A CERTAIN FINITENESS PROPERTY OF PISOT NUMBER SYSTEMS

SHIGEKI AKIYAMA, HUI RAO AND WOLFGANG STEINER

Abstract. In the study of substitutative dynamical systems and Pisot number systems, an algebraic condition, which we call 'weak finiteness', plays a fundamental role. It is expected that all Pisot numbers would have this property. In this paper, we prove some basic facts about 'weak finiteness'. We show that this property is valid for cubic Pisot units and for Pisot numbers of higher degree under a dominant condition.

1. Introduction

Let $\beta > 1$ be a real number. The *β-transformation* is a piecewise linear transformation on $[0, 1)$ defined by

$$T_\beta : x \longrightarrow \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \xi \rfloor$ is the largest integer not exceeding ξ. By iterating this map and considering its trajectory

$$x \xrightarrow{x_1} T_\beta(x) \xrightarrow{x_2} T_\beta^2(x) \xrightarrow{x_3} \dots$$

with $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$, we obtain the greedy expansion of x:

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} \dots = .x_1 x_2 x_3 \dots .$$

For any real number $x > 0$, there is an $m > 0$ such that $\beta^{-m-1}x \in [0, 1)$. Thus we can express each x in the form

$$x = x_{-m}\beta^m + \dots + x_{-1}\beta + x_0 + \frac{x_1}{\beta} + \dots = x_{-m} \dots x_{-1}x_0.x_1x_2x_3 \dots,$$

which is called the *beta expansion.* If there is an integer k such that $x_i = 0$ for $i > k$, then we say that the β-expansion of x is finite and we occasionally omit writing zeros in the tail like: $x = x_{-m}x_{-m+1} \dots x_{k-1}x_k$.

Formally we may consider the trajectory of 1:

$$1 \xrightarrow{a_1} T_\beta(1) \xrightarrow{a_2} T_\beta^2(1) \xrightarrow{a_3} \dots .$$

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We call $a_1a_2a_3\dots$ the *expansion of one* and denote it by $d_\beta(1)$. Define

$$d^*_\beta(1) = \begin{cases} \overline{d_\beta(1)} & \text{if } d_\beta(1) \text{ is not finite} \\ \overline{a_1 \dots a_{d-1}(a_d - 1)} & \text{if } d_\beta(1) = a_1 \dots a_d \text{ with } a_d \neq 0, \end{cases}$$

where $\overline{x_1 \dots x_k}$ stands for the periodic expansion $x_1 \dots x_k x_1 \dots x_k \dots$. Then a sequence (finite or infinite) over the alphabet $\{0, 1, 2, \dots\}$ is said to be *admissible* if all its right truncations are lexicographically less than $d^*_\beta(1)$. A sequence is the β-expansion of some real number if and only if it is admissible. (See Parry [18] and Ito-Takahashi [17] for details). Let $\mathrm{Fin}(\beta)$ be the set of non-negative real numbers with finite β-expansion. Denote by $\mathbb{Z}[\beta]$ the minimal ring containing $\mathbb{Z}$ and β and by $\mathbb{Z}[\beta]_{\geq 0}$ the nonnegative elements of $\mathbb{Z}[\beta]$. We say that the number β has the *finiteness property* or the property (F) if

(F): $\quad \mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta]_{\geq 0}$

holds. This property was introduced by Frougny-Solomyak [14]. They showed that it implies that β is a Pisot number, i.e. a real algebraic integer greater than 1 with all conjugates lying strictly inside the unit circle, and they found the following class of Pisot numbers satisfying this property. Here a root of a polynomial is called *dominant*, if it has the maximal modulus of all roots.

Theorem A. (Frougny-Solomyak [14]) *If β is the dominant root of the polynomial $x^d - b_1x^{d-1} - b_2x^{d-2} - \dots - b_d \in \mathbb{Z}[x]$ with $b_1 \geq b_2 \geq \dots \geq b_d > 0$, then β is a Pisot number and has the property (F).*

Another class of Pisot numbers with (F) was found by Hollander.

Theorem B. (Hollander ([15])) *If β is the dominant root of the polynomial $x^d - b_1x^{d-1} - b_2x^{d-2} - \dots - b_d \in \mathbb{Z}[x]$ with $b_1 > \sum_{i=2}^d b_i$ and $b_i \geq 0$ $(1 \leq i \leq d)$, then β is a Pisot number and has the property (F).*

An alternative proof of Theorem B is given in §6. Of particular interest are *Pisot units*, which are Pisot numbers as well as algebraic units. Akiyama-Sadahiro [5] and Akiyama [1] used Pisot units β with the property (F) to construct tilings of $\mathbb{R}^{d-1}$ (where d is the degree of β). Praggastis [19] showed that such tiling gives rise to a Markov partition of the torus when β satisfies (F). The idea of these constructions is due to Thurston [27]. Note that a tiling close to these was originally obtained by Rauzy [20] in connection with substitutative dynamical systems. Arnoux-Ito [6] gave a further generalization of this 'Rauzy fractal' and described the relation with Markov partitions of toral automorphisms. A lot of applications of this theory are found (cf. Ito-Rao [13], Steiner [26]).

Note that there are Pisot numbers without the property (F), in particular all numbers with infinite expansions of one. A classification of cubic Pisot units with (F) was established in Akiyama [2] (see also Proposition 1).

The first author [3] also showed that the origin is an 'exclusive' inner point of the central tile if and only if (F) holds. For the tiling property, he showed that the condition (F) can be relaxed. Namely, for a Pisot unit β, Thurston's construction gives a tiling if and only if

(W): For any $x \in \mathbb{Z}[1/\beta]_{\geq 0}$ and any positive ε, there exist $y, z \in \mathrm{Fin}(\beta)$ that $x = y - z$ and $z < \varepsilon$.

holds. We call this condition *weak finiteness property* or (W) in this paper.

This property was first studied by Hollander [15]. He tried to show that a substitutative dynamical system associated to beta expansions has purely discrete spectrum by reducing this problem to showing (W). Sidorov [24] used this property to construct an almost conjugacy between the beta shift and a related toral automorphism. He also found another application of (W) for Bernoulli convolutions [25].

To study the tilings rising from Rauzy fractal, Ito-Rao [16] introduced the super-coincidence condition of a substitution. (W) is equivalent to the super-coincidence condition if we restrict to substitutions coming from β-numeration systems (see Ei-Ito-Rao [12]).

The present paper is devoted to the study of the property (W).

A Salem number is a real algebraic integer greater than 1 such that all its conjugates lie inside the closed unit disk and at least one conjugate lies on the unit circle. First we show

Theorem 1. *If β has the property (W), then it must be a Pisot or a Salem number.*

However, we are not able to prove (W) for any Salem number. Second, we derive an easier criterion for the property (W).

Theorem 2. *The property (W) is equivalent to:*

(W'): *For any $x \in \mathbb{Z}[1/\beta] \cap [0, 1)$, there exist $y, z \in \mathrm{Fin}(\beta)$ such that $x = y - z$ with $y < \beta$ and $z < 1$.*

This will be used to prove Theorem 4 in §5 and Proposition 3 in §6. It is easy to show that quadratic Pisot numbers β satisfy this weakly finiteness (see §2). In [3] it is conjectured that the property (W) holds for all Pisot units, in [25] that it should hold even for all Pisot numbers. We give partial answers to this conjecture.

Theorem 3. *If β is a cubic Pisot unit, then β satisfies (W).*

We do not know whether all cubic Pisot numbers satisfy (W).

Theorem 4. *Let β be the dominant root of $x^d - b_1 x^{d-1} - \cdots - b_d \in \mathbb{Z}[x]$. If $b_1 > \sum_{j=2}^{d} |b_j|$ (and $(b_1, b_2) \neq (2, -1)$), then β satisfies (W).*

Hereafter we refer to the inequality $b_1 > \sum_{j=2}^{d} |b_j|$ as *dominant condition.*

The paper is organized as follows. In §2, we review known results and also prove Theorems 1 and 2. If we knew the set $\mathcal{P}$ of purely periodic orbits of T_β, then we could show (W) without difficulty. In §3 the set $\mathcal{P}$ is given for cubic Pisot units by using an idea of [15]. Thus we can show Theorem 3 in §4. In §5, we prove Theorem 4. In §6, we discuss an alternative approach by using a branching beta expansion. This gives an efficient algorithm to confirm (F) or (W) in practice.

2. General criteria for weak finiteness

First we prove the necessary condition for numbers satisfying (W) given in Theorem 1.

Proof of Theorem 1. (W) implies that β is an algebraic integer, since we have an expression $\beta - \lfloor\beta\rfloor = y - z$ with $y, z \in \mathrm{Fin}(\beta)$ and $z < y < 1$. Assume that there is a conjugate γ of β with $|\gamma| > 1$. Take a positive integer m. From (W) we infer

$$\beta^m - \lfloor\beta^m\rfloor = \sum_{i=1}^{\ell} c_i \beta^{-i}$$

with $c_i \in (-\beta, \beta) \cap \mathbb{Z}$. Thus we have

$$\gamma^m - \lfloor\beta^m\rfloor = \sum_{i=1}^{\ell} c_i \gamma^{-i} \leq \frac{\lfloor\beta\rfloor}{1 - |\gamma|}.$$

This is absurd since the left side is not bounded when $m \to \infty$. $\square$

Now we turn to sufficient conditions for (W). It is obvious that (F) implies (W). In [3], it is shown that the Pisot numbers with the following property satisfy (W):

- **(PF):** For each polynomial $P(x)$ with non negative integer coefficients, $P(\beta) \in \mathrm{Fin}(\beta)$.

