On the beta-expansions of 1 and algebraic numbers for a Salem number beta *

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Abstract

We study the digits of β-expansions in the case where β is a Salem number. We introduce new upper bounds for the numbers of occurrences of consecutive 0's in the expansion of 1. We also give lower bounds for the numbers of nonzero digits in the β-expansions of algebraic numbers. As applications, we give criteria for transcendence of the values of power series at certain algebraic points.

1 Main results

Rényi [23] introduced representations of real numbers in a real base $\beta > 1$. The representations are called β-expansions. We recall the definition of β-expansions. We use the following notation throughout this paper. Let $\mathbb{N}$ be the set of nonnegative integers and let $\mathbb{Z}^+$ the set of positive integers. Let x be a real number. We denote the integral and fractional parts of x by $\lfloor x \rfloor$ and $\{x\}$, respectively. Moreover, we denote the minimal integer not less than x by $\lceil x \rceil$. We use the Landau symbols o, O and the Vinogradov symbols $\gg, \ll$ with their regular meanings. We denote $f \sim g$ if the ratio f/g tends to 1. We recall that a Pisot number is an algebraic integer greater than 1 whose conjugates except itself have absolute values less than 1. Moreover, a Salem number is an algebraic integer greater than 1 whose conjugates except itself have absolute value at most 1 with at least one conjugate having absolute value 1. If K is a subfield of a field L, then $[L : K]$ denotes the degree of the field extension L/K.

The β-transformation $T_\beta : [0,1] \to [0,1)$ is defined by

$$T_\beta(x) := \{\beta x\}$$

for $x \in [0,1]$. Let η be a real number with $0 \le \eta \le 1$. If $\beta = b$ is a rational integer, then suppose further that $\eta < 1$. Set $t_n(\beta;\eta) := \lfloor \beta T_\beta^{n-1}(\eta) \rfloor$ for each positive integer n. Then we have $t_n(\beta;\eta) \in \mathbb{Z} \cap [0,\beta)$. The β-expansion of η is

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written as

$$\eta = \sum_{n=1}^{\infty} t_n(\beta;\eta)\beta^{-n}. \tag{1.1}$$

Put

$$d_\beta(\eta) := t_1(\beta;\eta)t_2(\beta;\eta)\ldots.$$

Since the β-expansion of 0 is trivial, we only consider the β-expansions of positive real numbers. If $\beta = b \in \mathbb{Z}$ and if $\eta = 1$, then set

$$d_b(1) = t_1(b;1)t_2(b;1)\ldots := (b-1)(b-1)\ldots.$$

If $\eta = 1$, then the β-expansion of 1 is simply called the expansion of 1. We call $d_\beta(\eta)$ infinite if $t_n(\beta;\eta) \neq 0$ for infinitely many n's. Verger-Gaugry [27] say that $d_\beta(\eta)$ is gappy if $t_m(\beta;\eta) = 0$ for infinitely many m's and if $t_n(\beta;\eta) \neq 0$ for infinitely many n's.

The sequence $d_\beta(1)$ plays a crucial role for studying the β-shifts. In particular, Blanchard [7] classified the β-shifts, using the expansion of 1. For instance, $d_\beta(1)$ is ultimately periodic if and only if the β-shift is sofic. If β satisfies these properties, then β is called a Parry number. Parry [22] studied the periodicity of $d_\beta(1)$. He showed that any Pisot number is a Parry number. However, it is still unknown whether there is a non-Parry Salem number. Boyd [9] proved that any Salem number of degree 4 is a Parry number. On the other hand, his numerical experiments [10] implies that there are possible examples of non-Parry Salem numbers of degree greater than 4.

The sequence $d_\beta(1)$ is mysterious. It is generally difficult to decide whether $d_\beta(1)$ is gappy. Dubickas [15] proved that $d_\beta(1)$ is gappy in the case where β is a rational number with $1 < \beta < 2$. The number of occurrences of consecutive 0's in $d_\beta(1)$ is important in Blanchard's classification. Put

$$\{n \in \mathbb{Z}^+ \mid t_n(\beta;1) \neq 0\} =: \{v_1(\beta) < v_2(\beta) < \cdots < v_m(\beta) < \cdots\}.$$

If $d_\beta(1)$ is periodic, then the sequence $v_{m+1}(\beta) - v_m(\beta)$ $(m = 1, 2, \cdots)$ is bounded. For an algebraic number β, we denote its minimal polynomial by $A_d X^d + A_{d-1}X^{d-1} + \cdots + A_0 \in \mathbb{Z}[X]$, where $A_d > 0$. Let $\beta_1, \ldots, \beta_d$ be the conjugates of β. Then the Mahler measure of β is defined by

$$M(\beta) = A_d \prod_{i=1}^{d} \max\{1, |\beta_i|\}.$$

Verger-Gaugry [27] showed for any algebraic number $\beta > 1$ that

$$\limsup_{m\to\infty} \frac{v_{m+1}(\beta) - v_m(\beta)}{v_m(\beta)} \le \frac{\log M(\beta)}{\log \beta} - 1. \tag{1.2}$$

In particular, if β is a Salem number, then

$$\lim_{m\to\infty} \frac{v_{m+1}(\beta) - v_m(\beta)}{v_m(\beta)} = 0. \tag{1.3}$$

We give new upper bounds for $v_{m+1}(\beta) - v_m(\beta)$ as follows:

THEOREM 1.1. *Let β be a Salem number of degree d. Then there exists an effectively computable positive constant $C_1(\beta)$, depending only on β, such that*

$$v_{m+1}(\beta) - v_m(\beta) \le d \log_\beta m$$

for any m with $m \ge C_1(\beta)$, where $\log_\beta m = \log m / \log \beta$.

Note that Theorem 1.1 is stronger than (1.3) because $v_m(\beta) \ge m$. The sequence $d_\beta(\eta)$ is interesting also in the case where η is an algebraic number with $0 < \eta < 1$. For instance, if $\beta = b$ is an integer greater than 1, then $d_b(\eta)$ denotes the base-b expansion of η. Borel [8] conjectured that every algebraic irrational number is normal in any integral base b.

