Constructions of Pisot and Salem numbers with flat palindromes

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Abstract

This paper introduces explicit conditions for some natural family of polynomials to define Pisot or Salem numbers, and reviews related topics as well as their references.

1   Introduction

A *Pisot* (resp. *Salem*) number is an algebraic integer $\theta > 1$ whose Galois conjugates other than θ have moduli less than 1. (resp. less than or equal to 1 and at least one conjugate lies on the unit circle.) These algebraic numbers unexpectedly or exceptionally appeared in number of quite different branches of mathematics. A comprehensive survey is found in the book [11]. However, related areas are still steadily expanding. To give convenient pointers to the reader, we only mention some of such areas with surveys/recent papers :

- Number theory (Uniform distribution [24], β-expansion [8, 1], Lehmer's problem [12, 45]),

- Harmonic analysis (Salem-Zygmund Theorem, Bernoulli convolution [41, 47, 19], Wavelet, Meyer set [28, 29, 32]),

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- Discrete dynamical systems (Symbolic dynamics [42], Pisot conjecture [6, 7], Tiling [30, 46, 3], Jacobi-Perron algorithm [18, 39]),

- Mathematical physics (Quasi-crystals, Aperiodic structures [34, 4, 5]).

This list is not intended to be exhaustive. You find many important references therein. An essential reason why such algebraic numbers have some importance may come from the fact that Pisot numbers behave like rational integers in many situations.

In this paper we give a new easy construction of Pisot and Salem numbers. Here the word 'easy' means that it is given by a simple arrangement of coefficients of a defining polynomial, not necessarily minimal in degree. To give the reader an easy access to the related references, we review in §2 and §3, related known constructions of them as well. Our target is to examine polynomials of the form,

$$f(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \cdots - a_1x + 1 \in \mathbb{Z}[x],$$

where the word $a_{d-1}a_{d-2}\cdots a_1$ is a *palindrome*, i.e., $a_i = a_{d-i}$ for all $i = 1, 2, \ldots, d-1$. First, we derive some sufficient conditions on the coefficients for the polynomial f to give a Salem number. More precisely, we show that under this condition, f is the minimal polynomial of a Salem number (possibly) times a cyclotomic polynomial (not necessarily irreducible).

Next, we observe the slightly modified polynomial,

$$g(x) := (f(x) - 1)/x = x^{d-1} - a_{d-1}x^{d-2} - a_{d-2}x^{d-3} - \cdots - a_2x - a_1.$$

This kind of polynomial was in fact considered before in [26] to study some algebraic integers arising from β-expansions. In the present paper, we get a sufficient condition for the polynomial g to be the minimal polynomial of a Pisot number.

Two conditions above for f and g are similar in spirit to each other. In detail, a word $b^{d-1} := \underbrace{bb\cdots b}_{d-1 \text{ times}}$ for some $b \in \mathbb{Z}$ is perturbed to get the word $a_{d-1}a_{d-2}\cdots a_1$, but still maintaining the palindromicity of $a_{d-1}a_{d-2}\cdots a_1$. We will show that if the perturbation is small enough then f gives a Salem number and g a Pisot number. For example, as a consequence of Theorem 3.2 in §3, we prove

Theorem 1.1. *Let $b \geq 2$ be an integer. If $a_{d-i} = a_i$ and $a_i \in \{b, b-1\}$ for every $i = 1, 2, \ldots, d-1$ with $b \geq \lfloor (d-1)/2 \rfloor$ then, $x^{d-1} - a_{d-1}x^{d-2} - a_{d-2}x^{d-3} - \cdots - a_2x - a_1$ gives a Pisot number.*

which settles affirmatively Question 1 in [27] and justifies the term "flat palindromes" in the title.

