CR Yamabe problem and CR Paneitz operator

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1. Yamabe problem

(N^n, g) : closed conn. Riem. mfd $(n \geq 3)$

$[g] := \{ u^{\frac{4}{n-2}} g \mid u \in C^\infty_+ \}$: conformal class

$$E(\tilde{g}) := \frac{\int_N \mathrm{Scal}_{\tilde{g}} \, d\mathrm{vol}_{\tilde{g}}}{\left(\int_N d\mathrm{vol}_{\tilde{g}} \right)^{\frac{n-2}{n}}} \quad (\tilde{g} \in [g])$$

$$E(u) := E\left(u^{\frac{4}{n-2}} g\right)$$

$$= \frac{4\frac{n-1}{n-2}\int_N |du|^2 dvol_g + \int_N Scal_g \cdot u^2 dvol_g}{\left(\int_N u^{\frac{2n}{n-2}} dvol_g\right)^{\frac{n-2}{n}}}$$

Yamabe problem

$\exists$ minimizer of E ?

This problem has been solved affirmatively!

"Solution"

Take $u_i \in C^\infty_+$ s.t.

$$\|u_i\|_{L^{\frac{2n}{n-2}}} = 1, \quad E(u_i) \to \inf_u E(u) =: Y(N, [g])$$

We may suppose $u_i \longrightarrow {}^\exists u$ weakly in $W^{1,2}$

Then u is a "minimizer of E".

Difficulty

$W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$: NOT cpt $\rightsquigarrow$ $\|u\|_{L^{\frac{2n}{n-2}}}$ may be 0.

Strategy

(i) $Y(N,[g]) \leq Y(S^n,[g_{std}]) =: Y_0$

(ii) $Y(N,[g]) < Y_0 \Longrightarrow {}^\exists$minimizer of E

(iii) ${}^\exists$minimizer of E on (S^n, g_{std})

(iv) $(N,[g]) \overset{conf}{\not\cong} (S^n,[g_{std}]) \Longrightarrow Y(N,[g]) < Y_0$

positive mass theorem,
Green kernel, ...

2. CR manifolds

X : $(m+1)$-$\dim_{\mathbb{C}}$ cpx mfd

$M \subset X$: real hypersurface

$\rightsquigarrow \quad T^{1,0}M := T^{1,0}X|_M \cap (TM \otimes \mathbb{C})$: $\operatorname{rank}_{\mathbb{C}} = m$

- $T^{1,0}M \cap \overline{T^{1,0}M} = 0$

- $[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M)$

Def

- $(M^{2m+1}, T^{1,0}M)$: CR manifold

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot\ T^{1,0}M \subset TM \otimes \mathbb{C} : \mathrm{rank}_{\mathbb{C}}\ m \ \text{w/}\ T^{1,0}M \cap \overline{T^{1,0}M} = 0 \\ \cdot\ [\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M) \end{cases}$$

- $(M, T^{1,0}M)$: embeddable

$$\overset{\text{def}}{\Longleftrightarrow} \ {}^{\exists} f : M \to \mathbb{C}^N : \text{embedding w/ } f_* T^{1,0}M \subset T^{1,0}\mathbb{C}^N$$

Def

- $(M, T^{1,0}M)$: strictly pseudoconvex (spc)

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} {}^\exists \theta \in \Omega^1(M) \text{ s.t. } 0 \neq {}^\forall Z \in T^{1,0}M, \\ \qquad \theta(Z) = 0, \ -\sqrt{-1}\, d\theta(Z, \overline{Z}) > 0 \end{cases}$

Such a θ is called a contact form.

Another contact form $\widetilde{\theta}$ is of the form

$\widetilde{\theta} = u^{\frac{2}{m}} \theta \quad (u \in C^\infty_+)$.

3. CR Yamabe problem

$(M^{2m+1}, T^{1,0}M)$: closed conn spc CR mfd

$$\mathcal{E}(\hat{\theta}) := \frac{\int_M \mathrm{Scal}_{\hat{\theta}}\, \hat{\theta} \wedge (d\hat{\theta})^m}{\left(\int_M \underbrace{\hat{\theta} \wedge (d\hat{\theta})^m}_{\text{volume form}}\right)^{\frac{m}{m+1}}} \quad (\hat{\theta} : \text{contact form})$$

Tanaka-Webster scalar curvature

This is a CR analogue of E.

$$\mathcal{E}(u) := \mathcal{E}(u^{\frac{2}{m}}\theta)$$

$du|_{\ker\theta}$

$$= \frac{\frac{2(m+1)}{m}\int_M |d_b u|^2\, \theta\wedge(d\theta)^m + \int_M \mathrm{Scal}_\theta \cdot u^2\, \theta\wedge(d\theta)^m}{\left(\int |u|^{\frac{2(m+1)}{m}}\, \theta\wedge(d\theta)^m\right)^{\frac{m}{m+1}}}$$

CR Yamabe problem

$\exists$ minimizer of $\mathcal{E}$?