We consider the number of nonzero digits written as

$$\nu_\beta(\eta; N) := \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \le N,\ t_n(\beta;\eta) \ne 0\}$$

where N is a positie integer and Card denotes the cardinality. We denote the number of digit changes by

$$\gamma_\beta(\eta; N) := \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \le N,\ t_n(\beta;\eta) \ne t_{n+1}(\beta;\eta)\},$$

where N is a positive integer. Bugeaud [12, 13] introduced the function $\gamma_\beta(\eta; N)$ to investigate the complexity of the sequence $d_\beta(\eta)$. Observe that

$$\nu_\beta(\eta; N) \ge \frac{1}{2}\gamma_\beta(\eta; N) + O(1). \tag{1.4}$$

Let again $\beta = b$ be an integer greater than 1. If Borel's conjecture is true, then, for any algebraic irrational $\eta \in (0,1)$,

$$\lim_{N\to\infty} \frac{\nu_b(\eta; N)}{N} = \frac{b-1}{b}.$$

However, little is known on the conjecture. It is still unproved whether

$$\limsup_{N\to\infty} \frac{\nu_b(\eta; N)}{N} > 0.$$

Now we introduce known results on lower bounds for $\nu_b(\eta; N)$. Let $D \ge 2$ be the degree of η. Bailey, Borwein, Crandall, and Pomerance [5] showed that if $b = 2$, then there exist positive constants $C_2(\eta)$ and $C_3(\eta)$ such that

$$\nu_2(\eta; N) \ge C_2(\eta) N^{1/D} \tag{1.5}$$

for any integer N with $N \ge C_3(\eta)$. Note that $C_2(\eta)$ is computable but $C_3(\eta)$ is not. Rivoal [24] improved $C_2(\eta)$ for certain classes of algebraic irrational η. Changing $C_2(\eta)$ and $C_3(\eta)$ by suitable positive constants $C_2(b,\eta)$ and $C_3(b,\eta)$, respectively, we can prove (1.5) for any integral base b in the same way as the case of $b = 2$. Moreover, modifying the proof of (1.5), Adamczewski, Faverjon [4], and Bugeaud [11] independently gave effective versions of the lower bounds for general integer $b \ge 2$. Here we introduce the results by Bugeaud as follows: We denote the minimal polynomial of $1 + \{\eta\} (= 1 + \eta)$ by

$$P(X) = A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0 \in \mathbb{Z}[X],$$

where $A_D > 0$. We denote the height of $P(X)$ by H_P. Namely, H_P is the maximal absolute value of the coefficients of $P(X)$. Then, for any integer N with $N > (20b^D D^2 H_P)^{2D}$, we have

$$\nu_b(\eta; N) \geq \frac{1}{b-1} \left(\frac{N}{2(D+1)A_D} \right)^{1/D}. \tag{1.6}$$

We return to the case where $\beta > 1$ is a general real number. We review known results on the periodicity of β-expansions. Schmidt [25] proved that if each rational number η with $0 < \eta < 1$ has an ultimately periodic β-expansion, then β is a Pisot or Salem number. Next we suppose that β is a Pisot number. Then Bertrand [6] and Schmidt [25] independently showed for $\eta \in (0, 1]$ that $d_\beta(\eta)$ is ultimately periodic if and only if $\eta \in \mathbb{Q}(\beta)$. Schmidt [25] also conjectured that if β is a Salem number, then the β-expansion of any rational number is ultimately periodic, which is still not proved. In what follows, we study the sequence $d_\beta(\eta)$ in the case where β is a Pisot or Salem number and η is an algebraic number. In this section the implied constants in the symbol $\gg$ are effectively computable positive ones depending only on β and η. Moreover, we say that certain property (A) holds for any sufficiently large N if (A) is true for any integer N with $N \geq N_0$, where N_0 is an effectively computable positive constant depending only on β and η.

Bugeaud [13] gave lower bounds for $\gamma_\beta(\eta; N)$ as follows: Let β be a Pisot or Salem number and η an algebraic number with $0 < \eta \leq 1$. Assume that $t_n(\beta; \eta) \neq t_{n+1}(\beta; \eta)$ for infinitely many n's. Then we have

$$\gamma_\beta(\eta; N) \gg (\log N)^{3/2} (\log \log N)^{-1/2} \tag{1.7}$$

for any sufficiently large N. The author [16, 17] improved lower bounds for $\gamma_\beta(\eta; N)$ in the case where $\beta = b$ is an integer greater than 1. Namely, if η satisfies certain assumptions on its minimal polynomial, then

$$\gamma_b(\eta; N) \gg N^{1/D}$$

for any sufficiently large N, where D is the degree of η.

We now consider lower bounds for $\nu_\beta(\eta; N)$ in the case where β is a Pisot or Salem number. If $t_n(\beta; \eta) \neq t_{n+1}(\beta; \eta)$ for infinitely many n's, then using (1.4) and (1.7), we obtain

$$\nu_\beta(\eta; N) \gg (\log N)^{3/2} (\log \log N)^{-1/2} \tag{1.8}$$

for any sufficiently large N. On the other hand, using Theorem 2.1, we improve (1.8) as follows:

THEOREM 1.2. *Let β be an Pisot or Salem number and $\eta \in (0, 1]$ an algebraic number. Let $D := [\mathbb{Q}(\beta, \eta) : \mathbb{Q}(\beta)]$. Suppose that $d_\beta(\eta)$ is infinite. Then there exist effectively computable positive constants $C_4(\beta, \eta)$ and $C_5(\beta, \eta)$, depending only on β and η, such that*

$$\nu_\beta(\eta; N) \geq C_4(\beta, \eta) N^{1/(2D-1)} (\log N)^{-1/(2D-1)}$$

for any integer N with $N \geq C_5(\beta, \eta)$.

In particular, we consider the case where β is a Salem number and $\eta \in (0,1]$ is a rational number such that $d_\beta(\eta)$ is infinite. Suppose that the Schmidt's conjecture is true, namely, $d_\beta(\eta)$ is periodic. Then there exist positive constants C_6, C_7 such that

$$\nu_\beta(\eta; N) \geq C_6 N$$

for any N with $N \geq C_7$. On the other hand, using Theorem 1.2, we obtain partial results for the Schmidt conjecture as follows:

COROLLARY 1.3. *Let β be a Salem number and $\eta \in (0,1]$ a rational number. Suppose that $d_\beta(\eta)$ is infinite. Then there exist effectively computable positive constants $C_8(\beta,\eta)$ and $C_9(\beta,\eta)$, depending only on β and η, such that*

$$\nu_\beta(\eta; N) \geq C_8(\beta,\eta)\frac{N}{\log N}$$

for any integer N with $N \geq C_9(\beta,\eta)$.

It is well-known that $d_\beta(1)$ is infinite. Thus, Corollary 1.3 implies that

$$\nu_\beta(1; N) \gg \frac{N}{\log N}$$

for any sufficiently large N. We note that Dubickas [15] estimated lower bounds for $\nu_\beta(1;N)$ in the case where β is a transcendental number satisfying certain Diophantine assumptions.