2 Construction of Salem numbers.

Given a polynomial $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ with $a_d \neq 0$, another polynomial f^* is defined by

$$f^*(x) := x^d f(1/x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_{d-1}x + a_d.$$

If $f = f^*$, then f is said to be a *reciprocal polynomial*. A Salem number $\beta > 1$ is a zero of a reciprocal polynomial of even degree greater than two (see [43]). In this paper, the defining polynomial of β could be reducible and we do not restrict ourselves to even d. Suppose that $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ is reciprocal with $a_d \neq 0$. Then we may write

$$f(x) = x^{d/2}g(x).$$

One notes that $g(x) \in \mathbb{R}[x + x^{-1}]$ if d is even, and that $g(x) \in \mathbb{R}[x^{1/2} + x^{-1/2}]$ otherwise. If d is odd, then we take coherently their *principal values* for the complex exponents. Therefore, we have, whether d is even or not, $g(x) = g(x^{-1})$ and thus

$$g(e^{\theta\sqrt{-1}}) \in \mathbb{R} \ \text{ for any } \theta \in \mathbb{R}.$$

Let us recall the well known characterization of Salem numbers (see, e.g., [43]). As d is even, put $g(x) = G(y) \in \mathbb{R}[y]$ with $y = x + 1/x$. Let $f(x)$ be an irreducible reciprocal polynomial of even degree greater than 2. Then $f(x)$ is the minimal polynomial of a Salem number if and only if $f(1) < 0$ and $G(y)$ has $d/2 - 1$ distinct zeros in $(-2, 2)$. Note that the condition ' $f(1) < 0$ and $G(y)$ has $d/2 - 1$ zeros in $(-2, 2)$' does not imply that f is irreducible. If it is reducible, then there is an irreducible factor which does not have any zero outside the unit circle. Denote by ζ one of its zeros. Such a factor must be a cyclotomic polynomial since the sequence ζ^n $(n = 1, 2, \dots)$ is bounded for all conjugates and hence eventually periodic[1]. Here we summarize known constructions of Salem numbers:

- The above characterization by $G(y)$ gives a practical method. However, the set of such $G(y)$'s is not easy to handle.

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[1]This is due to Kronecker. Hereafter we refer to this fact as Kronecker's Theorem.

- The polynomial $x^m f(x) \pm f^*(x)$ is reciprocal for a monic polynomial $f(x)$. This can be applied to prove that every Pisot number is an accumulation point of the set of Salem numbers (see [43]). This idea was further developed by [9, 13, 10, 36, 37]. The essential idea of their construction is summarized as *interlacing property*.

- Chinburg [16] showed that every Salem number is the exponential of a rational multiple of the derivative at s = 0 of an Artin L-function. This is intimately related to the Stark conjecture on units in number fields. See also [23].

- Inspired by the growth rate study of Coxeter groups [21, 20, 40], the construction of Salem numbers using graphs was explored [31, 38, 35, 25].

Our construction of Salem numbers falls into the second category. Especially the interlacing property plays an essential rôle in this paper. See the paragraph after the proof of Theorem 2.1. We shortly discuss this property at the end of this section as well.

Let $h(\theta) := g(e^{\theta\sqrt{-1}})$. Then $h : [0, 2\pi] \to \mathbb{R}$ satisfies $h(\theta) = h(2\pi - \theta)$. If d is odd, then $f(-1) = g(-1) = h(\pi) = 0$. Although $f(z)z^{-d/2}$ is ramified at 0 when d is odd, h is well-defined and continuous in any case, because $h(\pi) = 0$. This is a small trick to avoid tedious notations of Riemann surface of $\sqrt{z}$.

Given a reciprocal $f(x) \in \mathbb{R}[x]$, let us denote the above real function h by

$$\mathcal{T}f(\theta) := g(e^{\theta\sqrt{-1}}).$$

Then one observes that if d is even then

$$f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) = (-1)^k \cdot \mathcal{T}f\left(\frac{2k\pi}{d}\right)$$

for every $k = 1, 2, \ldots, d-1$, and that if d is odd then

$$f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) = \begin{cases} (-1)^k \cdot \mathcal{T}f\left(\frac{2k\pi}{d}\right), & k = 1, 2, \ldots, \frac{d-1}{2}, \\ (-1)^{k+1} \cdot \mathcal{T}f\left(\frac{2k\pi}{d}\right), & k = \frac{d+1}{2}, \frac{d+3}{2}, \ldots, d-1. \end{cases}$$

We derive a sufficient condition for Salem numbers. We use in the next theorem, and will use hereafter minus signs for coefficients. This setting will be repeatedly convenient for our purpose.

