# Weak convergence of the partial sum of $I(d)$ process to a fractional Brownian motion in finite interval representation

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**ABSTRACT**

An integral transformation which changes a fractional Brownian motion to a process with independent increments has been given. A representation of a fractional Brownian motion through a standard Brownian motion on a finite interval has also been given. On the other hand, it is known that the partial sum of the discrete time fractionally integrated process ($I(d)$ process) weakly converges to a fractional Brownian motion in infinite interval representation. In this talk we derive the weak convergence of the partial sum of $I(d)$ process to a fractional Brownian motion in finite interval representation.

## 1 Introduction

Stochastic analysis for FBM has been developed by Decreusefond and Üstünel (1997) using Malliavin calculus. Norros et al. (1999) showed that many basic results can be obtained more directly with rather elementary arguments and computations. Norros et al. (1999) considered a normalized fractional Brownian motion (FBM) $(Z_t)_{g\geq 0}$ with self-similarity parameter $H \in (0,1)$ which is characterized by the following properties.

**(i)** $Z_t$ has stationary increments.

**(ii)** $Z_0 = 0$ and $E\{Z_t\} = 0$ for all $t$.

**(iii)** $E\left(Z_t^2\right) = |t|^{2H}$ for all $t$.

**(iv)** $Z_t$ is Gaussian.

**(v)** $Z_t$ has continuous sample paths.

Mandelbrot and Van Ness (1968) defiend the process more constructively as the integral

$$Z_t - Z_s = c_H \left( \int_s^t (t-u)^{H-1/2}\, dW_u + \int_{-\infty}^t \left\{ (t-u)^{H-1/2} - (s-u)^{H-1/2} \right\} dW_u \right),$$

where $W_t$ is the standard Brownian motion. The normalization $E\left(Z_1^2\right) = 1$ is achieved with the choice

$$c_H = \left( \frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma(2-2H)} \right)^{1/2},$$

where $\Gamma(\cdot)$ denotes the Gamma function.

## 1.1 The fundamental martingale $M$

Norros et al. (1999) considered the following process. Let $w(t,s)$ be the function

$$w(t,s) = \begin{cases} c_1 s^{1/2-H} (t-s)^{1/2-H}, & \text{for } s \in (0,t), \\ 0, & \text{for } s \notin (0,t), \end{cases}$$

where

$$c_1 = \left\{ 2HB\left(\frac{1}{2} - H, H + \frac{1}{2}\right)\right\}^{-1}$$

and $B$ is the beta funtion

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

Then, the centered Gaussian process

$$M_t = \int_0^t w(t,s)\, dZ_s$$

has independent increments and variance function

$$E\left(M_t^2\right) = c_2^2 t^{2-2H},$$

where

$$c_2 = \frac{c_H}{2H(2-2H)^{1/2}}.$$

In particular, $M$ is a martingale.

Note that the process

$$W_t = \frac{2H}{c_H} \int_0^t s^{H-1/2} dM_s$$

is a standard Brownian motion.

Fot sll $0 \le s \le t$, we have

$$\mathrm{Cov}(Z_s, M_t) = s.$$

As a consequence, the increment $M_t - M_s$ is independent of $\mathcal{F}_s$.

It is easier to proceed by considering, instead of $Z$, the process

$$Y_t = \int_0^t s^{\frac{1}{2}-H} dZ_s,$$

It is obvious that we have the inverse relationship $Z_t = \int_0^t s^{H-\frac{1}{2}} dY_s$; in paticular $Y$ generates the same filtration $(\mathcal{F}_t)$ as $Z$.

The process $Y$ has the integral representation

$$Y_T = 2H \int_0^T (T-t)^{H-1/2}\, dM_t$$

and we have the prediction formula

$$E\left[Y_T \mid \mathcal{F}_t\right] = 2H \int_0^t (T-s)^{H-1/2}\, dM_s.$$

As we noted, the process

$$W_t = \frac{2H}{c_H} \int_0^t s^{H-1/2} dM_s$$

is a standard Brownian motion. We also have the inverse relationship

$$M_t = \frac{c_H}{2H} \int_0^t s^{1/2-H} dW_s.$$

Now we have a sequence of simple representation formulae which allow us to proceed from process to process in the order $W \to M \to Y \to Z$.