We also introduce the Diophantine exponents which are measures of the periodicity of sequences. We give notation on words. Let $\mathcal{W} := \mathbb{Z} \cap [0,\beta)$ and V a finite nonempty word on the alphabet $\mathcal{W}$ with length $|V|$. For any positive real number x, put

$$V^x := \underbrace{V \ldots V}_{\lfloor x \rfloor} V',$$

where V' is the prefix of V with length $\lfloor \{x\}|V| \rfloor$. Let $\rho \geq 1$ and let $\boldsymbol{t} = (t_n)_{n=1}^\infty$ be an infinite word on the alphabet from $\mathcal{W}$. We say that $\boldsymbol{t}$ satisfies Condition $(*)_\rho$ if there exist two sequences of finite words $(U_n)_{n=1}^\infty$, $(V_n)_{n=1}^\infty$ and a sequence of positive real numbers $(\tau_n)_{n=1}^\infty$ such that:

1. For any $n \geq 1$, the word $U_n V_n^{\tau_n}$ is a prefix of $\boldsymbol{t}$;
2. For any $n \geq 1$, we have $|U_n V_n^{\tau_n}|/|U_n V_n| \geq \rho$;
3. The sequence $(|V_n^{\tau_n}|)_{n=1}^\infty$ is strictly increasing.

We then define the Diophantine exponent $\mathrm{Dio}(\boldsymbol{t})$ of $\boldsymbol{t}$ by the supremum of the real numbers ρ for which $\boldsymbol{t}$ satisfies Condition $(*)_\rho$. It is easily seen that $1 \leq \mathrm{Dio}(\boldsymbol{t}) \leq \infty$. Moreover, if $\boldsymbol{t}$ is ultimately periodic, then $\mathrm{Dio}(\boldsymbol{t}) = \infty$. Adamczewski and Bugeaud [2] showed the following: Let β be an algebraic number greater than 1. Let $\boldsymbol{t} = d_\beta(\eta)$ for an algebraic $\eta \in (0,1]$. Then $\boldsymbol{t}$ is ultimately periodic or

$$\mathrm{Dio}(\boldsymbol{t}) \leq \frac{\log M(\beta)}{\log \beta}. \tag{1.9}$$

The Diophantine exponents are applicable to the study of subwords in infinite sequences. For instance, (1.9) implies (1.2). Moreover, consider the case where β is a Pisot or Salem number and η is an algebraic number in $(0,1)\backslash\mathbb{Q}(\beta)$. Then, applying the Diophantine exponents, Adamczewski and Bugeaud [3] estimated lower bounds for the number of distinct blocks of n digits occurring in $\boldsymbol{t} = d_\beta(\eta)$. We also introduce that Dubickas [15] gave upper bounds for $\mathrm{Dio}(d_\beta(1))$ in the case where β is a transcendental number satisfying certain Diophantine assumptions.

Note that (1.1) is regarded as a special value of the power series

$$\sum_{n=1}^{\infty} t_n(\beta;\eta)X^n.$$

In this paper, we also discuss arithmetical properties of the values of power series. In Section 2 we give new criteria for transcendence of the values of power series. In Section 3 we review the transcendence of the values of gap and lacunary series. In Section 4 we prove the theorems stated in Sections 1 and 2.

2 New criteria for transcendence of the values of power series

Transcendence of the special values of power series at algebraic points has been investigated by various mathematicians. In Sections 2 and 3 we study the values of power series in one variable. In what follows, $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ is a bounded sequence of integers such that $s_n \neq 0$ for infinitely many $n \in \mathbb{N}$. Put

$$f(\boldsymbol{s};X) := \sum_{n=0}^{\infty} s_n X^n.$$

We define the sequence $(w(\boldsymbol{s};m))_{m=0}^{\infty}$ by

$$\{n \in \mathbb{N} \mid s_n \neq 0\} =: \{w(\boldsymbol{s};0) < w(\boldsymbol{s};1) < \cdots < w(\boldsymbol{s};m) < \cdots\}. \tag{2.1}$$

In Section 3 we review the transcendence of $f(\boldsymbol{s};z)$ for algebraic z under the assumption that

$$\limsup_{m\to\infty} \frac{w(\boldsymbol{s};m+1)}{w(\boldsymbol{s};m)} > 1.$$

Little is known on the transcendence of $f(\boldsymbol{s};z)$ for algebraic z with $0 < |z| < 1$ in the case of

$$\lim_{m\to\infty} \frac{w(\boldsymbol{s};m+1)}{w(\boldsymbol{s};m)} = 1. \tag{2.2}$$

In this section we introduce new criteria for the transcendence of $f(\boldsymbol{s};\beta^{-1})$, where β is a Pisot or Salem number. The criteria are applicable even to the values $f(\boldsymbol{s},\beta^{-1})$, where $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ are certain sequences satisfying (2.2). In the last of this section we give such examples. In the rest of this section we assume that $s_n \geq 0$ for any nonnegative integer n. We take an integer $b \geq$

2. We consider the case of $X = b^{-1}$. Namely, applying (1.6), we investigate arithmetical properties of $f(\boldsymbol{s}; b^{-1})$. Put

$$\Gamma(\boldsymbol{s}) := \{n \in \mathbb{N} \mid s_n \neq 0\}.$$

Let $\mathcal{A}$ be a nonempty subset of $\mathbb{N}$. Set

$$\lambda(\mathcal{A}; N) := \mathrm{Card}\{n \in \mathbb{N} \mid n \leq N,\ n \in \mathcal{A}\}$$

for $N \in \mathbb{N}$. We consider the base-b expansion of $f(\boldsymbol{s}; b^{-1})$. Since $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ is a bounded sequence of nonnegative integers, we have

$$\nu_b(f(\boldsymbol{s}; b^{-1}); N) \ll \lambda(\Gamma(\boldsymbol{s}); N).$$

Consequently, we get the following: Suppose for a positive integer D that $\boldsymbol{s}$ satisfies

$$\liminf_{N\to\infty} \frac{\lambda(\Gamma(\boldsymbol{s}); N)}{N^{1/D}} = 0.$$

Then $f(\boldsymbol{s}; b^{-1})$ is not an algebraic number with degree at most D. Namely,

$$[\mathbb{Q}(f(\boldsymbol{s}; b^{-1})) : \mathbb{Q}] > D. \tag{2.3}$$

In fact, $f(\boldsymbol{s}; b^{-1})$ is irrational because its base-b expansion is not ultimately periodic. Thus, (2.3) follows from (1.6). In particular, (2.3) implies the following criteria for transcendence, which were essentially proved by Bailey, Borwein, Crandall, and Pomerance [5]: Assume for any positive real number ε that

$$\liminf_{N\to\infty} \frac{\lambda(\Gamma(\boldsymbol{s}); N)}{N^{\varepsilon}} = 0.$$

Then $f(\boldsymbol{s}; b^{-1})$ is transcendental.