This condition was studied in [14] and proved for β where the expansion of one $a_1 a_2 a_3 \ldots$ has decreasing digits. Quadratic Pisot numbers satisfy either (F) by Theorem A or (PF) by the above criterion. Hence each quadratic Pisot number has the property (W).

For Pisot numbers, it is sufficient to test a finite set of $\mathbb{Z}[1/\beta]$:

Bertrand [9] and Schmidt [23] proved independently that every element of $\mathbb{Q}(\beta)$, so in particular every element of $\mathbb{Z}[1/\beta]$, has eventually periodic β-expansion if β is a Pisot number. (For Salem numbers, this is unknown.) Therefore we study the set

$$\mathcal{P} = \{x \in \mathbb{Z}[\beta]_{\geq 0} \mid T_\beta^m(x) = x \text{ for some } m > 0\}.$$

(The periodic points of $\mathbb{Z}[1/\beta]$ are always in $\mathbb{Z}[\beta]$, since we can choose n large such that $x = \beta^n x - P(\beta)$, and both $\beta^n x$ and $P(\beta)$ belong to $\mathbb{Z}[\beta]$.) It is easily seen that $\mathcal{P}$ is a finite set and gives the set of all possible periodic tails of beta expansions (cf. [3] Lemma 2). Therefore for Pisot numbers, (W) is equivalent to

(P): For any $x \in \mathcal{P}$ and any positive ε, there exist $y, z \in \mathrm{Fin}(\beta)$ such that $x = y - z$ and $z < \varepsilon$.

Furthermore, in [15], it is implicitly noted that

Lemma 1. *The property (W) is equivalent to*

(H): *For any $x \in \mathcal{P}$, there exist $y, z \in \mathrm{Fin}(\beta)$ such that $x = y - z$ with $y < 1$ and $z < 1$.*

For the convenience of the reader, we give the proof due to Hollander.

Proof. For each $x \in \mathbb{Z}[1/\beta]$ and for a sufficiently large n, we have the beta expansion

$$x = x_{-m} \dots x_{-1} x_0 . x_1 \dots x_n + \beta^{-n-1}\tau$$

with $\tau = .\overline{c_1 \dots c_\ell} \in \mathcal{P}$. We may assume that $\tau \neq 0$. Since this expansion is less than $d_\beta(1)$ at any starting point, there exists n so that $x_{-m} \dots x_{-1} x_0 . x_1 \dots x_{n-1}(x_n + 1)$ is admissible. Express $\tau = y - z$ by (H). Then, as finite words, the beta expansion of $x_{-m} \dots x_n + \beta^{-n-1} y$ coincides with the concatenation of $x_{-m} \dots x_n$ and the beta expansion of y. This means

$$x = (x_{-m} \dots x_{-1} x_0 . x_1 \dots x_n + \beta^{-n-1} y) - \beta^{-n-1} z$$

gives a desired expression. Thus β has the property (W). □

Now we turn to the equivalent condition for (W) which is needed in §5 and §6. Although we do not have any example, the following proof is valid even for Salem numbers.

Proof of Theorem 2. Clearly, (W) implies (W'). We are going to prove the other direction.

Let $d^*_\beta(1) = a^*_1 a^*_2 \dots$ be the infinite representation of 1. Pick $x \in \mathbb{Z}[1/\beta]$ with infinite greedy expansion $x = .x_1 x_2 \dots$ and let $x = .B_1 B_2 \dots$ be its *free block decomposition*, which is recursively given by $B_1 = x_1 \dots x_{k_1}$ such that $x_1 \dots x_{k_1 - 1} = a^*_1 \dots a^*_{k_1 - 1}$ and $x_{k_1} < a^*_{k_1}$, $B_2 = x_{k_1+1} \dots x_{k_2}$ such that $x_{k_1+1} \dots x_{k_2 - 1} = a^*_1 \dots a^*_{k_2 - k_1 - 1}$ and $x_{k_2} < a^*_{k_2 - k_1}$ and so on.

We distinguish four cases:

i) There exists arbitrarily large j such that $x_{k_j} < a^*_{k_j - k_{j-1}} - 1$. In this case, we consider

$$\eta = x - .x_1 \dots x_{k_j} \in \mathbb{Z}[\beta^{-1}] \cap [0, \beta^{-k_j}).$$

Then $\beta^{k_j}\eta$ is in $\mathbb{Z}[\beta^{-1}] \cap [0,1)$ and has, by assumption (W'), a representation

$$\beta^{k_j}\eta = y_0 . y_1 y_2 \dots y_J - .z_1 z_2 \dots z_J$$

with $y_0 \in \{0,1\}$. Hence

$$\begin{aligned} x &= .x_1 \dots x_{k_j-1}(x_{k_j}+y_0)y_1 \dots y_J - .0^{k_j} z_1 z_2 \dots z_J \\ &= y - z, \end{aligned}$$

where $y, z \in \mathrm{Fin}(\beta)$ and $z < \beta^{-k_j}$. Since k_j can be arbitrarily large, we get the desired representation.

ii) There exists arbitrarily large j such that $k_{j+1} - k_j > k_j - k_{j-1}$. Then we first claim that

$$x_{k_{j-1}+1} \dots x_{k_j-1}(x_{k_j}+1)0^{k_{j+1}-k_j-2}1$$

is admissible. By the definition of the free block decomposition, $x_{k_{j-1}+1} \dots x_{k_j-1}(x_{k_j}+1)$ is admissible. Hence the only possibility that the claim is false would be

$$x_{k_{j-1}+1} \dots x_{k_j-1}(x_{k_j}+1)0^{k_{j+1}-k_j-1} = a_1^* \dots a_{k_{j+1}-k_{j-1}-1}^*,$$

hence $a_{k_j-k_{j-1}+1}^* = \cdots = a_{k_{j+1}-k_{j-1}-1}^* = 0$, in particular $a_{k_{j+1}-k_j}^* = 0$, but this contradicts $a_{k_{j+1}-k_j}^* > x_{k_{j+1}}$ and the claim is proved. So set

$$\begin{aligned} \eta &= .x_1 \dots x_{k_j-1}(x_{k_j}+1) - x \\ &< x_1 \dots x_{k_j-1}(x_{k_j}+1) - .x_1 \dots x_{k_j} a_1^* \dots a_{k_{j+1}-k_j-1}^* \\ &< \beta^{-k_j} - a_1^*\beta^{-k_j-1} - \cdots - a_{k_{j+1}-k_j-1}^*\beta^{-k_{j+1}+1} < \beta^{-k_{j+1}+1}, \end{aligned}$$

By assumption (W'), we get

$$\beta^{k_{j+1}-1}\eta = y_0.y_1 y_2 \dots y_J - .z_1 z_2 \dots z_J$$

and thus

$$\begin{aligned} x &= .x_1 \dots x_{k_j-1}(x_{k_j}+1)0^{k_{j+1}-k_j-1} z_1 z_2 \dots z_J - .0^{k_{j+1}-2} y_0 y_1 \dots y_J \\ &= y - z, \end{aligned}$$

where $y, z \in \mathrm{Fin}(\beta)$ and $z < \beta^{-k_{j+1}+2}$.

It remains to deal with the case $k_{j+1} - k_j = k_j - k_{j-1} = \ell$ for all sufficiently large j where the assumption of i) fails, i.e. x is eventually periodic with period $a_1^* \dots a_{\ell-1}^*(a_\ell^*-1)$.

iii) Let $x = .\overline{a_1^* \dots a_{\ell-1}^*(a_\ell^*-1)}$.

Let $\kappa \geq 0$ be the integer such that $d_\beta^*(1) = .a_1^* \dots a_\ell^* 0^\kappa a_{\ell+\kappa+1}^* \dots$ and $a_{\ell+\kappa+1}^* > 0$. Let, for arbitrary $j > 0$,

$$\begin{aligned} \eta &= .(a_1^* \dots a_{\ell-1}^*(a_\ell^*-1))^{j-1} a_1^* \dots a_\ell^* - x \\ &= \beta^{-j\ell} - \frac{(a_1^*\beta^{-1} + \cdots + a_{\ell-1}^*\beta^{-\ell+1} + (a_\ell^*-1)\beta^{-\ell})\beta^{-j\ell}}{1-\beta^{-\ell}} \\ &= \frac{a_{\ell+\kappa+1}^*\beta^{-\ell-\kappa-1} + a_{\ell+\kappa+2}^*\beta^{-\ell-\kappa-2} + \cdots}{1-\beta^{-\ell}}\beta^{-j\ell} < \frac{\beta^{-j\ell-\ell-\kappa}}{1-\beta^{-\ell}}. \end{aligned} \tag{1}$$

From $a_\ell^* > 0$ we infer that $d_\beta^*(1)$ is lexicographically larger than $10^{\ell-2}1$. Hence $1 > \beta^{-1} + \beta^{-\ell}$ and $(1-\beta^{-\ell})^{-1} < \beta$. This together with (1) implies $\eta < \beta^{-j\ell-\ell-\kappa+1}$.