Theorem 2.1. *Let $f(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \cdots - a_1 x + 1 \in \mathbb{Z}[x]$ be a reciprocal polynomial, and assume*

$$f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) \geq 0$$

for all $k = 1, 2, \ldots, d-1$. If f is non-cyclotomic, then there is a zero $\beta > 1$ of f such that $1/\beta$ is also a zero and all the other zeros have moduli 1, whence β is a Salem number.

Proof. Let us start with the case $f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) > 0$ for all k. Assume at first that d is even. Since $(-1)^k \cdot \mathcal{T}f(2k\pi/d) > 0$ for all $k = 1, 2, \ldots, d-1$, the Intermediate Value Theorem yields $d-2$ zeros θ_i $(i = 1, 2, \ldots, d-2)$ of $\mathcal{T}f$ such that

$$\frac{2\pi}{d} < \theta_1 < \frac{4\pi}{d} < \theta_2 < \cdots < \frac{2(d-2)\pi}{d} < \theta_{d-2} < \frac{2(d-1)\pi}{d}. \qquad (1)$$

Consequently, all $e^{\theta_i\sqrt{-1}}$, $i = 1, 2, \ldots, d-2$, are the zeros of f.

In case d is odd, we apply the Intermediate Value Theorem twice for the real function $\mathcal{T}f$ — one on the interval $[0, \pi]$ to get the zeros $\theta_1 < \ldots < \theta_{(d-3)/2}$, and the other on $[\pi, 2\pi]$ to get the zeros $\theta_{(d+1)/2} < \ldots < \theta_{d-1}$. Hence we can arrange the zeros including $\theta_{(d-1)/2} = \pi$ as in (1).

For the general case $f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) \geq 0$, put $f_\varepsilon(x) := f(x) + \varepsilon(x^d + 1)$ for a real $\varepsilon > 0$. Taking a small ε, the above argument guarantees that f_ε has zeros with the arrangement (1). The same is valid for $f = \lim_{\varepsilon\downarrow 0} f_\varepsilon$ by a well known fact that each zero of a polynomial is a continuous function of its coefficients as long as its leading coefficient is nonzero [17].

Suppose $f(1) > 0$. Then there exists another zero $\theta_0 \in (0, 2\pi/d)$ of $\mathcal{T}f$ because $\mathcal{T}f(2\pi/d) < 0$. So $e^{\pm\theta_0\sqrt{-1}}$ are also the zeros of f. By Kronecker's theorem, f is a cyclotomic polynomial. If $f(1) = 0$, then the remaining zero of f should be on the unit circle since f is reciprocal. And f is again a cyclotomic polynomial. We thus obtain $f(1) < 0$ and then a zero $\beta > 1$ of f, which is followed by $f(1/\beta) = 0$. The set $\{\beta, 1/\beta\} \cup \{e^{\theta_i\sqrt{-1}} \mid i = 1, 2, \ldots, d-2\}$ exhausts the whole set of zeros of f.

$\square$

The above situation of (1) is said that the zeros of f on the unit circle *interlace* the zeros of $(x^d - 1)/(x - 1)$. Here we adopted the interlacing

property to construct Salem numbers.