We now consider the case of $X = \beta^{-1}$, where β is a Pisot or Salem number.

THEOREM 2.1. *Let β be a Pisot or Salem number and ξ an algebraic number with $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$. Let $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ be a sequence of integers with $0 \leq s_n \leq B$ for any n, where B is a positive integer independent of n. Suppose that $s_n \neq 0$ for infinitely many n's and that $\xi = f(\boldsymbol{s}; \beta^{-1})$. Then there exist effectively computable positive constants $C_{10}(\beta, \xi, B)$ and $C_{11}(\beta, \xi, B)$, depending only on β, ξ and B, such that*

$$\lambda(\Gamma(\boldsymbol{s}); N) \geq C_{10}(\beta, \xi, B) N^{1/(2D-1)} (\log N)^{-1/(2D-1)}$$

for any integer N with $N \geq C_{11}(\beta, \xi, B)$.

Applying Theorem 2.1, we investigate the arithmetical properties of $f(\boldsymbol{s}; \beta^{-1})$.

COROLLARY 2.2. *Let $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ be a bounded sequence of nonnegative integers. Let D be a positive integer. Assume that*

$$\liminf_{N\to\infty} \lambda(\Gamma(\boldsymbol{s}); N) \cdot \frac{(\log N)^{1/(2D-1)}}{N^{1/(2D-1)}} = 0.$$

Let β be a Pisot or Salem number. Then $f(\boldsymbol{s}; \beta^{-1})$ satisfies

$$\left[\mathbb{Q}\left(\beta, f(\boldsymbol{s}; \beta^{-1})\right) : \mathbb{Q}(\beta)\right] > D.$$

We give examples of Corollary 2.2. Let D be a positive integer and t a real number with $t > 2D-1$. Let β be a Pisot or Salem number. Put

$$\zeta_t(\beta) := \sum_{m=0}^{\infty} \beta^{-\lfloor m^t \rfloor}.$$

Then Theorem 2.1 implies that

$$[\mathbb{Q}(\beta, \zeta_t(\beta)) : \mathbb{Q}(\beta)] > D. \tag{2.4}$$

In fact, we define $\boldsymbol{s}^{(t)} = (s_n^{(t)})_{n=0}^{\infty}$ by

$$\{\lfloor m^t \rfloor \mid m = 0, 1, \ldots\} =: \{0 = s_0^{(t)} < s_1^{(t)} < \cdots\}.$$

Then we have

$$\lambda\left(\Gamma\left(\boldsymbol{s}^{(t)}\right); N\right) \sim N^{1/t} = o\left(N^{1/(2D-1)}(\log N)^{-1/(2D-1)}\right).$$

If $\beta = b$ is an integer greater than 1, then Corollary 2.2 is weaker than (2.3). In particular, (2.4) holds under the weaker assumption, namely, $t > D$.

Using Corollary 2.2, we generalize the criteria for transcendence by Bailey, Borwein, Crandall, and Pomerance [5] as follows:

COROLLARY 2.3. *Let $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ be a bounded sequence of nonnegative integers. Assume for any positive real number ε that*

$$\liminf_{N\to\infty} \frac{\lambda(\Gamma(\boldsymbol{s}); N)}{N^{\varepsilon}} < \infty.$$

Let β be a Pisot or Salem number. Then $f(\boldsymbol{s}; \beta^{-1})$ is transcendental.

In particular, we have the following:

COROLLARY 2.4. *Let $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ be a bounded sequence of nonnegative integers. Let $(w(\boldsymbol{s}; m))_{m=0}^{\infty}$ be defined by (2.1). Suppose for any positive real number A that*

$$\limsup_{m\to\infty} \frac{w(\boldsymbol{s}; m)}{m^A} = \infty.$$

Let β be a Pisot or Salem number. Then $f(\boldsymbol{s}; \beta^{-1})$ is transcendental.

We give examples of Corollary 2.4. For any positive integer m, let

$$\mu(m) := m^{\log m} = \exp\left((\log m)^2\right).$$

Moreover, for any integer $m \geq 3$, put

$$\iota(m) := m^{\log\log m} = \exp\left(\log m \cdot \log\log m\right).$$

Then we have

$$\lim_{m\to\infty} \frac{\mu(m)}{m^A} = \infty, \ \lim_{m\to\infty} \frac{\iota(m)}{m^A} = \infty$$

for any positive real A. Using Corollary 2.4, we deduce for any Pisot or Salem number β that

$$\sum_{m=1}^{\infty} \beta^{-\lfloor \mu(m) \rfloor}, \quad \sum_{m=3}^{\infty} \beta^{-\lfloor \iota(m) \rfloor} \tag{2.5}$$

are transcendental numbers. Note that

$$\lim_{m\to\infty} \frac{\lfloor \mu(m+1) \rfloor}{\lfloor \mu(m) \rfloor} = 1, \quad \lim_{m\to\infty} \frac{\lfloor \iota(m+1) \rfloor}{\lfloor \iota(m) \rfloor} = 1.$$

Thus, the numbers in (2.5) are the special values of the power series satisfying (2.2).

3 Transcendence of the values of gap and lacunary series

Let again $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ be a bounded sequence of (not necessarily nonnegative) integers such that $s_n \neq 0$ for infinitely many $n \in \mathbb{N}$. Let $f(\boldsymbol{s}; X)$ and $(w(\boldsymbol{s}; m))_{m=0}^{\infty}$ be defined as in Section 2. We call $f(\boldsymbol{s}; X)$ a gap series if

$$\limsup_{m\to\infty} \frac{w(\boldsymbol{s}; m+1)}{w(\boldsymbol{s}; m)} = \infty.$$

For example, $\varphi(X) := \sum_{n=0}^{\infty} X^{n!}$ is a gap series. We say that $f(\boldsymbol{s}; X)$ is a lacunary series if

$$\liminf_{m\to\infty} \frac{w(\boldsymbol{s}; m+1)}{w(\boldsymbol{s}; m)} > 1. \tag{3.1}$$

Let k be an integer greater than 1. Then $\psi_k(X) := \sum_{n=0}^{\infty} X^{k^n}$ is a lacunary series. We introduce known results on the transcendence of $f(\boldsymbol{s}; z)$, where z is an algebraic number with $0 < |z| < 1$, in the case where $f(\boldsymbol{s}; X)$ is a gap or lacunary series.