If $\ell > 1$, then $\eta < \beta^{-j\ell-\kappa-1}$, hence

$$\eta = .0^{j\ell+\kappa} y_0 y_1 \dots y_J - .0^{j\ell+\kappa+1} z_1 \dots z_J$$

and

$$x = .(a_1^* \dots a_{\ell-1}^*(a_\ell^*-1))^{k-1} a_1^* \dots a_\ell^* 0^{\kappa+1} z_1 \dots z_J - .0^{j\ell+\kappa} y_0 y_1 \dots y_J$$

gives the desired representation.

The case $\ell = 1$, i.e. $x = .\overline{(a_1^*-1)}$ (with $a_1^* \geq 2$) is more complicated. The formula (1) becomes

$$\eta = \frac{a_{\kappa+2}^*\beta^{-1} + a_{\kappa+3}^*\beta^{-2} + \cdots}{1-\beta^{-1}} \beta^{-j-\kappa-1} \tag{2}$$

and thus $\eta < \beta^{-j-\kappa}$. If $\eta < \beta^{-j-\kappa-1}$, then the argument for $\ell > 1$ still works here. So we may assume $\beta^{-j-\kappa-1} < \eta < \beta^{-j-\kappa}$. From (W'), we have

$$\eta = .0^{j+\kappa-1} y_0 y_1 \dots y_J - .0^{j+\kappa} z_1 \dots z_J.$$

Hence

$$x = .(a_1^*-1)^{j-1} a_1^* 0^\kappa z_1 \dots z_J - .0^{j+\kappa-1} y_0 y_1 \dots y_J.$$

If $.z_1 z_2 \dots < .a_{\kappa+2}^* a_{\kappa+3}^* \dots$, then we already have the desired representation. So we assume

$$.z_1 z_2 \dots > .a_{\kappa+2}^* a_{\kappa+3}^* \dots. \tag{3}$$

We are going to show

$$\eta < (1+\beta^{-1})\beta^{-j-\kappa-1}. \tag{4}$$

If this holds, then we have $\eta - \beta^{-j-\kappa-1} \in (0, \beta^{-j-\kappa-2})$. Hence

$$\eta - \beta^{-j-\kappa-1} = .0^{j+\kappa+1} y_0' y_1' \dots y_{J'}' - .0^{j+\kappa+2} z_1' \dots z_{J'}',$$

and

$$x = .(a_1^*-1)^{j-1} a_1^* 0^{\kappa+2} z_1' \dots z_{J'}' - .0^{j+\kappa} 1 y_0' y_1' \dots y_{J'}'$$

is a desired representation.

We may assume $y_1 = 0$ because of $z_1 > 0$. (Otherwise, decrease both y_1 and z_1.) Hence we have

$$y_0.y_1 \dots < 1.0 a_1^* a_2^* \dots = 1 + \beta^{-1}.$$

This together with (3) implies

$$\eta < (1 + \beta^{-1} - a_{\kappa+2}^*\beta^{-1} - a_{\kappa+3}^*\beta^{-2} - \dots)\beta^{-j-\kappa}.$$

Substituting η by its expression in (2), we get

$$\frac{a_{\kappa+2}^*\beta^{-2} + a_{\kappa+3}^*\beta^{-3} + \dots}{1-\beta^{-1}} < 1 + \beta^{-1} - a_{\kappa+2}^*\beta^{-1} - a_{\kappa+3}^*\beta^{-2} + \dots,$$

thus

$$a_{\kappa+2}^*\beta^{-1} + a_{\kappa+3}^*\beta^{-2} + \cdots < (1+\beta^{-1})(1-\beta^{-1}).$$

Using (2) once again, we get (4).

iv) Finally we consider $x = .x_1 \dots x_m \overline{a_1^* \dots a_{\ell-1}^*(a_\ell^* - 1)}$. By iii), there exist $.y_1 \dots y_J$ and $.z_1 \dots z_J \in \mathrm{Fin}(\beta)$ such that

$$.\overline{a_1^* \dots a_{\ell-1}^*(a_\ell^* - 1)} = .y_1 \dots y_J - .z_1 \dots z_J.$$

Hence for any j,

$$x = .x_1 \dots x_m(a_1^* \dots a_{\ell-1}^*(a_\ell^* - 1))^j y_1 \dots y_J - .0^{m+j\ell} z_1 \dots z_J$$

is a desired representation. This completes the proof of the theorem. □

3. Purely periodic orbits

In this section, we determine the set $\mathcal{P}$, the purely periodic expansions in $\mathbb{Z}[\beta]$ for cubic Pisot units. Geometrically this set $\mathcal{P}$ corresponds to dual tiles sharing the origin (cf. [3]).

We first review briefly the idea of [15] to interpret T_β as a shift on a symbolic space. Let $\beta > 1$ be an algebraic integer. Let $1 = 0.b_1 b_2 \dots b_d$ be an arbitrary expression of 1 in base β, where b_i are integers. (We do not consider the admissibility and also allow b_i to be negative.) Let

$$r_i = 0.b_{i+1} \dots b_d, \qquad 0 \le i \le d-1.$$

It is easy to check that $\{r_0, r_1, \dots, r_{d-1}\}$ spans $\mathbb{Z}[\beta]$. Hence for any $x \in \mathbb{Z}[\beta] \cap [0, 1)$, there are integers $z_1, z_2, \dots, z_d$ such that

$$x = z_1 r_{d-1} + z_2 r_{d-2} + \dots + z_d r_0$$

and a sequence $(z_{d+1}, z_{d+2}, \dots)$, such that for each $i \ge 1$,

$$0 \le z_i r_{d-1} + z_{i+1} r_{d-2} + \dots + z_{i+d-1} r_0 < 1. \tag{5}$$

Then the sequence in the above formula is uniquely determined by initial values $z_1, z_2, \dots, z_{d-1}$ and we call it a *carry sequence* of x. Let $x_i = b_1 z_{d+i-1} + \dots + b_d z_i$, then it is easy to check that the β-expansion of x is $0.x_1 x_2 \dots$. Hence

Lemma 2. ([15]) *Let $x \in \mathbb{Z}[\beta] \cap [0, 1)$. Then $x \in \mathcal{P}$ if and only if a carry sequence of x is purely periodic.*

Let $\beta > 1$ be the dominant root of the polynomial

$$f(x) = x^3 - ax^2 - bx - c.$$

Then β is a Pisot number if and only if

$$|b - c| < a + c \text{ and } c^2 - b < \mathrm{sgn}(c)(1 + ac)$$

holds. When $c = 1$, β is a Pisot number if and only if $-a+1 \le b \le a+1$. When $c = -1$, β is a Pisot number if and only if $-a + 3 \le b \le a - 1$. (cf. [2])

Proposition 1. *Let β be a cubic Pisot unit. Then the set $\mathcal{P}$ is given by the following table.*

$c=1$	$a=1$	$b=0,1,2$	0
	$a\geq 2$	$-1\leq b\leq a+1$	0
	$a\geq 3$	$-a+1\leq b\leq -2$	$\overline{(va+vb)}, v\geq 0$, *admissible*
$c=-1$	$a\geq 1$	$b=a-1$	$0, \overline{(a-1)(b-1)}, \overline{(b-1)(a-1)}, \overline{a(b-2)}, \overline{(b-2)a}$
	$a\geq 4$	$2\leq b\leq a-2$	$0, \overline{(a-1)(b-1)}, \overline{(b-1)(a-1)}$
	$a\geq 2$	$b=1$	$0, \overline{(a-1)0}, \overline{0(a-1)}, \overline{(a-1)}$
	$a\geq 3$	$-a+3\leq b\leq 0$	$\overline{(av+bv-2v)}, v\geq 0$, *admissible*

Proof. For $a\leq 6$, this is checked by a theoretic bound on $\mathcal{P}$ (cf. [3], Lemma 2). In the following, we assume $a\geq 6$. Let $r_0=1$, $r_1=\frac{b}{\beta}+\frac{c}{\beta^2}$ and $r_2=\frac{c}{\beta}$. Then $\{r_0,r_1,r_2\}$ is a basis of $\mathbb{Z}[\beta]$. Suppose $(z_i)_{i\geq 1}$ is a purely periodic carry sequence other than 0. Then

$$0\leq z_ir_2+z_{i+1}r_1+z_{i+2}<1. \tag{6}$$

We denote

$$z_{\min}=\min\{0,\min_{z_i\leq 0} z_i\},\ z_{\max}=\max\{0,\max_{z_i\geq 0} z_i\}.$$

i) c=1, b=a+1.

This case has been treated by [2]. Here we give a simpler proof. Let θ and θ' (assume $\theta>\theta'$) be the roots of

$$r_2X^2+r_1X+1=0.$$

Then $\theta'<\theta<-1$. Let $y_i=z_i-\theta z_{i+1}$ for $i\geq 0$, then we have

$$z_ir_2+z_{i+1}r_1+z_{i+2}=y_ir_2+y_{i+1}(r_1+\theta r_2).$$

Since $r_1+\theta r_2>0$ and $r_2>0$, by (6) we have

$$\begin{aligned} & y_{\min}r_2+y_{\max}(r_1+\theta r_2)<1,\\ 0< \ & y_{\max}r_2+y_{\min}(r_1+\theta r_2). \end{aligned}$$

These two formulas imply that

$$-\frac{r_2}{(r_1+\theta r_2)^2-r_2^2}<y_{\min}\leq y_{\max}<\frac{r_1+\theta r_2}{(r_1+\theta r_2)^2-r_2^2}.$$

When $a\geq 6$, we have $r_1>1$, $r_2<\frac{1}{6}$ and $-\frac{4}{3}<\theta<-1$. Hence from the above formula we have $-\frac{1}{3}<y_{\min}<y_{\max}<\frac{4}{3}$. So for any $i>1$, $-\frac{1}{3}<z_{i-1}-\theta z_i<\frac{4}{3}$ holds.

