum_{i=0}^{N} (a_i - b_i - c_i)(\beta^{(j)})^i \quad \text{with } |a_i - b_i - c_i| \leq 2\lfloor\beta\rfloor,$$

and the announced bound on the moduli of the conjugates of x holds true. □

Example 2. Let β be a quadratic Pisot unit, then the set U' contains 5 points.

5 A first reduction of the cardinality of the sets containing F

In order to reduce the size of the sets containing F we study the properties of the elements of F.

Lemma 5. *Let β be a Pisot number and let $F \subset (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$. If $f \in F$ there exist a nonnegative integer N, and two finite words $b_N \cdots b_0$ and $a_N \cdots a_0$ respectively admissible for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ and $\mathbb{Z}_\beta$ such that*

$$f_0 = f, \quad \forall\, 0 \leq i \leq N \quad f_{i+1} = \frac{f_i - (b_i - a_i)}{\beta} \quad \text{and} \quad f_{N+1} = 0.$$

Proof. An element f in F can be written as $f = \sum_{i=0}^{N} (b_i - a_i)\beta^i$ with $x = \sum_{i=0}^{N} a_i\beta^i \in \mathbb{Z}_\beta$, $a_N \cdots a_0$ being admissible for $\mathbb{Z}_\beta$, and $y = \sum_{i=0}^{N} b_i\beta^i \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$, $b_N \cdots b_0$ being admissible for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$. Note that leading 0's are allowed.

With these notations we get for all $0 \leq i \leq N$, $f_i = \sum_{j=0}^{N-i} (b_{j+i} - a_{j+i})\beta^j$ and $f_{N+1} = 0$. □

Let $V = \left\{x \in \mathbb{Z}[\beta] \mid |x| < \frac{2\lfloor\beta\rfloor}{\beta - 1}, \text{ and } \forall\, 2 \leq j \leq d,\ |x^{(j)}| < \frac{2\lfloor\beta\rfloor}{1 - |\beta^{(j)}|}\right\}$. It is a finite set, with the following property that for all $f \in ((\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta) \cap U'$, the elements $f_0, \ldots, f_N$ of any sequence associated with f according to Lemma 5 belong to V. Indeed, from Lemmas 4 and 5, when $F \subset U'$, for all i, $|b_i - a_i| \leq 2\lfloor\beta\rfloor$. So for $0 \leq i \leq N$ and $2 \leq j \leq d$, the conjugates $f_i^{(j)}$ of f_i satisfy $|f_i^{(j)}| \leq 2\lfloor\beta\rfloor/(1 - |\beta^{(j)}|)$. Moreover the smallest C such that $|x| < C$ implies $|(x - (b - a))/\beta| < C$ is $C = 2\lfloor\beta\rfloor/(\beta - 1)$.

Following [7], we define a directed graph G whose set of vertices is the set V and having an edge $x \xrightarrow{(b,a)} y$ labelled by (b, a) if $y = (x - (b - a))/\beta$.

Lemma 6. *Let $F \subset U'$ be a minimal set satisfying $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Let V_0 be the subset of V of vertices connected to 0 in G. Then $F \subset V_0$.*

From each vertex f of G which is in U' we look for a path from f to 0 in G which is successful in $\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times \mathcal{A}_{\mathbb{Z}_\beta}$. Note that in G words are processed least significant digit first, contrarily to the automata for $\mathbb{Z}_\beta$ and $\mathbb{Z}_\beta - \mathbb{Z}_\beta$, where words are processed most significant digit first (*i.e.* from left to right). So we first define an automaton $\mathcal{G}_f$ having as underlying transition graph G with reversed edges, 0 as initial state and f as terminal state. We then compute the intersection automaton $\mathcal{I}_f = (\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times \mathcal{A}_{\mathbb{Z}_\beta}) \cap \mathcal{G}_f$. The following result then holds true.

Proposition 5. *An element f of U' is in V_0 if and only if the language recognized by $\mathcal{I}_f$ is nonempty.*

Remark 2. The number of states of the automaton $\mathcal{I}_f$ constructed above is $\mathcal{O}\left(K^3 \times |V|\right)$ where K is the number of states of $\mathcal{A}_{\mathbb{Z}_\beta^+}$ and $|V|$ is the number of vertices of G.

6 Minimization of the cardinality of the set F

The finite sets $U' \cap V_0$ obtained by the previous construction are not minimal. An element $y \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$ can be close to two different points of $\mathbb{Z}_\beta$, for example such that $x < y < x'$ with $x, x' \in \mathbb{Z}_\beta$ and $y = x + f = x' + f'$ with $f, f' \in U' \cap V_0$.

Theorem 1. *A minimal set $F \subset U' \cap V_0$ can be computed by an algorithm exponential in time and space. It consists in building a transducer which rewrites a representation of an element of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ into its representation in $\mathbb{Z}_\beta + F$.*

Proof. To find a minimal set $F \subset U' \cap V_0$ we proceed in two steps.

First for each $f \in U' \cap V_0$, we define a deterministic automaton $\mathcal{A}_f$ that recognizes the set of admissible words for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ that appear as the first component of the labels of the successful paths in $\mathcal{I}_f$. The automaton $\mathcal{A}_f$ is obtained by erasing the second component of the labels (that belongs to $\mathbb{Z}_\beta$) of the edges of $\mathcal{I}_f$ and determinizing the automaton defined in this way. The determinization of automata is based on the so-called subset construction (see [4]), which is exponential in space, and the automaton $\mathcal{A}_f$ has $\mathcal{O}(2^{Q_{\mathcal{I}_f}})$ states.

Next we look amongst all subsets of $U' \cap V_0$ for the smallest set F such that the language recognized by $\cup_{f \in F} \mathcal{A}_f$ contains an admissible representation of each element of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$. To test the inclusion, we compute the complement $\mathcal{C}_F$ of $\cup_{f \in F} \mathcal{A}_f$. Then the language recognized by $\cup_{f \in F} \mathcal{A}_f$ contains an admissible representation of each element of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ if and only if the intersection of $\mathcal{C}_F$ and $\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta}$ is empty. Note that the complexity of the search amongst all subsets of $U' \cap V_0$ is exponential in time.

From the set F obtained above, we define a transducer that provides, given $y = \sum_{i=0}^{N} b_i \beta^i \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$ where $b_N \ldots b_0$ is admissible for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$, a decomposition $(a_N \ldots a_0, f)$ where $a_N \ldots a_0$ is admissible for $\mathbb{Z}_\beta$, $f \in F$ and $y = \sum_{i=0}^{N} a_i \beta^i + f$.

Consider the intersection automaton $\mathcal{I}_F = (\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times \mathcal{A}_{\mathbb{Z}_\beta}) \cap \mathcal{G}_F$ (F is the set of terminal states of $\mathcal{G}_F$). For any element y admissible for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ there exists $f \in F$ such that y is the first component of the label of a successful path w ending in (s, f) where s is any state of $(\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta}) \times \mathcal{A}_{\mathbb{Z}_\beta}$ (by construction all states are terminal). Consequently we get $y = x + f$ where x is the second component of the label of the same path w and so is admissible for $\mathbb{Z}_\beta$.

More generally the first component of the labels of the edges in $\mathcal{I}_F$ can be interpreted as the inputs admissible for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ of the transducer, the second component as the corresponding outputs admissible for $\mathbb{Z}_\beta$. The associated element of F is given by the first component of the label of the state where the path ends. □

To conclude, the method used here for determining minimal sets F could be generalized to more general Meyer sets related with integral matrices having β as spectral radius.

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