Liouville [18, 19] investigated transcendence of the values of gap series at certain rational points. For instance, he proved for any integer b greater than 1 that $\varphi(b^{-1}) = \sum_{n=0}^{\infty} b^{-n!}$ is transcendental, which is one of the first examples of transcendental numbers. The proof is based on the theory of approximations of algebraic numbers by rational numbers. We extract results on Diophantine approximations from the book by Shidlovskii [26]. Recall that H_P denotes the height of P. Moreover, we denote the total degree of P by $\deg_{\underline{X}} P(X_1, \ldots, X_m)$. Then Theorem 11 in [26, p. 34] implies the following: Let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers and $\delta := [\mathbb{Q}(\alpha_1, \ldots, \alpha_m) : \mathbb{Q}]$. Then there exists an effectively computable positive constant $C_{12} = C_{12}(\alpha_1, \ldots, \alpha_m)$, depending only on $\alpha_1, \ldots, \alpha_m$, satisfying the following: For any polynomial $P(X_1, \ldots, X_m) \in \mathbb{Z}[X_1, \ldots, X_m]$ with $H_P \le H$ and $\deg_{\underline{X}} P(X_1, \ldots, X_m) \le k$, we have

$$P(\alpha_1, \ldots, \alpha_m) = 0 \text{ or } |P(\alpha_1, \ldots, \alpha_m)| \ge \frac{C_{12}^k}{H^{\delta-1}}. \tag{3.2}$$

Using (3.2), we obtain the following:

PROPOSITION 3.1. *Let z and ξ be algebraic numbers with $0<|z|<1$. Let $\boldsymbol{s}=(s_n)_{n=0}^{\infty}$ be a sequence of integers with $|s_n|\le B$ for any n, where B is a positive integer independent of n. Assume that*

$$f(\boldsymbol{s};z)=\xi$$

and that

$$\sum_{n=0}^{M} s_n z^n \neq \xi \tag{3.3}$$

for any nonnegative integer M. Then there exist effectively computable positive constants $C_{13}(z,\xi,B)$ and $C_{14}(z,\xi,B)$, depending only on z,ξ, and B, such that

$$\frac{w(\boldsymbol{s};m+1)}{w(\boldsymbol{s};m)}<C_{13}(z,\xi,B)$$

for any integer m with $m\ge C_{14}(z,\xi,B)$, where $(w(\boldsymbol{s};m))_{m=0}^{\infty}$ is defined by (2.1).

Proof. For simplicity, we put $w(m):=w(\boldsymbol{s};m)$ $(m=0,1,\ldots)$. Then we have

$$\xi=\sum_{m=0}^{\infty} s_{w(m)} z^{w(m)}.$$

We apply (3.2) with $\alpha_1=z$ and $\alpha_2=\xi$. Put $\delta:=[\mathbb{Q}(z,\xi):\mathbb{Q}]$. We may assume that $C_{12}=C_{12}(z,\xi)<1$. Let

$$P(X_1,X_2):=X_2-\sum_{i=0}^{m} s_{w(i)} X_1^{w(i)}$$

for positive integer m. Then we get $H_P\le B$ and $\deg_{\underline{X}} P(X_1,X_2)=w(m)$. Moreover, (3.3) implies that

$$0\neq P(z,\xi)=\sum_{i=m+1}^{\infty} s_{w(i)} z^{w(i)}.$$

Thus, using (3.2), we get

$$\begin{aligned}\frac{C_{12}^{w(m)}}{B^{\delta-1}} &\le |P(z,\xi)|\le \sum_{i=m+1}^{\infty} B|z|^{w(i)}\\ &\le B\sum_{n=w(m+1)}^{\infty}|z|^n=\frac{B}{1-|z|}|z|^{w(m+1)}.\end{aligned}$$

Taking the logarithm of the inequality above, we deduce Proposition 3.1. □

We now consider transcendence of the values of lacunary series. Mahler [20] proved transcendence of the values of power series satisfying certain kinds of functional equations. For instance, let k be an integer greater than 1. Then $\psi_k(X)$ fulfills

$$\psi_k(X^k)=\sum_{n=0}^{\infty} X^{k^{n+1}}=\sum_{n=0}^{\infty} X^{k^n}-X=\psi_k(X)-X.$$

Mahler proved for any algebraic z with $0 < |z| < 1$ that $\psi_k(z)$ is transcendental. For more details on Mahler's method, see, for instance, [21].

Using the Schmidt Subspace Theorem, Corvaja and Zannier [14] showed for any algebraic z with $0 < |z| < 1$ that if $f(\boldsymbol{s}; X)$ is lacunary, then $f(\boldsymbol{s}; z)$ is transcendental. Under weaker assumptions than (3.1), Adamczewski and Bugeaud [3] studied transcendence of $f(\boldsymbol{s}; \beta^{-1})$, where β is a Pisot or Salem number, as follows: Assume that $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ satisfies

$$\limsup_{m\to\infty} \frac{w(\boldsymbol{s}; m+1)}{w(\boldsymbol{s}; m)} > 1. \tag{3.4}$$

Then, for any Pisot or Salem number β, we have that $f(\boldsymbol{s}; \beta^{-1})$ either belongs to $\mathbb{Q}(\beta)$, or is transcendental. In particular, consider the case where $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ fulfills (3.4) and

$$s_n \in \{0, 1\} \text{ for any } n \in \mathbb{N}. \tag{3.5}$$

Namely, we have $f(\boldsymbol{s}; X) = \sum_{m=0}^{\infty} X^{w(\boldsymbol{s};m)}$. Adamczewski [1] showed for any Pisot or Salem number β that if $\boldsymbol{s}$ satisfies (3.4) and (3.5), then $f(\boldsymbol{s}; \beta^{-1})$ is transcendental.

4 Proof of the theorems in Sections 1 and 2

We see that Theorem 1.2 follows from Theorem 2.1. Hence, we only verify Theorems 1.1 and 2.1.