For a vector $\mathbf{u} = (u_1, u_2, \ldots, u_{d-1}) \in \mathbb{R}^{d-1}$, we consider the following:

$$L(\mathbf{u}) := \min_{y \in \mathbb{R}} \sum_{i=1}^{d-1} |u_i - y|.$$

The exact value of $L(\mathbf{u})$ is attained when y is the median of the coordinates u_i. To be more precise we may assume $u_1 \leq u_2 \leq \cdots \leq u_{d-1}$. Then it is easy to see that $L(\mathbf{u}) = \sum_{i=1}^{d-1} |u_i - y|$ for all y satisfying $u_{\lfloor d/2 \rfloor} \leq y \leq u_{\lceil d/2 \rceil}$. Here, a number $m(\mathbf{u})$ is defined to be the largest such y, i.e., $m(\mathbf{u}) := u_{\lceil d/2 \rceil}$.

The next lemma follows immediately from the identity,

$$\frac{x^d - 1}{x - 1} = x^{d-1} + x^{d-2} + \cdots + 1.$$

Lemma 2.2. *Let $f(x) = x^d - b(x^{d-1} + x^{d-2} + \cdots + x) + 1$. Then we have*

$$f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) = b + 2$$

for every $k = 1, 2, \ldots, d-1$.

Assume that $f(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \cdots - a_1 x + 1 \in \mathbb{Z}[x]$ is reciprocal and $\mathbf{a} = (a_1, \ldots, a_{d-1}) \in \mathbb{Z}^{d-1}$. If we put $b = m(\mathbf{a})$, then one finds that

$$\left| f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) - (m(\mathbf{a}) + 2) \right| \leq L(\mathbf{a}) \tag{2}$$

for all $k = 1, 2, \ldots, d-1$. Accordingly, we have another sufficient condition for Salem numbers. The following theorem is easier to check than Theorem 2.1. It refers to the coefficients only.

Theorem 2.3. *Let $f(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \cdots - a_1 x + 1 \in \mathbb{Z}[x]$ be a reciprocal polynomial, and $\mathbf{a} = (a_1, a_2, \ldots, a_{d-1}) \in \mathbb{Z}^{d-1}$. Suppose $m(\mathbf{a}) \geq L(\mathbf{a}) - 2$. If f is non-cyclotomic, then f gives a Salem number as in Theorem 2.1.*

Proof. The hypothesis implies that $f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) \geq 0$ for every $k = 1, 2, \ldots, d-1$. $\qquad\square$

Two theorems in this section do not specify the irreducibility of f. But if it is reducible, then Kronecker's theorem implies that it should be a product of the minimal polynomial of a Salem number and a cyclotomic polynomial.

Example 1. Let $\mathbf{a} = (a_1, a_2, \ldots, a_9) = (4, 4, 5, 4, 4, 4, 5, 4, 4)$ and $f(x) = x^{10} - a_9 x^9 - a_8 x^8 - \cdots - a_1 x + 1$. Then we have $m(\mathbf{a}) = 4$ and $L(\mathbf{a}) = 2$. Therefore, Theorem 2.3 shows that f gives a Salem number. But f is factored into

$$f(x) = (x^8 - 5x^7 - 4x^4 - 5x + 1)(x^2 + x + 1).$$

Note that the polynomial $x^8 - 5x^7 - 4x^4 - 5x + 1$ does not satisfy the condition of Theorem 2.3. □

It is not known whether there are Salem numbers arbitrarily close to 1. The smallest Salem number ever found [33] is the zero of its minimal polynomial

$$l(x) := x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Example 2. Let $\mathbf{b} = (b_1, b_2, \ldots, b_9) = (-1, 0, 1, 1, 1, 1, 1, 0, -1)$. Since $m(\mathbf{b}) = 1$ and $L(\mathbf{b}) = 6$, Theorem 2.3 does not work for $l(x)$. But we cannot exclude the case where there exists $d(x)$, a cyclotomic polynomial such that $d(x)l(x)$ satisfies the condition of Theorem 2.3. A calculation shows that

$$\min_{1 \leq k \leq 9} l\left(\exp\left(\frac{k\pi\sqrt{-1}}{5}\right)\right) = l\left(\exp\left(\frac{3\pi\sqrt{-1}}{5}\right)\right) > 0.14.$$