First we claim that $z_i\leq 0$ implies $z_{i-1}\geq |z_i|$. Since $-\frac{1}{3}<z_{i-1}-\theta z_i$ and $\theta<-1$, we have $z_{i-1}>\theta z_i-\frac{1}{3}\geq |z_i|-\frac{1}{3}$. Hence the claim is true because z_i are integers.

Second, we assert that $z_{i-1}\leq -z_i$ when $z_i>0$. From $z_i>0$ we infer $z_{i-1}\leq 0$, since we had $z_{i-1}-\theta z_i>1-\theta>\frac{4}{3}$ otherwise. Moreover, this implies $z_{i-2} \geq -z_{i-1}(\geq 0)$ by the above claim. Suppose our assertion is false, i.e. $-z_i+1 \leq z_{i-1} \leq 0$. This together with $z_{i-2} \geq -z_{i-1}$ implies

$$\begin{array}{rl} & z_{i-2}r_2 + z_{i-1}r_1 + z_i \\ \geq & -z_{i-1}r_2 + z_{i-1}r_1 + (1 - z_{i-1}) \\ \geq & -z_{i-1}(r_2 - r_1 + 1) + 1 \geq 1. \end{array}$$

This contradicts (6) and establishes our assertion. Hence in any case we have $|z_{i-1}| \geq |z_i|$. Since (z_i) is purely periodic, so $(z_i) = \overline{z(-z)}$ for some constant $z \geq 0$. Now by the left side of (6) we have $0 \leq z(-r_2+r_1-1)$, which implies $z = 0$. Therefore the only element of $\mathcal{P}$ is 0.

ii) $c = 1, 1 \leq b \leq a$.
This case follows from Theorem A.

iii) $c = 1, b = 0$.
This case has follows from Theorem B.

iv) $c = 1, -a+2 \leq b \leq -1$.
In this case $r_1 < 0$ and $r_2 > 0$. It is easy to check that

$$|r_1| + |r_2| < 1. \tag{7}$$

We assert that

$$|z_{\min}| \leq |z_{\max}| - 1 \tag{8}$$

holds. Setting $z_i = z_{\min}$ in $r_2 z_{i-2} + r_1 z_{i-1} + z_i < 1$, we get

$$0 \leq r_2 z_{i-2} + r_1 z_{i-1} + z_{\min} \leq z_{\max} r_2 + z_{\min} r_1 + z_{\min}.$$

Hence $(z_{\max} - z_{\min})r_2 \geq z_{\min}(r_2 - r_1 - 1)$. So no matter $z_{\min} = 0$ or not, we have (8).

We claim that $z_i = z_{\max}$ implies $z_{i-1} = z_{\max}$. Otherwise, setting $z_i = z_{\max}$ in $r_2 z_{i-2} + r_1 z_{i-1} + z_i < 1$, we get

$$r_2 z_{\min} + r_1(z_{\max} - 1) + z_{\max} \leq r_2 z_{i-2} + r_1 z_{i-1} + z_{\max} < 1.$$

This together with (8) implies $(z_{\max} - 1)(-r_2 + r_1 + 1) < 0$. Hence $z_{\max} = 0$ and $z_i = 0$ for any i. This is a contradiction and our claim is proved. Hence $(z_i = z)$ is a constant word, and the β-expansion of x is $0.x_1x_2\ldots$ with $x_i = z(a+b)$ for any $i \geq 1$. The proposition is proved in this case.

v) $c = 1, b = -a+1$.
Let $\theta > \theta'$ be the roots of $r_2X^2 + r_1X + 1 = 0$. Then $\theta > 1 > \theta' > 0$. Using the same argument as in i), we have

$$y_i r_2 + y_{i+1}(r_1 + \theta r_2) < 1$$

where $y_i = z_i - \theta z_{i+1}$. When $a \geq 6$ we have $r_1 + \theta r_2 < 0$. So setting $y_{i+1} = y_{\min}$ we get

$$y_{\min} r_2 + y_{\min}(r_1 + \theta r_2) < 1,$$

which implies $z_i - \theta z_{i+1} > y_{\min} > \frac{1}{r_1 + (\theta+1)r_2} > -\frac{3}{2}$ when $a \geq 6$.

If $z_i < -1$, then clearly $z_{i+1} < 0$ by the above inequality. If $z_i = -1$, then $z_{i+1} \le 0$, but $z_{i+1} = 0$ will lead to $z_k = 1$ for $k \ge i+2$ by direct calculation and hence is impossible. So we conclude that $z_i < 0$ implies $z_{i+1} < 0$. From (6), it is easy to show that there is at least one z_i such that $z_i \ge 0$. Hence we conclude that $z_i \ge 0$ for any i.

Setting $z_i = z_{\max}$ in $r_2 z_{i-2} + r_1 z_{i-1} + z_i < 1$, it is clear that z_{i-1} must be $z_{\max}$ also. Hence z_i is a constant sequence and this case is settled.

vi) $c = -1, 1 \le b \le a-1$.

In this case $r_1 > 0$ and $r_2 < 0$. Since $|r_1| + |r_2| < 1$, we have $|z_{\min}| \le z_{\max} - 1$ by the same argument as in (iv). Setting $z_{i+2} = z_{\max}$ in (6), we get

$$\begin{aligned} 1 &> z_i r_2 + z_{i+1} r_1 + z_{\max} \\ &> z_{\max} r_2 + z_{\min} r_1 + z_{\max} \\ &\ge z_{\max} r_2 - (z_{\max} - 1) r_1 + z_{\max}. \end{aligned}$$

Hence

$$z_{\max} < 1 + \frac{-r_2}{1 - r_1 + r_2}. \tag{9}$$

vi-1) $b = a-1$. In this case $z_{\max} \le 2$ by (9) when $a \ge 6$. So $z_i \in \{-1, 0, 1, 2\}$. If $z_0 = -1$ and $z_1 = -1$, then by (6) we have $z_2 = 1$, $z_3 = 0$, $z_4 = 1$, $z_5 = 0$. Hence it has purely periodic tail $\overline{10}$, which means that a purely periodic carry sequence cannot start with $(-1)(-1)$. By checking all the possibilities of z_0 and z_1, one can show that the purely periodic carry sequences are $\overline{0}, \overline{10}, \overline{01}, \overline{2(-1)}, \overline{(-1)2}$. Hence $\mathcal{P}$ is determined.

vi-2) $2 \le b \le a-2$. In this case $z_{\max} = 1$ and hence $z_i \in \{0, 1\}$. Calculations show that the purely periodic carry sequences are $\overline{0}, \overline{01}, \overline{10}$.

vi-3) $b = 1$. In this case $z_{\max} \le 1$ and $z_i \in \{0, 1\}$. The purely periodic carry sequences are $\overline{0}, \overline{10}, \overline{10}, \overline{1}$.

vii) $c = -1, -a+3 \le b \le 0$.

In this case $r_1 < 0$, $r_2 < 0$ and $|r_1| + |r_2| < 1$.

If (z_i) is a purely periodic carry sequence with $z_{\max} = 1$, then it is easy to check that $(z_i) = \overline{1}$.

Assume that $z_{\max} \ge 2$. We claim that $z_i = z_{\max}$ implies $z_{i-1} = z_{\max}$. Otherwise, setting $z_i = z_{\max}$ in $r_2 z_{i-2} + r_1 z_{i-1} + z_i < 1$, we get

$$z_{\max} r_2 + (z_{\max} - 1) r_1 + z_{\max} < 1,$$

which implies $z_{\max} < \frac{1+r_1}{1+r_1+r_2} < 2$ when $a \ge 6$. This contradicts our assumption and the claim is proved. Hence (z_i) is a constant sequence. This settles vii). $\square$

4. Weak finiteness of cubic Pisot units

We wish to show Theorem 3 by using the result in the previous section. Let us give an example to illustrate the idea. Set $a = 3, b =$

$-2, c = 1$. Then by Proposition 1, we have $.\overline{1} = .111\ldots \in \mathcal{P}$. As $d_\beta(1) = .20111\ldots$, we have

$$.111\ldots = .111\ldots - .(-1)20111\ldots = .2(-1)1 = (2\beta^{-1} + \beta^{-3}) - \beta^{-2},$$

which shows (H). First we recall the table of the expansion of one. The following result is due to [2] and Bassino [7].

Lemma 3.

$c = 1$	
b	$d_\beta(1)$
$-a+1 \le b \le -2$	$(a-1)(a+b-1)\overline{(a+b)}$
$b = -1$	$(a-1)(a-1)\,0\,1$
$0 \le b \le a$	$a\,b\,1$
$b = a+1$	$(a+1)\,0\,0\,a\,1$
$c = -1$	
b	$d_\beta(1)$
$-a+3 \le b \le 0$	$(a-1)(a+b-1)\overline{(a+b-2)}$
$1 \le b \le a-1$	$a\,\overline{(b-1)(a-1)}$

Proof of Theorem 3. If $\mathcal{P} = \{0\}$ then the the stronger condition (F) holds and we have nothing to prove. We will prove that the property (H) holds for those β with $\mathcal{P} \neq \{0\}$.

i) $c = 1$ and $-a+1 \le b \le -2$.