Proof of Theorem 1.1. For simplicity, we put

$$v_h(\beta) =: v_h,\ t_{v_h}(\beta; 1) =: \alpha_h$$

for $h = 1, 2, \ldots$. Then we have

$$1 = \sum_{h=1}^{\infty} \alpha_h \beta^{-v_h},$$

where

$$\alpha_h \in \mathbb{Z} \cap [1, \beta) \text{ for } h = 1, 2, \ldots. \tag{4.1}$$

For any positive integer m, put

$$A_m := \beta^{v_m} - \sum_{h=1}^{m} \alpha_h \beta^{v_m - v_h} = \sum_{h=m+1}^{\infty} \alpha_h \beta^{v_m - v_h}.$$

Note that A_m is an algebraic integer because β is a Salem number. Using (4.1), we have

$$\begin{aligned} 0 &< A_m < \sum_{h=m+1}^{\infty} \beta \cdot \beta^{v_m - v_h} \\ &\le \sum_{n=0}^{\infty} \beta \cdot \beta^{v_m - v_{m+1} - n} = \frac{\beta}{1 - \beta^{-1}} \beta^{v_m - v_{m+1}}. \end{aligned} \tag{4.2}$$

Let $\sigma_1, \ldots, \sigma_d$ be the conjugate embeddings of $\mathbb{Q}(\beta)$ into $\mathbb{C}$, where $\sigma_1(\gamma) = \gamma$ for any $\gamma \in \mathbb{Q}(\beta)$. Set $\sigma_i(\beta) =: \beta_i$ for $2 \leq i \leq d$. Then $|\beta_i| \leq 1$ because β is a Salem number. Thus, using (4.1), we get, for $2 \leq i \leq d$,

$$|\sigma_i(A_m)| \leq \left| \beta_i^{v_m} - \sum_{h=1}^{m} \alpha_h \beta_i^{v_m - v_h} \right| \leq 1 + m\beta.$$

Since $A_m > 0$ and since A_m is an algebraic integer, we obtain

$$1 \leq |A_m| \left| \prod_{i=2}^{d} \sigma_i(A_m) \right| \leq A_m (1 + m\beta)^{d-1}. \tag{4.3}$$

Combining (4.2) and (4.3), we deduce that

$$\beta^{v_{m+1} - v_m} \leq \frac{\beta}{1 - \beta^{-1}} (1 + m\beta)^{d-1}.$$

Hence, there exists an effectively computable positive constant $C_1(\beta)$ depending only on β such that

$$\beta^{v_{m+1} - v_m} \leq m^d$$

for any m with $m \geq C_1(\beta)$, whch implies Theorem 1.1. □

Proof of Theorem 2.1. Without loss of generality, we may assume that $0 \in \Gamma(\boldsymbol{s})$. In what follows, the implied constants in the symbol $\ll$ and the constants $C_{15}, C_{16}, \ldots$ are effectively computable positive ones depending only on β, ξ and B. Moreover, let $\mathcal{A}$ be a subset of $\mathbb{N}$. We say that certain property (A) holds for any sufficiently large $N \in \mathcal{A}$ if (A) is true for any $N \in \mathcal{A}$ with $N \geq N_0$, where N_0 is an effectively computable positive constant depending only on β, ξ and B. Since β is positive and since $s_n \neq 0$ for infinitely many n's, we get

$$\sum_{n=0}^{M} s_n \beta^{-n} \neq \xi$$

for any nonnegative integer M. Thus, Proposition 3.1 implies that there exist C_{15}, C_{16} satisfying, for any real number x with $x \geq C_{15}$,

$$\Gamma(\boldsymbol{s}) \cap [x, C_{16} x) \neq \emptyset. \tag{4.4}$$

Since $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$, there exists a polynomial $P(X) = A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0$, where $A_D, A_{D-1}, \ldots, A_0 \in \mathbb{Z}[\beta]$ and $A_D > 0$, such that $P(\xi) = 0$. For simplicity, we put

$$\Gamma := \Gamma(\boldsymbol{s}), \ \lambda(N) := \lambda(\Gamma; N).$$

We calculate ξ^k for any k with $1 \le k \le D$. We get

$$\begin{aligned}
\xi^k &= \left(\sum_{m\in\Gamma} s_m \beta^{-m}\right)^k \\
&= \sum_{m_1,\ldots,m_k\in\Gamma} s_{m_1}\cdots s_{m_k}\beta^{-m_1-\cdots-m_k} \\
&= \sum_{m=0}^{\infty} \beta^{-m} \sum_{\substack{m_1,\ldots,m_k\in\Gamma \\ m_1+\cdots+m_k=m}} s_{m_1}\cdots s_{m_k} \\
&=: \sum_{m=0}^{\infty} \beta^{-m}\rho(k;m),
\end{aligned}$$

where

$$\rho(k;m) = \sum_{\substack{m_1,\ldots,m_k\in\Gamma \\ m_1+\cdots+m_k=m}} s_{m_1}\cdots s_{m_k}.$$

Let m be a nonnegative integer. Then $\rho(k;m)$ is also a nonnegative integer by its definition. Put

$$k\Gamma := \{m_1 + \cdots + m_k \mid m_1, \ldots, m_k \in \Gamma\}.$$

Since $0 \in \Gamma$, we have

$$\Gamma \subset 2\Gamma \subset \cdots \subset (D-1)\Gamma \subset D\Gamma. \tag{4.5}$$

Moreover, we get

$$\begin{aligned}
\lambda(k\Gamma; N) &= \mathrm{Card}([0,N]\cap k\Gamma) \\
&\le \mathrm{Card}([0,N]\cap \Gamma)^k = \lambda(N)^k.
\end{aligned}\tag{4.6}$$

Observe that $\rho(k;m)$ is positive if and only if $m \in k\Gamma$. We estimate upper bounds for $\rho(k;m)$ as follows:

$$\rho(k;m) \le B^k \sum_{\substack{m_1,\ldots,m_k\in\Gamma \\ m_1+\cdots+m_k=m}} 1 \le B^k(m+1)^k. \tag{4.7}$$

Let $R \in \mathbb{N}$. Using

$$\begin{aligned}
0 &= P(\xi) = A_0 + \sum_{k=1}^{D} A_k \xi^k \\
&= A_0 + \sum_{k=1}^{D} A_k \sum_{m=0}^{\infty} \beta^{-m}\rho(k;m),
\end{aligned}$$

we obtain

$$\begin{aligned}
0 &= A_0\beta^R + \sum_{k=1}^{D} A_k \sum_{m=0}^{\infty} \beta^{-(m-R)}\rho(k;m) \\
&= A_0\beta^R + \sum_{k=1}^{D} A_k \sum_{m=-R}^{\infty} \beta^{-m}\rho(k;m+R).
\end{aligned}$$

Put

$$
\begin{aligned}
Y_R &:= \sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m+R), \\
&= -A_0\beta^R - \sum_{k=1}^{D} A_k \sum_{m=-R}^{0} \beta^{-m} \rho(k; m+R), \qquad (4.8)
\end{aligned}
$$

which implies that Y_R is an algebraic integer since β is a Pisot or Salem number. Bailey, Borwein, Crandall, and Pomerance [5] called Y_R BBP tails in the case of $\beta = 2$. For the proof of $P(\xi) \neq 0$, we consider BBP tails in the case where β is a Pisot or Salem number. We give lower bounds for $|Y_R|$ in the case of $Y_R \neq 0$.