Thus Theorem 2.1 tells us that l gives a Salem number. □

We briefly discuss the interlacing property appeared in the previous works. Let f be the minimal polynomial of a Salem number. Then Boyd [13] proved that if, for $c(x)$ a cyclotomic polynomial, $c(x)f(x)$ has no multiple root, then it is of the form

$$c(x)f(x) = xp(x) \pm p^*(x),$$

where $p(x)$ is a power of x times the minimal polynomial of a Pisot number. Later, Bertin and Boyd [10] generalized the choices for p and defined the subsets of Salem numbers, A_q and B_q according to the specific distribution of zeros of p and its constant term $q = |p(0)|$. They observed that the set T of Salem numbers is equal to $\bigcup_{q \geq 2} A_q = \bigcup_{q \geq 0} B_q$, and presented a necessary

and sufficient condition for each membership of A_q and B_q using the interlacing property. In terms of our context, they showed that for the minimal polynomial f of a Salem number, there exists a cyclotomic polynomial $c(x)$ and a reciprocal polynomial L such that the zeros of $c(x)f(x)$ interlace the zeros of L on the unit circle. In view of these results, one may say that the set of Salem numbers is not so easy to grasp and the converse of Theorem 2.3 (by finding a suitable cyclotomic factor $d(x)$ so that $d(x)f(x)$ satisfies the requirement) is unlikely to hold. However, for the moment, the relationship to our construction is unclear, since we do not have an algorithm to find the cyclotomic factors $c(x)$ and $d(x)$. Therefore it is hard to tell whether or not a given Salem number belongs to a fixed A_q or B_q.

It turned out that this interlacing property is a nice tool to construct Salem numbers. McKee and Smyth [37] constructed Salem numbers of any prescribed integer trace by this idea.

3   Construction of Pisot numbers.

First we review known methods of constructing Pisot numbers. Put $g(x) = x^d - c_{d-1}x^{d-1} - \cdots - c_0$.

- $c_{d-1} > \sum_{i=0}^{d-2} |c_i| + 1$ and $c_0 \neq 0$ then $g(x)$ is the minimal polynomial of a Pisot number (cf. Chapter 5.2 of [11]).

- $c_{d-1} \geq c_{d-2} \geq \cdots \geq c_0 > 0$ then $g(x)$ is the minimal polynomial of a Pisot number (cf. [15, 22]).

- There is a sufficient condition written by explicit inequalities of $c_0, \ldots, c_{d-1}$. This condition is also necessary if the degree is less than 5. It is conjectured to be a characterization for all degrees ([2]).

- A main theme of [11] is to study the distribution of Pisot numbers on the real line using complex/real analytic techniques: Hardy spaces and Schur's algorithm. Deeper results on Pisot and Salem numbers are derived along these lines ([14, 37]).

Using the idea of the previous section, we derive a sufficient condition for Pisot numbers.

Theorem 3.1. *Let* $f(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \cdots - a_1x + 1 \in \mathbb{Z}[x]$ *with* $f(1) < 1$ *and* $a_i = a_{d-i}$, $i = 1, 2, \ldots, d-1$. *Put* $g(x) = f(x) - 1$.

(a) *Suppose*

$$f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) > 1$$

for all $k = 1, 2, \ldots, d-1$. *Then* g *is a power of* x *times the minimal polynomial of a Pisot number. If* $a_1 \neq 0$, *then* $g(x)/x$ *is the minimal polynomial of a Pisot number.*

(b) *Suppose that the same values as Part (a) are greater than or equal to* 1 *for all* $k = 1, 2, \ldots, d-1$, *and that the equality holds at least once. If* $a_1 \neq 0$, *then* $g(x)/x$ *is a product of the minimal polynomial of a Pisot number and a cyclotomic polynomial.*

Proof. The condition $f(1) < 1$ guarantees a zero $\beta > 1$ of g.