Let v be an integer such that $\overline{(v(a+b))}$ belongs to $\mathcal{P}$. Then $v(a+b) < a-1$ as it has to be less than the expansion of one. First consider the case $v = 1$. Lemma 3 shows:

$$.\overline{(a+b)} = \beta^2 - (a-1)\beta - (a+b-1) = 1\,0\,0. - (a-1)(a+b-1).$$

Therefore

$$\begin{aligned} .\overline{(a+b)} &= .(a+b)(a+b)(a+b) + .0\,0\,0\,\overline{(a+b)} \\ &= .(a+b+1)(a+b)(a+b) - .0\,(a-1)(a+b-1) \\ &= .(a+b+1)(a+b-1)\,1 - .0\,(a-2), \end{aligned}$$

which gives a desired expression. We do induction on v. If $v(a+b)+1 < a-1$ then by adding the expansion of one, we see

$$.\overline{(v(a+b))} = .(v(a+b)+1)(v(a+b))((v-1)(a+b)+1)\,\overline{((v-1)(a+b))} - .0\,(a-1)$$

thus the problem is reduced to $\overline{((v-1)(a+b))}$. Similarly if $v(a+b)+1 = a-1$ and $a+b > 1$, then the same expression gives

$$.\overline{(v(a+b))} = .(v(a+b)+1)\,0\,((v-1)(a+b)+1)\,\overline{((v-1)(a+b))} - .0\,1$$

as desired. It remains to consider the case $v(a+b)+1 = a-1$ and $a+b = 1$ with $v \ge 2$. This implies $a \ge 4$. In this case, $d_\beta(1) =$

$.(a-1)\,0\,\overline{1}$. Adding two expansions of one after shifting, we have

$$\begin{aligned} .\overline{(a-2)} &= .\overline{(a-2)} + .1\,(-a+1)\,0\,\overline{(-1)} + .0\,1\,(-a+1)\,0\,\overline{(-1)} \\ &= .(a-1)\,0\,0\,(a-3)\,\overline{(a-4)} - .0\,0\,1. \end{aligned}$$

This reduces the problem to $\overline{(a-4)}$ which was discussed already. We finished this case.

ii) $c=-1, 1 \le b \le a-1$

In this case there are $\overline{(a-1)(b-1)}$ and $\overline{(b-1)(a-1)}$ in $\mathcal{P}$. Thus

$$\begin{aligned} \overline{(b-1)(a-1)} &= .(b-1)(a-1)\overline{(b-1)(a-1)} \\ &= b\beta^{-1} - \beta^{-2} = .b - .0\,1 \end{aligned}$$

and

$$\begin{aligned} \overline{(a-1)(b-1)} &= .(a-1)\,\overline{(b-1)(a-1)} \\ &= .(a-1)\,b - .0\,0\,1 \end{aligned}$$

hold and we have done. When $b=1$ and $b=a-1$ there are some other elements in $\mathcal{P}$, which will be treated in iii) and iv).

iii) $c=-1$ and $b=1$

In this case, we additionally have $\overline{(a-1)} \in \mathcal{P}$. Using ii), we have

$$\overline{0\,(a-1)} = .1 - .0\,1 \quad \text{and} \quad \overline{(a-1)\,0} = .(a-1)\,1 - .0\,0\,1.$$

So $\overline{(a-1)} = .a\,1 - .0\,1\,1 = .a - .0\,0\,1$

iv) $c=-1$ and $b=a-1$

Adding three expansions of one after shifting, we have the formal expression

$$\begin{aligned} 0 &= .(-1)\,a\,\overline{(a-2)(a-1)} - .0\,(-1)\,a\,\overline{(a-2)\,(a-1)} \\ &\quad +.0\,0\,(-1)\,a\,\overline{(a-2)\,(a-1)} \\ &= .(-1)\,(a+1)\,(-3)\,(a+1)\,\overline{(a-3)\,a}. \end{aligned}$$

Thus

$$\begin{aligned} .\overline{(a-3)\,a} &= .(a-3)\,a\,(a-3)\,a\,\overline{(a-3)\,a} \\ &= .(a-2)\,(-1)\,a\,(-1) = .(a-2)\,0\,a - .0\,1\,0\,1 \end{aligned}$$

and

$$.\overline{a\,(a-3)} = .a\,(a-2)\,0\,a - .0\,0\,1\,0\,1$$

give the required expressions.

v) $c=-1$ and $-a+3 \le b \le 0$

Set $\kappa = a+b-2$. Let v be an integer that $\overline{(v\kappa)}$ is admissible. Using the expansion of one, we see

$$.\overline{\kappa} = .\kappa\,\kappa\,\kappa\,\overline{\kappa} = .(a+b-1) - .0\,(1-b)\,1,$$

which shows the case $v = 1$. We proceed in the same manner as in i). As $\overline{(v\kappa)}$ is admissible, we have $v\kappa < a - 1$. By using

$$
\begin{aligned}
&.\overline{(v\kappa)} \\
&= .\overline{(v\kappa)} - .(-1)\,(a-1)\,(a+b-1)\,\overline{\kappa} \\
&= .(v\kappa+1)\,0\,((v-1)\kappa-1)\,\overline{((v-1)\kappa)} - .0\,(a-1-v\kappa),
\end{aligned}
$$

we can reduce the case to $\overline{((v-1)\kappa)}$ and have confirmed all cases. $\square$

5. Dominant condition

We shall proof Theorem 4 in this section. The essential idea is to use the sum of digits as in [14].

Let

$$\chi(x) = x^d - b_1 x^{d-1} - b_2 x^{d-2} - \cdots - b_d$$

and set $b_j = 0$ for all $j > d$. Let β be the dominant root of $\chi(x)$, and let $d_\beta(1) = a_1 a_2 \ldots$ when $\beta > 1$. We need the following lemma for the proof of Theorem 4.

Lemma 4. *Let $\beta > 1$ be the dominant root of $\chi(x)$ with $b_1 > \sum_{j=2}^d |b_j|$. If, for some $\ell > 0$, $a_j = b_1 + b_2 + \cdots + b_j - 1$ for all $j < \ell$, then $a_\ell \in \{b_1 + b_2 + \cdots + b_\ell - 1, b_1 + b_2 + \cdots + b_\ell\}$.*

Proof. i) For $\ell = 1$, we have to show $a_1 = \lfloor \beta \rfloor \in \{b_1 - 1, b_1\}$. This holds because we have $\chi(b_1 - 1) < 0$ and $\chi(b_1 + 1) > 0$, hence $\beta \in (b_1 - 1, b_1 + 1)$.

ii) For $1 < \ell \le d$, consider the following addition, where all lines are zero:

$$
\begin{array}{cccccccc}
(-1)\,b_1 & \ldots & b_{\ell-1}\,. & b_\ell & \ldots & b_d & & \\
\ddots\ \ddots & & \ddots\ . & \ddots & \ddots & \ddots & \ddots & \\
 & (-1) & b_1\ . & b_2 & \ldots & b_{d-\ell} & \ldots & b_d \\
 & & (-1). & b_1 & \ldots & b_{d-\ell-1} & \ldots & \ldots\, b_d \\
\hline
(-1)\,a_1 & \ldots & a_{\ell-1}\,. & (b_1 + \cdots + b_\ell) & \ldots & (b_{d-\ell-1} + \cdots + b_d) & & \ldots\, b_d
\end{array}
$$

Hence

$$a_\ell = \Big\lfloor (b_1 + \cdots + b_\ell).(b_2 + \cdots + b_{\ell+1}) \ldots (b_{d-\ell-1} + \cdots + b_d) \ldots b_d \Big\rfloor.$$

By the dominant condition $b_1 > \sum_{j=2}^d |b_j|$, we have

(10)

$$|b_2 + \cdots + b_{\ell+1}| \le b_1 - 1, \ldots, |b_{d-\ell-1} + \cdots + b_d| \le b_1 - 1, \ldots, |b_d| \le b_1 - 1.$$

If one of these inequalities is an equality, then $\sum_{j=2}^d |b_j| = b_1 - 1$ and all b_j must have the same sign. If all b_j are positive, then $\beta > b_1$, which contradicts the assumption $a_1 = b_1 - 1$. So all b_j are negative and $-\sum_{j=2}^d b_j = b_1 - 1$. Hence we can factorize $\chi(x)$ as

$$\chi(x) = (x-1)(x^{d-1} - (b_1-1)x^{d-1} - (b_1+b_2-1)x^{d-2} - \cdots - (b_1 + \cdots + b_{d-1} - 1)).$$

Clearly $b_1 - 1 \geq b_1 + b_2 - 1 \geq \cdots \geq b_1 + \cdots + b_{d-1} - 1 > 0$ and hence $d_\beta(1) = (b_1 - 1)(b_1 + b_2 - 1) \dots (b_1 + \cdots + b_{d-1} - 1)$. The lemma is proved in this case.