The process $Z$ has the following integral representation in terms of $W$;

$$Z_t = \int_0^t z(t,s)\, dW_s,$$

where

$$z(t,s) = c_H \left[ \left(\frac{t}{s}\right)^{H-1/2} (t-s)^{H-1/2} - \left(H - \frac{1}{2}\right) s^{1/2-H} \int_s^t u^{H-3/2} (u-s)^{H-1/2}\, du \right].$$

For $H > \frac{1}{2}$ we have also the slightly simple expression

$$\begin{aligned} z(t,s) &= \left(H - \frac{1}{2}\right) c_H s^{1/2-H} \int_s^t u^{H-1/2} (u-s)^{H-3/2}\, du \\ &= c_H (t-s)^{H-1/2}\, {}_2F_1\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, 1 - \frac{t}{s}\right), \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Let

$$\kappa = H - \frac{1}{2},$$

so that the range $H \in (0,1)$ now corresponds to the range $\kappa \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Consider the interval $[0,a]$ and let $s \in [0,a]$. An integral over $[0,s]$ is called left-sided and one over $[s,a]$ is called right-sided. The right-sided fractional integral of order $\alpha > 0$ on a interval $[0,a]$ of a function $f \in L^1[0,a]$ is defined by

$$\left(I^{\alpha}_{a-} f\right)(s) = \frac{1}{\Gamma(a)} \int_0^a f(u) (u-s)_+^{\alpha-1}\, du = \frac{1}{\Gamma(a)} \int_s^a f(u)(u-s)^{\alpha-1}\, du, \quad s \in (0,a).$$

Pipiras and Taqqu (2001) showed that the following unversal result. Let $a > 0$ and $B^{\kappa}$ be a standard FBM with parameter $\kappa \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Then

$$\{B^{\kappa}(t)\}_{t \in [0,a]} \overset{d}{=} \left\{ \sigma_1(\kappa) \int_0^a s^{-\kappa} \left(I^{\kappa}_{a-} u^{\kappa} 1_{[0,t)}(u)\right)(s)\, dB^0(s) \right\}_{t \in [0,a]},$$

where

$$\sigma_1(\kappa)^2 = \frac{\Gamma(\kappa)^2 \kappa (2\kappa+1)}{B(\kappa, 1-2\kappa)} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)}.$$

## 1.2 $I(d)$ model $(0 < d < 1/2)$

First, we define $I(d)$ process $\{z_t\}$ with $0 < d < 1/2$, which is stationary and invertible, as

$$
\begin{aligned}
& (1-L)^d z_t = \varepsilon_t \\
\Leftrightarrow\ & z_t = (1-L)^{-d} \varepsilon_t \\
& = \frac{1}{\Gamma(d)} \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)} \varepsilon_{t-j} := \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \qquad (1)
\end{aligned}
$$

where $\{\varepsilon_t\} \overset{i.i.d.}{\sim} \left(0, \sigma^2\right)$ and $E\left(\varepsilon_t^4\right) < \infty$. The coefficients satisfy $\psi_j = O\left(j^{d-1}\right)$, so that the degree of decresing is quite slow as $j \to \infty$. Let

$$x_t := \frac{1}{\sigma T^{d+\frac{1}{2}}} z_t$$

and define the partial sum process $\{X_T(u)\}$ on $[0, 1]$ as

$$X_T\left(\frac{s}{T}\right) := \sum_{t=1}^{s} x_t = \frac{1}{\sigma T^{d+\frac{1}{2}}} \sum_{t=1}^{s} z_t = \frac{1}{\sigma T^{d+\frac{1}{2}}} \sum_{t=1}^{s} \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad \text{and} \quad X_T(u) := X_T\left(\frac{[uT]}{T}\right).$$

To describe the FCLT result for $I(d)$ process, we introduce the following Type I fBm $B_H$ $(0 < H < 1)$:

$$B_H(t) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \left[ \int_{-\infty}^{0} \left\{(t-s)^{H-1/2} - (-s)^{H-1/2}\right\} dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s) \right],$$

where $\{B(t)\, a\}$ is a standard Brownian motion on $(-\infty, 1]$. Then, we have the following FCLT result.

$$X_T(u) \Rightarrow B_{d+\frac{1}{2}}(u).$$

The Type I fBm $\{B_H(t)\}$ is mean 0 Gaussian process. It corresponds to the standard Brownian motion when $H = \frac{1}{2}$. On the other hand, when $H \neq \frac{1}{2}$, we have

$$V\{B_H(t)\} = \frac{t^{2d+1} A(d)}{\Gamma^2(d+1)}$$

and

$$V\{B_H(t) - B_H(s)\} = \frac{(t-s)^{2d+1} A(d)}{\Gamma^2(d+1)},$$

where

$$A(d) = \int_0^{\infty} \left\{(1+u)^d - u^d\right\}^2 du + \frac{1}{2d+1},$$

so that $\{B_H(t)\}$ has stationary increments. However, the increments are not independent.