Consider a closed curve $\mathcal{C} = \{f(e^{\theta\sqrt{-1}}) \mid \theta \in [0, 2\pi]\}$ in the complex plane $\mathbb{C}$. Put $\theta_0 = 0$ and $\theta_{d-1} = 2\pi$, and let $\theta_1, \ldots, \theta_{d-2}$ be given as in (1). Since

$$f(e^{\theta\sqrt{-1}}) = (e^{\theta\sqrt{-1}})^{d/2} \cdot \mathcal{T}f(\theta)$$

and we take the principal values, we deduce

$$\text{Im} f(e^{\theta\sqrt{-1}}) < 0 \quad \text{for} \quad \theta_{m-1} < \theta < \frac{2m\pi}{d} \tag{3}$$

$$\text{Im} f(e^{\theta\sqrt{-1}}) > 0 \quad \text{for} \quad \frac{2m\pi}{d} < \theta < \theta_m \tag{4}$$

for $m = 1, 2, \ldots, d-1$. The case where d is odd is demonstrated in the example below.

Visualizing the result, one finds that $\mathcal{C}$ starts from $f(1) < 1$ in $\mathbb{C}$, moves anticlockwise, and passes through the following points:

$$f(1) \to f\left(\exp\left(\frac{2\pi\sqrt{-1}}{d}\right)\right) \to 0 \to f\left(\exp\left(\frac{4\pi\sqrt{-1}}{d}\right)\right) \to 0 \to \cdots$$

$$\to f\left(\exp\left(\frac{2(d-1)\pi\sqrt{-1}}{d}\right)\right) \to f(1).$$

Recall here that $f(e^{\theta_i\sqrt{-1}}) = 0$ for $i = 1, \ldots, d-2$, where θ_i come from (1) in Theorem 2.1. Moreover $\{f(1), 0\} \cup \{f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) \mid k = 1, 2, \ldots, d-1\}$ is

the set of all the intersections between $\mathcal{C}$ and the real axis. We thus conclude that $\mathcal{C}$ goes $d-1$ times around 1, i.e., the winding number of f at 1 is $d-1$. Now the Argument Principle tells us that $f(x) - 1 = g(x)$ has $d-1$ zeros strictly inside the unit circle. This proves Part (a).

Let ε be in the interval $(f(1), 1)$, and put $f_\varepsilon(x) := f(x) - \varepsilon$. Then f_ε has $d-1$ zeros strictly inside the unit circle. Again by the continuity of zeros of the polynomial [17], all zeros of $g(x) = \lim_{\varepsilon \uparrow 1} f_\varepsilon(x)$ other than $\beta > 1$ have moduli less than or equal to 1. It remains to exclude the case where $g(x)/x$ gives a Salem number. If so, then $g(x)/x$ would be a reciprocal polynomial. This implies

$$-1 = a_1 = a_{d-1} = a_2 = \cdots,$$

i.e., $g(x)/x = (x^d - 1)/(x - 1)$, a contradiction. $\qquad \square$

With the help of Theorem 3.1, we also state a condition on the coefficients to get Pisot numbers.

Theorem 3.2. *Let* $g_1(x) = x^{d-1} - a_{d-1}x^{d-2} - a_{d-2}x^{d-3} - \cdots - a_2 x - a_1 \in \mathbb{Z}[x]$ *with* $g_1(1) < 0$, $a_1 \neq 0$ *and* $a_i = a_{d-i}$, $i = 1, 2, \ldots, d-1$. *Put* $\mathbf{a} = (a_1, a_2, \ldots, a_{d-1}) \in \mathbb{Z}^{d-1}$.

(a) *If* $m(\mathbf{a}) \geq L(\mathbf{a})$, *then* g_1 *is the minimal polynomial of a Pisot number.*

(b) *If* $m(\mathbf{a}) = L(\mathbf{a}) - 1$, *then* g_1 *is either the minimal polynomial of a Pisot number, or a product of the minimal polynomial of a Pisot number and a cyclotomic polynomial.*

Proof. (a) Let $f(x) := xg_1(x) + 1$. From (2), it follows that

$$f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{d}\right)\right) > 1$$

for all $k = 1, 2, \ldots, d-1$.
(b) In the inequality above, the equality may (or may not) occur. The reducibility of g_1 depends on the occurrence of the equality. $\qquad \square$

Theorem 1.1 immediately follows from Theorem 3.2, since $m(\mathbf{a}) = b$ or $b - 1$ and $L(\mathbf{a}) \leq \lfloor (d-1)/2 \rfloor$.