If all the inequalities in (10) are strict, then

$$\left| .(b_2 + \cdots + b_{\ell+1}) \dots b_d \right| \leq \frac{b_1 - 2}{\beta} + \cdots + \frac{b_1 - 2}{\beta^{d-1}} < \frac{b_1 - 2}{\beta - 1} < 1$$

and $a_\ell \in \{b_1 + \cdots + b_\ell - 1, b_1 + \cdots + b_\ell\}$.

iii) $\ell > d$ means that $a_j = b_1 + b_2 + \cdots + b_j - 1$ holds for all $1 \leq j \leq d$. Then it is easy to check that

$$d_\beta(1) = (b_1 - 1)(b_1 + b_2 - 1) \dots (b_1 + \cdots + b_{d-1} - 1)\overline{(b_1 + \cdots + b_d - 1)}$$

and again the lemma holds. □

Proof of Theorem 4. By Theorem 2, it suffices to show (W'). As we have shown in the proof of Lemma 4, in case $-1 + b_1 + \cdots + b_d = 0$ and $(b_1, b_2) \neq (2, -1)$, the conditions of Theorem A are satisfied and β has the property (F). We have to exclude $(b_1, b_2) = (2, -1)$ because this means $\chi(x) = x^2 - 2x + 1$ and $\beta = 1$. So in the following we assume $-1 + b_1 + \cdots + b_d > 0$.

Consider $x \in \mathbb{Z}[1/\beta] \cap [0, 1)$. This means that x has a representation $x = .x_1 x_2 \dots x_J$ where x_i may be negative. Set $x_k^+ = \max(x_k, 0)$, $x_k^- = \max(-x_k, 0)$. Then

$$x = .x_1^+ x_2^+ \dots x_J^+ - .x_1^- x_2^- \dots x_J^-.$$

We extend the notion of admissibility to integer sequences $y_1 y_2 \dots$ with possibly negative entries by calling $y_1 y_2 \dots$ admissible if and only if $y_1^+ y_2^+ \dots$ and $y_1^- y_2^- \dots$ are admissible.

If $x_1 x_2 \dots x_J$ is admissible, then we are done. If $x_1 x_2 \dots x_J$ is not admissible, we define an algorithm which changes it to a new representation $x = x'_0.x'_1 x'_2 \dots x'_{J'}$. The idea is to decrease $\sum_{j=0}^{\infty} |x_j|$ by adding or subtracting digitwisely $.0^{k-1}1(-b_1) \dots (-b_d)$ for some $k \geq 0$. Then, we show that after a finite number of iterations, we obtain an admissible (finite) representation of x.

Algorithm. Assume, w.l.o.g, that k is the smallest integer such that

$$x_{k+1}^+ x_{k+2}^+ \dots \geq a_1 a_2 \dots.$$

If $x_{k+1} \geq b_1$, which is always the case if $a_1 = b_1$, we digitwisely add $.0^{k-1}1(-b_1) \dots (-b_d)$ to $.x_1 x_2 \dots x_J$ and we obtain a new representation of x in the form

$$x'_0.x'_1 x'_2 \dots = 0.x_1 \dots x_{k-1}(x_k + 1)(x_{k+1} - b_1) \dots (x_{k+d} - b_d) x_{k+d+1} \dots.$$

(For $k=0$, read $x'_0.x'_1x'_2\ldots = 1.(x_1-b_1)\ldots(x_d-b_d)x_{d+1}\ldots$). By the dominant condition, we have

$$\sum_{j\geq 0}^{\infty}|x'_j| - \sum_{j\geq 0}^{\infty}|x_j| \leq 1-b_1+\sum_{j=2}^{d}|b_j| \leq 0.$$

The construction is finished. We remark that the left side of the above formula is strictly less than 0 in case $x_k<0$.

If $x_{k+1}=b_1-1$, then necessarily $a_1=b_1-1$ and $x^+_{k+2}x^+_{k+3}\ldots \geq a_2a_3\ldots$. Moreover, Lemma 4 gives $a_2\in\{b_1+b_2-1, b_1+b_2\}$.

If $x_{k+2}\geq b_1+b_2$, which is always the case if $a_2=b_1+b_2$, then we add $.0^{k-1}1(-b_1)\ldots(-b_d)$ and $.0^k1(-b_1)\ldots(-b_d)$ to $x_1x_2\ldots x_J$ and obtain

$$\begin{aligned} x'_0.x'_1x'_2\ldots = 0.x_1\ldots x_{k-1}(x_k+1)(x_{k+1}+1-b_1)(x_{k+2}-b_1-b_2)\\ (x_{k+3}-b_2-b_3)\ldots(x_{k+d}-b_{d-1}-b_d)(x_{k+d+1}-b_d)x_{k+d+2}\ldots\end{aligned}$$

and hence

$$\sum_{j=0}^{\infty}|x'_j| - \sum_{j=0}^{\infty}|x_j| \leq 1+1-b_1-b_1-b_2+\sum_{j=2}^{d}|b_j|+\sum_{j=3}^{d}|b_j| \leq 0.$$

In general, we look for the positive integer ℓ such that $x_{k+j}=b_1+\cdots+b_j-1$ for all $j<\ell$ and $x_{k+\ell}\geq b_1+\cdots+b_\ell$. We claim that such an ℓ always exists under the assumption $-1+b_1+\cdots+b_d>0$. Otherwise, $x^+_{k+1}x^+_{k+2}\cdots \geq a_1a_2\ldots$ and Lemma 4 imply recursively $x_{k+j}=a_j=b_1+\cdots+b_j-1>0$ for all $j\geq 1$, but this contradicts that $x_1x_2\ldots x_J$ is finite and the claim is proved.

Now, add $.0^{k-2+j}1(-b_1)\ldots(-b_d)$ to $x_1x_2\ldots x_J$ for $1\leq j\leq \ell$. We obtain $x=x'_0.x'_1x'_2\ldots$ of the form

$$\begin{aligned} 0.x_1\ldots x_{k-1}(x_k+1)(x_{k+1}+1-b_1)\ldots(x_{k+\ell-1}+1-b_1-\cdots-b_{k+\ell-1})\\ (x_{k+\ell}-b_1-\cdots-b_\ell)(x_{k+\ell+1}-b_2-\cdots-b_{\ell+1})\ldots\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=0}^{\infty}|x'_j| - \sum_{j=0}^{\infty}|x_j| \leq \\ &1+(1-b_1)+\cdots+(1-b_1-\cdots-b_{\ell-1})-(b_1+\cdots+b_\ell)+\sum_{i=2}^{\ell}\sum_{j=i}^{\ell}|b_j| \\ &\leq \ell-\ell b_1+\ell\sum_{j=2}^{d}|b_j|\leq 0.\end{aligned}$$

The construction is finished.

Hence we always obtain a new representation of x with $\sum_{j=0}^{\infty}|x'_j| \leq \sum_{j=0}^{\infty}|x_j|$. If $x^-_{k+1}x^-_{k+2}\ldots \geq a_1a_2\ldots$, a similar argument works by adding $.0^{k-1}(-1)b_1\ldots b_d$ to $.x_1x_2\ldots$. Furthermore, $\sum_{j=0}^{\infty}|x'_j| = \sum_{j=0}^{\infty}|x_j|$

is possible only for $x_k \geq 0$ if $x^+_{k+1}x^+_{k+2}\ldots \geq a_1a_2\ldots$ (and $x_k \leq 0$ if $x^-_{k+1}x^-_{k+2}\ldots \geq a_1a_2\ldots$).

Starting with $x^{(0)}_0.x^{(0)}_1x^{(0)}_2\ldots = 0.x_1x_2\ldots$, construct iteratively $x^{(i+1)}_0.x^{(i+1)}_1\ldots$ from $x^{(i)}_0.x^{(i)}_1\ldots$ as above by using the minimal $k \geq 0$ such that $x^{(i)+}_{k+1}x^{(i)+}_{k+2}\ldots \geq a_1a_2\ldots$ or $x^{(i)-}_{k+1}x^{(i)-}_{k+2}\ldots \geq a_1a_2\ldots$ and denote this k by k_i. We have

$$\sum_{j=0}^{\infty}|x^{(1)}_j| \geq \sum_{j=0}^{\infty}|x^{(2)}_j| \geq \sum_{j=0}^{\infty}|x^{(3)}_j| \geq \ldots$$

Our algorithm terminates once we get an admissible sequence $x^{(i)}_1x^{(i)}_2\ldots$. The admissibility implies $|.x^{(i)}_1x^{(i)}_2\ldots| < 1$ and $x^{(i)}_0 \in \{0,1\}$ because of $x \in [0,1)$. Hence $x^{(i)}_0x^{(i)}_1x^{(i)}_2\ldots$ is admissible if $\beta > 2$. The only possibility for $\beta < 2$ is $b_1\ldots b_d = 20^{d-2}(-1)$, but this is excluded by the assumption $-1+b_1+\cdots+b_d > 0$. Therefore $x^{(i)}_0x^{(i)}_1x^{(i)}_2\ldots$ is admissible and we have a representation $x = y - z$ with $y, z \in \mathrm{Fin}(\beta)$, $y < \beta$ and $z < 1$.

Suppose that the algorithm does not terminate in finitely many steps. Then $\sum_{j=0}^{\infty}|x_j|$ becomes a constant after some iterations. We take this sequence as the starting seqence and show that the $\{x^{(i)}_0.x^{(i)}_1x^{(i)}_2\ldots\}$ converges to an infinite sequence.