## 1.3 $I(d)$ model $(d > 1/2)$

Next, we consider $I(d)$ model with $(d > 1/2)$:

$$
\begin{aligned}
& (1-L)^d z_t = \varepsilon_t \\
\Leftrightarrow\ & z_t = (1-L)^{-d} \varepsilon_t,
\end{aligned}
$$

which is non-stationary. In this case we can not define the linear representetion like as (1). Instead, we assume that $\varepsilon_j = 0, (j \le 0)$ and define the trancated process:

$$z_t^* = \frac{1}{\Gamma(d)} \sum_{j=0}^{t-1} \frac{\Gamma(j+d)}{\Gamma(j+1)} \varepsilon_{t-j} := \sum_{j=0}^{t-1} \psi_j \varepsilon_{t-j}.$$

Let

$$x_t^* := \frac{1}{\sigma T^{d+\frac{1}{2}}} z_t^*$$

and define the partial sum process $\left\{X_T^*(u)\right\}$ on $[0,1]$ as

$$\begin{aligned} X_T^*\left(\frac{s}{T}\right) := \sum_{t=1}^{s} x_t^* = \frac{1}{\sigma T^{d+\frac{1}{2}}} \sum_{t=1}^{s} z_t^* = \frac{1}{\sigma T^{d+\frac{1}{2}}} \sum_{t=1}^{s} \sum_{j=0}^{t-1} \psi_j \varepsilon_{t-j} \\ = \sum_{r=1}^{s} \left( \sum_{j=0}^{s-r} \frac{1}{\sigma T^{d+\frac{1}{2}}} \psi_j \right) \varepsilon_r \\ := \sum_{r=1}^{s} a_{s,r} \varepsilon_r, \end{aligned}$$

where

$$a_{s,r} = \frac{1}{\sigma T^{d+\frac{1}{2}}} \sum_{j=0}^{s-r} \psi_j = \frac{1}{\sigma T^{d+\frac{1}{2}} \Gamma(d)} \sum_{j=0}^{s-r} \frac{\Gamma(j+d)}{\Gamma(j+1)}.$$

Furthermore, we introduce the following Type II fBm $W_H$ $(H > 0)$:

$$W_H(t) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^t (t-s)^{H-\frac{1}{2}} \, dW(s).$$

Then, if $\{\varepsilon_t\} \overset{i.i.d.}{\sim} \left(0, \sigma^2\right)$, $E\left(\varepsilon_t^4\right) < \infty$, we have the following FCLT result:

$$X_T^*(u) := X_T^*\left(\frac{[uT]}{T}\right) \Rightarrow W_{d+\frac{1}{2}}(u).$$

Note that Type II fBm does not have stationary increment. Moreover, we see that the variances o Type I fBm and Type II fBm are different, namely

$$\begin{aligned} E\left\{W_H^2(t)\right\} = \frac{t^{2H}}{2H\Gamma^2(H+1/2)} \\ < E\left\{B_H^2(t)\right\} = \frac{t^{2H}\Gamma(2-2H)}{2H\Gamma(H+1/2)\Gamma(3/2-H)}. \end{aligned}$$

# 2 The fundamental process

## 2.1 Uncorrelated increments process

Norros et al. (1999) developed a sequence of simple representation formulae which allow us to proceed from process to process from the heuristic idea from the discrete-time case. In this paper, we develope the discrete-time formulae from the same idea.

Let $\boldsymbol{z}_n = (z_1, \ldots, z_n)'$ be second order stationary process and $\boldsymbol{\Sigma}_n = \mathrm{Var}(\boldsymbol{z}_n)$. Define the centered process

$$M_n := \sum_{t=1}^{n} a_{n,t} z_t,$$

which satisfies, for $m \geq n$

$$\langle M_m, M_n \rangle := \mathrm{Cov}(M_m, M_n) = \mathrm{Cov}(M_n, M_n) = \langle M_n, M_n \rangle.$$

That is, $\{M_n\}$ is the uncorrelated increments process. Note that if $m > n$, i.e., $m - 1 \geq n$, we have

$$\langle M_m - M_{m-1}, M_n \rangle = 0.$$

Furthermore, define

$$DM_t := (1 - L) M_t = M_t - M_{t-1},$$

$$\alpha_t^2 := (1 - L) \langle M_t, M_t \rangle = \langle M_t, M_t \rangle - \langle M_{t-1}, M_{t-1} \rangle$$

and let

$$W_n := \sum_{t=1}^{n} \alpha_t^{-1} DM_t.$$

Then, we have, for $m \geq n$

$$\langle W_m, W_n \rangle = n = \langle W_n, W_n \rangle.$$

Therefore, $\{W_n\}$ is also (standardized) uncorrelated increments process.