LEMMA 4.1. *There exist positive integers C_{17} and C_{18} satisfying the following: If $R \geq C_{17}$, then we have*

$$Y_R = 0 \ \textit{or}\ |Y_R| \geq R^{-C_{18}}.$$

Proof. Put $d := \deg \beta$. Let $\sigma_1, \ldots, \sigma_d$ be the conjugate embeddings of $\mathbb{Q}(\beta)$ into $\mathbb{C}$, where $\sigma_1(\gamma) = \gamma$ for any $\gamma \in \mathbb{Q}(\beta)$. Set

$$C_{19} := \max\{|\sigma_i(A_j)| \mid i = 1, \ldots, d, j = 0, \ldots, D\}.$$

We put $\sigma_i(\beta) =: \beta_i$ for $2 \leq i \leq d$. Then (4.8) implies that

$$\sigma_i(Y_R) = -\sigma_i(A_0)\beta_i^R - \sum_{k=1}^{D} \sigma_i(A_k) \sum_{m=0}^{R} \beta_i^m \rho(k; -m+R). \qquad (4.9)$$

Recall that $|\beta_i| \leq 1$ because β is a Pisot or Salem number. Thus, combining (4.7) and (4.9), we get, for any sufficiently large R,

$$
\begin{aligned}
|\sigma_i(Y_R)| &\leq C_{19} + \sum_{k=1}^{D} C_{19} \sum_{m=0}^{R} \rho(k; -m+R) \\
&\leq C_{19} + \sum_{k=1}^{D} C_{19}(R+1)B^D(R+1)^D \leq R^{D+2}.
\end{aligned}
$$

Assume that $Y_R \neq 0$. Since Y_R is an algebraic integer, we obtain

$$1 \leq |Y_R| \prod_{i=2}^{d} |\sigma_i(Y_R)| \leq |Y_R| R^{(d-1)(D+2)}.$$

for any sufficienlty large $R \in \mathbb{N}$. Namely,

$$|Y_R| \geq R^{-C_{18}}.$$

□

Put

$$C_{20} := (\log \beta)^{1/(2D-1)} \Big(4(1 + C_{16})(D + 2C_{18}) \Big)^{-1/(2D-1)}$$

and

$$\Xi := \left\{ N \in \mathbb{N} \;\middle|\; N \geq 2, \lambda(N) < C_{20} N^{1/(2D-1)} (\log N)^{-1/(2D-1)} \right\}.$$

In what follows, we show that Ξ is bounded. Namely,

$$\lambda(N) \geq C_{20} N^{1/(2D-1)} (\log N)^{-1/(2D-1)}$$

for any sufficiently large $N \in \mathbb{N}$, which implies Theorem 2.1. We construct $J_2 \subset [0, N)$ such that $Y_R > 0$ for any $R \in J_2 \cap \mathbb{Z}$. If $D = 1$, then $Y_R > 0$ for any $R \in \mathbb{N}$ by the definition of Y_R and $\rho(k; m)$. So we put $J_2 := [0, N)$. Now we define J_2 in the case of $D \geq 2$. For any interval $I = [a, b)$, we denote its length by $|I| := b - a$. Put

$$[0, N) \cap (D-1)\Gamma =: \{0 = j_1 < j_2 < \cdots < j_\tau\},$$

where

$$\tau \leq \lambda(N)^{D-1} \tag{4.10}$$

by (4.6). Let $j_{1+\tau} := N$. Observe that $j_a \in (D-1)\Gamma$ for any a with $1 \leq a \leq \tau$ and that

$$\sum_{a=1}^{\tau} (j_{1+a} - j_a) = N. \tag{4.11}$$

There exists a p with $1 \leq p \leq \tau$ such that

$$j_{1+p} - j_p = \max_{1 \leq a \leq \tau} \{j_{1+a} - j_a\}.$$

Let $J_1 := [j_p, j_{1+p}) \subset [0, N)$ and $L_1 := |J_1| = j_{1+p} - j_p$. Then we have

$$j_p \in (D-1)\Gamma. \tag{4.12}$$

Moreover, for any k with $1 \leq k \leq D - 1$,

$$(j_p, j_{1+p}) \cap k\Gamma \subset (j_p, j_{1+p}) \cap (D-1)\Gamma (= \emptyset) \tag{4.13}$$

by (4.5). Combining (4.10), (4.11) and the definition of p, we get

$$L_1 = \max_{1 \leq a \leq \tau} \{j_{1+a} - j_a\} \geq \frac{N}{\tau} \geq \frac{N}{\lambda(N)^{D-1}}. \tag{4.14}$$

The definition of C_{20} and Ξ implies for any $N \in \Xi$ that

$$\frac{N}{\lambda(N)^{2D-1}} > 4(1 + C_{16})(D + 2C_{18}) \log_\beta N. \tag{4.15}$$

Thus, if $N \in \Xi$ satisfies $N > \beta^{C_{15}}$, then

$$\frac{L_1}{1 + C_{16}} > C_{15}$$

by (4.14) and (4.15). In particular, (4.4) implies that there exists $\theta_0 = \theta_0(N)$ with

$$\theta_0 \in \left[\frac{L_1}{1+C_{16}}, \frac{C_{16}L_1}{1+C_{16}}\right) \cap \Gamma.$$

In what follows, we assume that $N > \beta^{C_{15}}$. Let

$$M = M(N) := j_p + \theta_0 \in \left[j_p + \frac{L_1}{1+C_{16}}, j_p + \frac{C_{16}L_1}{1+C_{16}}\right) \tag{4.16}$$

and

$$J_2 := [j_p, M) \subset J_1.$$

Then $M \in D\Gamma$ by (4.12) and $\theta_0 \in \Gamma$. Hence, we defined J_2 in the case of $D \geq 2$. In the case of $D = 1$, we have $p = 1$, $j_1 = 0$ and $M = N$.

We observe that

$$|J_2| \geq \frac{N}{(1+C_{16})\lambda(N)^{D-1}}. \tag{4.17}$$

In fact, if $D = 1$, then (4.17) is clear. In the case of $D \geq 2$, we get

$$|J_2| = M - j_p \geq \frac{L_1}{1+C_{16}} \geq \frac{N}{(1+C_{16})\lambda(N)^{D-1}}$$

by (4.14) and (4.16).