Let $L = \min_{i\geq 0} k_i$ and h be the smallest number such that $k_h = L$. Assume, w.l.o.g., $\left(x^{(h)}_{L+1}\right)^+\left(x^{(h)}_{L+2}\right)^+\ldots \geq a_1a_2\ldots$.

First we argue that $x^{(h)}_L \geq 0$, for otherwise $\sum_{j\geq 0}|x^{(h+1)}_j| < \sum_{j\geq 0}|x^{(h)}_j|$. Second, the next time we come back to L, we cannot come back with a different sign. For if h' is the next time of coming back with

$$\left(x^{(h')}_{L+1}\right)^-\left(x^{(h')}_{L+2}\right)^-\ldots$$

non-admissible, then $x^{(h')}_L > 0$ and thus $\sum_{j=0}^{\infty}|x^{(h+1)}_j| < \sum_{j=0}^{\infty}|x^{(h)}_j|$. Hence we can return to L at most $\lfloor\beta\rfloor$ times.

By repeating the above argument, we have proved that $x^{(i)}_0.x^{(i)}_1x^{(i)}_2\ldots$ converges to an infinite sequence, which contradicts that $\sum_{j=0}^{\infty}|x_j|$ is a constant. Therefore our algorithm terminates and gives us an admissible representation of x. □

6. Branching beta expansion

There is another way to access this weak finiteness problem. The result in this section gives a practical way to show the property (F) or (W) for a fixed β. Before introducing the branching algorithm, we begin with an easier case, the property (F). Denote by $T_+ = T_\beta$ and define $T_-(x) = T_\beta(x) - 1 = \beta x - \lfloor\beta x + 1\rfloor$. Of course $T_-(x) \in [-1, 0)$.

We say that an element $x \in \mathbb{Z}[\beta]$ is β-finite if there is a positive integer ℓ that $T_\beta^\ell(x) = 0$. This means that x can be expanded in the form:

$$x = \sum_{i=1}^{\ell} x_i \beta^{-i} = .x_1 x_2 \dots x_\ell,$$

with $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. Note that the first digit $x_1 = \lfloor \beta x \rfloor$ is an integer without restriction but the remaining expansion $.0\, x_2 x_3 \dots$ is the beta expansion of $x - x_1/\beta$.

Proposition 2. *Assume that there exists a subset E of $\mathbb{Z}[\beta]$ which satisfies*

- $0 \in E$
- $T_+(E) \cup T_-(E) \subset E$.
- *Each element of E is β-finite.*

Then β has the property (F).

Originally this type of method was introduced by Brunotte [11] and Scheicher-Thuswaldner [22] independently for canonical number systems. The next proof is an analogy to Lemma 4.1 in [4].

Proof. Assume that ξ is β-finite and $\eta \in E$. We wish to show that $\xi + \eta$ is β-finite. Note that

$$T_\beta(\xi + \eta) - T_\beta(\xi) \equiv T_\beta(\eta) \pmod{\mathbb{Z}}.$$

Thus if $T_\beta(\xi + \eta) - T_\beta(\xi) \in [0, 1)$ then we have

$$T_\beta(\xi + \eta) = T_\beta(\xi) + T_+(\eta)$$

and if $T_\beta(\xi + \eta) - T_\beta(\xi) \in [-1, 0)$ then

$$T_\beta(\xi + \eta) = T_\beta(\xi) + T_-(\eta).$$

Thus we have shown that there exists an $\eta' \in E$ such that

$$T_\beta(\xi + \eta) = T_\beta(\xi) + \eta'.$$

Repeating this argument, we see that there exists an ℓ such that

$$T_\beta^\ell(\xi + \eta) = T_\beta^\ell(\xi) + \eta'' = \eta''$$

with $\eta'' \in E$. Using the assumption of the proposition, we have shown that $\xi + \eta$ is β-finite.

Since $T_-(0) = -1$, it is easy to see that $T_-^i(0)$ $(i = 1, 2, \dots, d)$ forms a basis of $\mathbb{Z}[\beta]$ as $\mathbb{Z}$-module, where d is the degree of β. Hence E contains a basis of $\mathbb{Z}[\beta]$. Using the additivity, each element of $\mathbb{Z}[\beta]$ is β-finite. Let x be an element of $\mathbb{Z}[1/\beta]$ and take a positive integer N that $\beta^N x \in \mathbb{Z}[\beta]$. Then $T_\beta^N(x) \in \mathbb{Z}[\beta] \cap [0, 1)$. This shows that x is β-finite. Reviewing the definition of the beta expansion, this also proves that each element $x \in \mathbb{Z}[\beta]_{\geq 0}$ has finite beta expansion. $\square$

Assume that $\beta > 1$ is the dominant root of the polynomial $x^d - b_1 x^{d-1} - b_2 x^{d-2} - \dots - b_d$. Let $r_i = .b_{i+1} b_{i+2} \dots b_d = \sum_{k=1}^{d-i} b_{i+k} \beta^{-k}$

for $i = 0, 1, \dots d-1$. Assume that $\sum_{i=1}^{d-1} |r_i| < 1$. Then we show that the set

$$E = \left\{ \sum_{i=0}^{d-1} z_{d-i} r_i \;\middle|\; z_i \in \{-1, 0, 1\} \right\}$$

satisfies first the first two conditions of Proposition 2. The first condition is clear. Using the carry sequence explained in §3, we have

$$T_{\pm}\left(\sum_{i=0}^{d-1} z_{d-1-i} r_i \right) = \sum_{i=0}^{d-1} z_{d-i} r_i$$

and z_d has two choices to satisfy

$$-1 \le z_1 r_{d-1} + z_2 r_{d-2} + \cdots + z_d r_0 < 1. \tag{11}$$

Thus to show the 2nd condition of Lemma 2, it suffices to show that if $z_i \in \{-1, 0, 1\}$ for $i = 1, \dots d-1$ then $z_d \in \{-1, 0, 1\}$. But this is clear from the condition $\sum_{i=1}^{d-1} |r_i| < 1$. Now we give an alternative proof of the results in [15].

The following corollary can be found implicity in [15].

Corollary 1. *If $r_1, r_2, \dots r_{d-1} > 0$ and $\sum_{i=1}^{d-1} r_i < 1$ then β has the property (F).*

Proof. We only need to show that each element x of E has finite beta expansion. Recalling (5), z_{i+d-1} is determined from $z_i, z_{i+1}, \dots z_{i+d-2}$ by T_β. Suppose that x does not have finite beta expansion. Then we may assume $x \in \mathcal{P}$. Using periodicity, the associated carry sequence (z_i) cannot take the value -1 since $z_{i+d-1} = -1$ causes a contradiction in (5). Thus z_i must be 0 or 1. This implies $z_{i+d-1} = z_{i+d} = \cdots = 0$ and thus the carry sequence falls into the 0 cycle. This is absurd. □

From this, Theorem B is easily shown, since $r_1 + \cdots + r_{d-1}$ equals

$$(b_2 + b_3 + \cdots + b_d)\beta^{-1} + (b_3 + b_4 + \cdots + b_d)\beta^{-2} + \cdots + b_d \beta^{-d+1},$$

which is easily seen to be less than 1.

Now, let us introduce the 'branching' beta expansion. Assume that x can be transformed by one of two maps $T_{\pm}$:

$$x = \xi_1 \xrightarrow{x_1} \xi_2 \xrightarrow{x_2} \xi_3 \xrightarrow{x_3} \dots$$

where $x_i = \beta \xi_i - T_{m_i}(\xi_i)$ and $m_i \in \{+, -\}$. Then we can expand x by

$$x = \sum_{i=1}^{\infty} x_i \beta^{-i} = .x_1 x_2 \dots \tag{12}$$

Take an integer $q < \beta$. We say that x is *q-expansible* if $|T_{m_k}(\xi_k)| < q/\beta$ for all k, and x is *q-finite* if additionally $T_{m_\ell}(\xi_\ell) = 0$ for some ℓ. If x is q-expansible, then x is expanded in a form (12) with $|x_i| \le q$ for $i \ge 2$. Note that x_1 may be large. The largest digit $|x_i| = q$ ($i \ge 2$) appears only when we change the 'branching direction', i.e., the signs m_{i-1} and m_i are different.

Conversely, if we have an expression (12) of x with $|x_i| \le q-1$ for $i \ge 2$ then one will see that x is q-expansible. In fact, taking m_i appropriately, we have

$$T_{m_k}(\xi_k) = \sum_{i=k+1}^{\infty} x_i \beta^{k-i}$$

with

$$\Big| T_{m_k}(\xi_k) \Big| \le \frac{q-1}{\beta-1} < \frac{q}{\beta} \ ,$$

where we used the assumption $q < \beta$.

If $q > \beta/2$, then each $x \in \mathbb{R}$ is q-expansible. This fact is seen by the *central beta transformation*:

$$U_\beta(x) = \beta x - \left\lfloor \beta x + \frac{1}{2} \right\rfloor$$

which acts on $[-1/2, 1/2) \subset (-q/\beta, q/\beta)$. This gives digits $x_i \in (-(\beta+1)/2, (\beta+1)/2) \cap \mathbb{Z}$ for $i \ge 2$. This is a deterministic algorithm and U_β coincides with T_+ or T_- depending on the applied value. In general, the above branching expansion is indeterministic and we have one or two choices of digits.