Example 3. Let $g_1(x) = x^4 - x^3 - 1$, $f(x) = xg_1(x) + 1$ and $\mathbf{a} = (1, 0, 0, 1)$. Then we have $f(1) < 1$, $m(\mathbf{a}) = 1$ and $L(\mathbf{a}) = 2$. Thus Theorem 3.2 shows
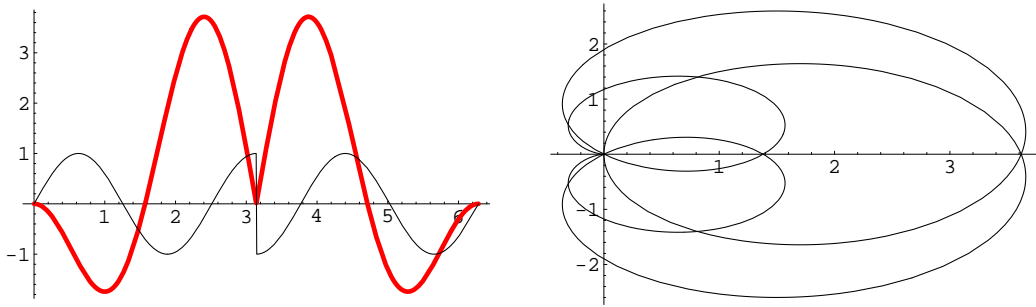
Figure 1: $\mathcal{T}f(\theta)$ and $\mathrm{Im}(e^{\theta\sqrt{-1}})^{5/2}$ (left), and the contour of $f(e^{\theta\sqrt{-1}})$ (right)

that g_1 gives a Pisot number, but does not tell us about its irreducibility. A calculation leads us to

$$\min_{1\le k\le 4} f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{5}\right)\right) = f\left(\exp\left(\frac{2\pi\sqrt{-1}}{5}\right)\right) > 1.38.$$

Now Theorem 3.1 says that g_1 is the minimal polynomial of a Pisot number. Actually, the real zero $\beta \approx 1.38028$ of g_1 is the second smallest among all Pisot numbers [44].

In the left of Figure 1, the thick line represents $\mathcal{T}f(\theta)$ and the thin line the imaginary part of the principal values of $(e^{\theta\sqrt{-1}})^{5/2}$. From this figure, we verify (3) and (4) in the proof of Theorem 3.1. The right one is the closed contour $\mathcal{C} = \{f(e^{\theta\sqrt{-1}}) \mid \theta \in [0, 2\pi]\}$. We observe that the winding number of f at 1 is 4. □

Example 4. The real zero $\beta \approx 1.32472$ of $x^3 - x - 1$ is known to be the smallest Pisot number [44]. Let $g_1(x) = x^5 - x^4 - 1$, $f(x) = xg_1(x) + 1$ and $\mathbf{a} = (1, 0, 0, 0, 1)$. Note $g_1(x) = (x^3 - x - 1)(x^2 - x + 1)$. We have $f(1) < 1$, $m(\mathbf{a}) = 0$ and $L(\mathbf{a}) = 2$. Thus Theorem 3.2 cannot be applied to g_1. We verify by a direct calculation,

$$f\left(\exp\left(\frac{2k\pi\sqrt{-1}}{6}\right)\right) \ge 1,$$

for every $k = 1, \dots, 5$. And equality holds when $k = 1, 5$. Now Theorem 3.1 (b) proves that g_1 is a product of the minimal polynomial of a Pisot number and a cyclotomic polynomial. Here the cyclotomic part is $x^2 - x + 1$. □

11

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