Rem: $S^{1,2} := \{u \in L^2 \mid d_b u \in L^2\} \hookrightarrow L^{\frac{2(m+1)}{m}}$: NOT cpt

Folland-Stein space

Known results

$$\mathcal{Y}(M, T^{1,0}M) := \inf_u \mathcal{E}(u)$$: CR Yamabe constant

(i) $\mathcal{Y}(M, T^{1,0}M) \leq \mathcal{Y}(S^{2m+1}, T^{1,0}S^{2m+1}) =: \mathcal{Y}_0$

(ii) $\mathcal{Y}(M, T^{1,0}M) < \mathcal{Y}_0 \Longrightarrow {}^{\exists}$ minimizer of $\mathcal{E}$

(iii) ${}^{\exists}$ minimizer of $\mathcal{E}$ on $(S^{2m+1}, T^{1,0}S^{2m+1})$

(Jerison-Lee 1987)

Higher dimensional case ($m \geq 2$)

- $m \geq 2$, $(M, T^{1,0}M)$: non-$\boxed{\text{spherical}}$
  loc. isom. to S^{2m+1}

$\Longrightarrow \mathcal{Y}(M, T^{1,0}M) < \mathcal{Y}_0$ (Jerison-Lee 1989)

- $m \geq 3$, $(M, T^{1,0}M) \overset{CR}{\not\cong} (S^{2m+1}, T^{1,0}S^{2m+1})$

$\Longrightarrow \mathcal{Y}(M, T^{1,0}M) < \mathcal{Y}_0$ (Cheng-Chiu-Yang 2014)

Three-dimensional case $(m=1)$

$^{\exists}(M^3, T^{1,0}M)$ s.t. $\nexists$ minimizer of $\mathcal{E}$.

small perturbation
of $(S^3, T^{1,0}S^3)$

(Cheng-Malchiodi-Yang 2019)

Problem

Find a "good" condition A s.t.

$$(M^3, T^{1,0}M) \overset{CR}{\ncong} (S^3, T^{1,0}S^3) + A \Rightarrow \mathcal{Y}(M, T^{1,0}M) < \mathcal{Y}_0$$

4. CR Paneitz operator and Main Theorem

$(M^3, T^{1,0}M)$: closed conn spc CR mfd

$\rightsquigarrow$ $P : C^\infty \longrightarrow \Omega^3$: CR Paneitz operator

Roughly speaking, P is defined by

$P = - d^c_{CR}\, d\, d^c_{CR}$, where d^c_{CR} : CR analogue of d^c

(c.f. On cpx mfds, $d = \partial + \bar\partial$, $d^c = \frac{\sqrt{-1}}{2}(\bar\partial - \partial)$)

Properties

- P : 4th order lin. diff. op.
- P is "formally self-adjoint".

$$\int_M v(Pu) = \int_M u(Pv) \quad (u, v \in C^\infty)$$

P : nonnegative $\overset{\text{def}}{\Longleftrightarrow}$ $\forall u \in C^\infty$, $\int_M u(Pu) \geq 0$

Thm (Cheng-Malchiodi-Yang 2017)

$(M^3, T^{1,0}M) \overset{CR}{\not\cong} (S^3, T^{1,0}S^3)$ + P: nonnegative

$\Longrightarrow \mathcal{Y}(M^3, T^{1,0}M) < \mathcal{Y}_0$ ($\Rightarrow {}^\exists$ minimizer of $\mathcal{E}$)

This follows from the CR positive mass theorem.

Main Theorem (T. 2020)

$(M^3, T^{1,0}M)$: embeddable $({}^{\exists} f : M \hookrightarrow \mathbb{C}^N$ w/ $f_* T^{1,0}M \subset T^{1,0}\mathbb{C}^N)$

$\Longrightarrow$ P is nonnegative

Cor

$(M^3, T^{1,0}M)$: embeddable $\Longrightarrow$ ${}^{\exists}$ minimizer of $\mathcal{E}$.

5. Proof of Main Theorem

Assume: $(M^3, T^{1,0}M)$: embeddable

$$\rightsquigarrow \begin{cases} {}^\exists X : \text{2-dim}_{\mathbb{C}} \text{ proj mfd} \\ {}^\exists \Omega \subset X : \text{domain} \end{cases} \quad \text{s.t.} \quad \partial\Omega \overset{CR}{\cong} M.$$

Lempert 1995

ω_+ : "good" Kähler form on Ω

$u \in C^\infty(\partial\Omega)$: fix

$\rightsquigarrow$ $\exists!\ \tilde{u} \in C^\infty(\Omega)$: harmonic ext of u

Epstein-Melrose-Mendoza 1991

$\Longleftrightarrow$ $dd^c \tilde{u} \wedge \omega_+ = 0$

Then we have $dd^c \tilde{u} \wedge dd^c \tilde{u} \leq 0$

(c.f. A : 2×2 Herm. matrix

$\mathrm{tr} A = 0 \Rightarrow \det A \leq 0$)

Stokes' theorem implies

$$0 \geq \int_\Omega dd^c \tilde{u} \wedge dd^c \tilde{u} = \int_\Omega d(d^c \tilde{u} \wedge dd^c \tilde{u})$$

$$= \int_{\partial\Omega} d^c \tilde{u} \wedge dd^c \tilde{u}$$

$$= - \int_{\partial\Omega} u \, (P \, u)$$

Therefore $\displaystyle\int_{\partial\Omega} u \, (P \, u) \geq 0$. $\blacksquare$

Appendix

$$(M^3, T^{1,0}M) : \text{CR mfd} \rightsquigarrow \begin{array}{c} (F_M^4, [g]) : \text{conf. mfd} \\ \pi \big\downarrow S^1 \\ M \end{array}$$

$\widetilde{P} = \Delta^2 + (\text{l.o.t})$: Paneitz operator on $(F_M^4, [g])$

CR Paneitz operator P is defined by

$$P =' \pi_* \widetilde{P} = \underbrace{\Delta_b}_{d_b^* d_b}{}^2 + (\text{l.o.t.})$$

Thm (Gamara 2001)

$^{\forall}(M^3, T^{1,0}M)$, $^{\exists}u$: critical point of $\mathcal{E}$

Cheng-Malchiodi-Yang's example means

$^{\exists}(M, T^{1,0}M)$ s.t $\mathcal{E}$ has a crtical point but no minimizer.