Proposition 3. *Assume that there exists a subset E of $\mathbb{Z}[\beta]$ which satisfies*

- $0 \in E$
- $T_+(E) \cup T_-(E) \subset E$
- *There exists $\beta/2 < q < \beta$ that each $x \in E$ is q-finite.*

Then each element of $\mathbb{Z}[1/\beta]$ is q-finite. If $q < \lfloor \beta \rfloor$ then β satisfies the property (W). The last inequality can be replaced by $q \le \lfloor \beta \rfloor$ when $a_2 = \lfloor \beta T_\beta(1) \rfloor > 0$.

Proof. Assume that ξ is q-finite and $\eta \in E$. We aim for showing that $\xi + \eta$ is q-finite. By the assumption,

$$\xi = \xi_1 \xrightarrow{x_1} \xi_2 \xrightarrow{x_2} \xi_3 \xrightarrow{x_3} \cdots \xrightarrow{x_{\ell-1}} \xi_\ell \xrightarrow{x_\ell} 0$$

where $x_i = \beta \xi_i - T_{m_i}(\xi_i)$ and $m_i \in \{+,-\}$ and $|T_{m_k}(\xi_k)| < q/\beta$ for all k. We claim that there exists an $\eta' \in E$ and $k_1 \in \{+,-\}$ such that

$$T_{k_1}(\xi+\eta) = T_{m_1}(\xi) + \eta' \text{ with } \Big| T_{k_1}(\xi+\eta) \Big| < \frac{q}{\beta} \ .$$

Note that

$$T_{m_1}(\xi+\eta) - T_{m_1}(\xi) = T_+(\xi+\eta) - T_+(\xi) \equiv T_+(\eta) \pmod{\mathbb{Z}}.$$

Thus if $T_+(\xi+\eta) - T_+(\xi) \in [0,1)$ then we have

$$T_{m_1}(\xi+\eta) = T_{m_1}(\xi) + T_+(\eta)$$

and if $T_+(\xi+\eta) - T_+(\xi) \in [-1, 0)$ then

$$T_{m_1}(\xi+\eta) = T_{m_1}(\xi) + T_-(\eta).$$

Thus if $|T_{m_1}(\xi+\eta)| < q/\beta$, then we take $k_1 = m_1$ and $\eta' = T_\pm(\eta) \in E$. Assume that $T_{m_1}(\xi+\eta) \in [-1, -q/\beta]$. Then we see $m_1 =$ '$-$'. As $|T_{m_1}(\xi)| < q/\beta$, the value $T_\pm(\eta)$ must be negative. So we have

$$T_-(\xi+\eta) = T_-(\xi) + T_-(\eta)$$

and thus

$$T_+(\xi+\eta) = T_-(\xi) + T_+(\eta)$$

and $T_+(\xi+\eta) \in [0, 1-q/\beta] \subset (-q/\beta, q/\beta)$. This shows that we can take $k_1 =$ '$+$' and $\eta' = T_+(\eta)$. The case $T_{m_1}(\xi+\eta) \in [q/\beta, 1)$ is done the same way and we have shown the claim.

Repeating this argument, we see that there exist k_i $(i = 1, \ldots, \ell)$ such that

$$T_{k_\ell} T_{k_{\ell-1}} \ldots T_{k_2} T_{k_1}(\xi+\eta) = T_{m_\ell} T_{m_{\ell-1}} \ldots T_{m_2} T_{m_1}(\xi) + \eta'' = \eta''$$

with $\eta'' \in E$ and $|T_{k_i} T_{k_{i-1}} \ldots T_{k_1}(\xi+\eta)| < q/\beta$ for each i. Using the assumption of the proposition, we have shown that $\xi+\eta$ is q-finite.

As in the proof of the previous proposition, E contains a basis of $\mathbb{Z}[\beta]$ and therefore each element of $\mathbb{Z}[\beta]$ is q-finite. Let x be an element of $\mathbb{Z}[1/\beta]$ and take a positive integer N such that $\beta^N x \in \mathbb{Z}[\beta]$. Iterating the central beta expansion we have $U_\beta^N(x) \in \mathbb{Z}[\beta] \cap [-1/2, 1/2)$ which is q-finite. This shows that x is q-finite.

Suppose $q < \lfloor \beta \rfloor$. As each element x of $\mathbb{Z}[1/\beta] \cap [-1/2, 1/2)$ is q-finite, we have $x = x_1 \ldots x_\ell = x_1^+ \ldots x_\ell^+ - x_1^- \ldots x_\ell^-$ with $|x_i| \le q \le \lfloor \beta \rfloor - 1$. Note that $x_1^+ \ldots x_\ell^+$ and $x_1^- \ldots x_\ell^-$ are admissible since we do not use the digit $\lfloor \beta \rfloor$. Take an element $x \in \mathbb{Z}[1/\beta] \cap [0, 1)$. Then x or $1-x$ is contained in $[-1/2, 1/2)$. In the latter case, the relation $1 - x = .x_1^+ \ldots x_\ell^+ - .x_1^- \ldots x_\ell^-$, shows $x = 1.x_1^- \ldots x_\ell^- - .x_1^+ \ldots x_\ell^+$. This shows that β has the property (W) by Proposition 2. If $a_2 > 0$, then one can take $q = \lfloor \beta \rfloor$ in the above proof since $x_i^m = \lfloor \beta \rfloor$ implies $x_{i+1}^m = 0$ where $m \in \{+, -\}$. Indeed the crucial digit $\lfloor \beta \rfloor$ appears only when we change the sign m_i in the branching beta expansion (12). $\square$

Propositions 2 and 3 give us an efficient way to show the properties (F) or (W) for a fixed β. Namely

- **(a):** Let $E_1 = \{0\}$.
- **(b):** Define inductively $E_n = E_{n-1} \cup T_+(E_{n-1}) \cup T_-(E_{n-1})$.
- **(c):** If $E_n = E_{n-1}$ then go to (d) otherwise go to (b).
- **(d):** For each element x of E_n, confirm that x is β-finite. If it is true, then β has the property (F).
- **(e):** If there exists $x \in E_n$ which is not β-finite, i.e., x gives an eventually periodic expansion, then we start over again and try to show that all elements of E_n are q-finite.

The emerging process (b) will terminate in a finite number of steps when β is a Pisot number. This is easily proved since $E_n \subset \mathbb{Z}[\beta]$ and by each Galois conjugate map, the image of E_n is bounded. Thus this gives an efficient algorithm to confirm that the property (F) holds or not. The process (e) is executed in the following way. Let $\mathcal{G}$ be a directed graph of vertices E_n and draw edges $a \to b$ between $a, b \in E_n$ when $T_\pm(a) = b$ and $|b| < q/\beta$. If we can walk along this $\mathcal{G}$ from each vertex to 0, then all elements of E_n are q-finite. However at this moment, this is not an established algorithm for (W). It is not known that each x in E_n must be q-finite even if $\beta > 2$ satisfies (W). By using Proposition 3, we can also give a sufficient condition of (W).

Corollary 2. *Let*

$$q = \begin{cases} \lfloor \beta \rfloor & \text{if } a_2 > 0 \\ \lfloor \beta \rfloor - 1 & \text{if } a_2 = 0 \end{cases}.$$

If $\sum_{i=1}^{d-1} |r_i| < q/\beta$ *then* β *has the property (W).*

Proof. As $\sum_{i=1}^{d-1} |r_i| < q/\beta < 1$, the set $E = \left\{ \sum_{i=0}^{d-1} z_{d-i} r_i \mid z_i \in \{-1, 0, 1\} \right\}$ satisfies the first two conditions of Proposition 3. It remains to show that each element of E is q-finite. As $|\sum_{i=1}^{d-1} z_{d-i} r_i| \le \sum_{i=1}^{d-1} |r_i| < q/\beta$, one can take $z_d = 0$ in the branching beta expansion keeping the q-expansible property. Continuing the same argument, we are able to take $z_d = z_{d+1} = \cdots = 0$ and thus each element of E is q-finite. □

This assertion is close to Theorem 4. In fact, $\sum_{i=1}^{d-1} |r_i| < (b_1 - 1)/\beta$ implies the dominant condition $b_1 > \sum_{i=2}^{d} |b_i|$ and conversely $b_1 > 1 + \sum_{i=2}^{d} |b_i|$ implies $\sum_{i=1}^{d-1} |r_i| < (b_1 - 1)/\beta$. However Corollary 2 occasionally exceeds Theorem 4. For example, the dominant root of $x^3 - 3x^2 - 2x + 1$ satisfies the inequality $\sum_{i=1}^{d-1} |r_i| < q/\beta$ with $q = 3$ but does not satisfy the dominant condition.

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(Shigeki Akiyama) Department of Mathematics, Faculty of Science, Niigata University, Ikarashi-2, 8050, Niigata 950-2181, Japan
E-mail address: `akiyama@math.sc.niigata-u.ac.jp`

(Hui Rao) Department of Mathematics, Faculty of Science, Wuhan University, 430072, Wuhan, P.R. China
E-mail address: `myhacone@yahoo.com`

(Wolfgang Steiner) Institut für Geometrie, TU Wien, Wiedner Hauptstrasse 8-10/113, 1040 Wien, Austria

E-mail address: steiner@geometrie.tuwien.ac.at