Defining

$$DW_t := W_t - W_{t-1},$$

let

$$\overline{M}_n := \sum_{t=1}^{n} \alpha_t DW_t.$$

Then, we have

$$\frac{1}{\sqrt{n}} W_{[nt]} = \frac{1}{\sqrt{n}} \sum_{s=1}^{[nt]} DW_s \Rightarrow W(t) = \int_0^t dW(s),$$

where $W(t)$ is standard Brownian motion. Furthermore, we have, for $m \geq n$

$$\langle \overline{M}_m, \overline{M}_n \rangle = \langle M_n, M_n \rangle = \langle \overline{M}_n, \overline{M}_n \rangle.$$

## 2.2 Expression formula

Let

$$Z_n := \sum_{t=1}^{n} z_t,$$

then, for $0 \le s \le t$, we have

$$\mathrm{Cov}\,(Z_s, M_t) = s.$$

Furthermore, let

$$Y_n := \sum_{t=1}^{n} \alpha_t z_t,$$

and

$$DY_t := (1 - L)\,Y_t = Y_t - Y_{t-1} = \alpha_t z_t,$$

then we have

$$\sum_{t=1}^{n} \alpha_t^{-1} DY_t = Z_n.$$

Here we consider the process of the form

$$\overline{Y}_n := \sum_{t=1}^{n} b_{n,t} DM_t,$$

which satisfies

$$\langle \overline{Y}_n, \overline{Y}_n \rangle = \langle Y_n, Y_n \rangle.$$

Moreover, define for $1 \le s \le n$,

$$\overline{Y}_{n|s} := \sum_{t=1}^{s} b_{n,t} DM_t,$$

then we have

$$\langle \overline{Y}_n - \overline{Y}_{n|s}, M_s \rangle = 0,$$

namely $\overline{Y}_n - \overline{Y}_{n|s}$ is orthogonal to $M_s$.

## 2.3 $W \to M \to Y \to Z$

Let

$$\widetilde{Y}_n := \sum_{t=1}^{n} b_{n,t} D\overline{M}_t,$$

where

$$D\overline{M}_t := (1-L)\,\overline{M}_t = \overline{M}_t - \overline{M}_{t-1} = \alpha_t DW_t,$$

that is

$$\widetilde{Y}_n = \sum_{t=1}^{n} b_{n,t}\alpha_t DW_t := \sum_{t=1}^{n} c_{n,t} DW_t,$$

with $c_{n,t} := b_{n,t}\alpha_t$ and $\widetilde{Y}_0 = 0$.

Furthermore, let

$$\widetilde{Z}_n := \sum_{t=1}^{n} \alpha_t^{-1} D\widetilde{Y}_t,$$

where

$$D\widetilde{Y}_t := (1-L)\,\widetilde{Y}_t = \widetilde{Y}_t - \widetilde{Y}_{t-1},$$

so that, we can rewrite

$$\widetilde{Z}_n = \sum_{t=1}^{n} \alpha_t^{-1} c_{t,t} DW_t + \sum_{t=1}^{n-1} \left\{ \sum_{s=t}^{n-1} \alpha_{s+1}^{-1} \left( c_{s+1,t} - c_{s,t} \right) \right\} DW_t.$$

## 3 Weak convergence of $I(d)$ process

Now, we obtain for $I(d)$ process

$$\begin{aligned} \frac{1}{\sigma n^{d+1/2}} \widetilde{Z}_{[nt]} &= \frac{1}{\sigma n^{d+1/2}} \sum_{s=1}^{[nt]} v_{s-1}^{1/2} DW_s + \frac{1}{\sigma n^{d+1/2}} \sum_{s=1}^{[nt]-1} \left\{ \sum_{u=s}^{[nt]-1} \theta_{u,u+1-s} v_{s-1}^{1/2} \right\} DW_s \\ &\Rightarrow \frac{1}{\Gamma(d)} \int_0^t s^{-d} \left\{ \int_s^t (u-s)^{d-1} u^d du \right\} dW(s) := \int_0^t dZ(s) = Z(t). \end{aligned}$$

# References


Mandelbrot B. B. and van Ness J. W. (1968). Fractional Brownian motions, fractional noises and applications *SIAM Review,* **10**, 422–437.

Norros, I. and Valkeila, E. and Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions *Bernoulli* **4**, 571–587.

Pipiras, V. and Taqqu, Murad S. (2001). Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli* **7**, 873–897.

Tanaka, K. (2013). Distributions of the maximum likelihood and minimum contrast estimators associated with the fractional Ornstein-Uhlenbeck process *Stat. Inference Stoch. Process.* **16**, 173–192.