LEMMA 4.2. *There exists a positive integer C_{21} with $C_{21} > \beta^{C_{15}}$ satisfying the following: If $N \in \Xi$ satisfies $N \geq C_{21}$, then $Y_R > 0$ for any $R \in J_2 \cap \mathbb{Z}$.*

Proof. We may assume that $D \geq 2$. We first show that there exists a positive integer C_{21} satisfying, for any $N \in \Xi$ with $N \geq C_{21}$,

$$Y_{M-1} > 0. \tag{4.18}$$

We have

$$\begin{aligned} Y_{M-1} &= A_D \sum_{m=1}^{\infty} \beta^{-m} \rho(D; m+M-1) \\ &\quad + \sum_{k=1}^{D-1} A_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m+M-1) \\ &=: S_1 + S_2. \end{aligned}$$

Thus, we get

$$S_1 \geq \frac{A_D}{\beta}\rho(D; M) \geq \frac{A_D}{\beta} > 0$$

because $M \in D\Gamma$. We now estimate upper bounds for $|S_2|$. Using (4.14), (4.15), and (4.16), we get

$$\begin{aligned} j_{1+p} - M &\geq j_{1+p} - j_p - \frac{C_{16}L_1}{1+C_{16}} \\ &= \frac{L_1}{1+C_{16}} \geq \frac{N}{(1+C_{16})\lambda(N)^{D-1}} > 2D \log_\beta N. \end{aligned} \tag{4.19}$$

Let $1 \le k \le D-1$. Take an integer m with $1 \le m < j_{1+p} - M + 1$. Then we get $j_p < M \le m + M - 1 < j_{1+p}$. Thus, (4.13) implies that $\rho(k; m + M - 1) = 0$. Hence, using (4.7), (4.19), and $M \le N$, we obtain

$$\begin{aligned}
|S_2| &\le \sum_{k=1}^{D-1} |A_k| \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + M - 1) \\
&= \sum_{k=1}^{D-1} |A_k| \sum_{m=j_{1+p}-M+1}^{\infty} \beta^{-m} \rho(k; m + M - 1) \\
&\le \sum_{k=1}^{D-1} |A_k| \sum_{m=j_{1+p}-M+1}^{\infty} \beta^{-m} B^D (m + M)^D \\
&\le \sum_{k=1}^{D-1} |A_k| B^D \sum_{m=j_{1+p}-M+1}^{\infty} \beta^{-m} (m + N)^D \\
&\ll \sum_{m=\lceil 2D \log_\beta N \rceil}^{\infty} \beta^{-m} (m + N)^D.
\end{aligned}$$

There exists C_{22} such that if $N \ge C_{22}$, then, for any $m \in \mathbb{Z}^+$,

$$\left(\frac{m + N + 1}{m + N} \right)^D \le \left(1 + \frac{1}{N + 1} \right)^D < \frac{\beta + 1}{2}. \tag{4.20}$$

Consequently,

$$\begin{aligned}
|S_2| &\ll \beta^{-\lceil 2D \log_\beta N \rceil} \left(\lceil 2D \log_\beta N \rceil + N \right)^D \sum_{m=0}^{\infty} \beta^{-m} \left(\frac{\beta + 1}{2} \right)^m \\
&\ll N^{-2D} \left(\lceil 2D \log_\beta N \rceil + N \right)^D.
\end{aligned}$$

In particular, we get

$$|S_2| < \frac{A_D}{2\beta}$$

for any sufficiently large $N \in \Xi$. Finally, taking a suitable constant C_{21} with $C_{21} > \max\{\beta^{C_{15}}, C_{22}\}$, we deduce that, for any $N \in \Xi$ with $N \ge C_{21}$,

$$Y_{M-1} = S_1 + S_2 \ge \frac{A_D}{2\beta} > 0,$$

which implies (4.18).

Next we show that if $N \ge C_{21}$, then $Y_R > 0$ for any R with $R \in [j_p, M) \cap \mathbb{Z}$ by induction on R. Assume that $Y_R > 0$ for certain R with $R \in (j_p, M) \cap \mathbb{Z}$. By (4.13) and $R \in (j_p, j_{1+p})$, we have $\rho(k; R) = 0$ for any k with $1 \le k \le D - 1$.

Consequently,

$$\begin{aligned}
Y_{R-1} &= \sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m}\rho(k; m+R-1) \\
&= \frac{1}{\beta}\sum_{k=1}^{D} A_k \rho(k; R) \\
&\quad + \frac{1}{\beta}\sum_{k=1}^{D} A_k \sum_{m=2}^{\infty} \beta^{-(m-1)}\rho(k; m-1+R) \\
&= \frac{1}{\beta} A_D \rho(D; R) + \frac{1}{\beta}\sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m}\rho(k; m+R) \\
&= \frac{1}{\beta} A_D \rho(D; R) + \frac{1}{\beta} Y_R \geq \frac{1}{\beta} Y_R > 0.
\end{aligned}$$

Therefore, we proved Lemma 4.2. □

In what follows, we assume that $N \in \Xi$ satisfies $N \geq C_{21}$. Observe that

$$\mathrm{Card}(J_2 \cap D\Gamma) \leq \lambda(D\Gamma; N) \leq \lambda(N)^D$$

by (4.6). Put

$$(J_2 \cap D\Gamma) \cup \{j_p\} =: \{j_p = i_1 < i_2 < \cdots < i_\omega\},$$

where

$$\omega \leq 1 + \lambda(N)^D \leq 2\lambda(N)^D. \tag{4.21}$$

Let $i_{1+\omega} := M$. Then (4.17) implies that

$$\sum_{h=1}^{\omega} (i_{1+h} - i_h) = |J_2| \geq \frac{N}{(1+C_{16})\lambda(N)^{D-1}}. \tag{4.22}$$

There exists a q with $1 \leq q \leq \omega$ such that

$$i_{1+q} - i_q = \max_{1 \leq h \leq \omega} \{i_{1+h} - i_h\}.$$

Put $I_1 := [i_q, i_{1+q}) \subset J_2$. Let $1 \leq k \leq D$. Then

$$(i_q, i_{1+q}) \cap k\Gamma \subset (i_q, i_{1+q}) \cap D\Gamma (= \emptyset) \tag{4.23}$$

by (4.5). Moreover, using (4.21), (4.22), and the definition of q, we obtain

$$\begin{aligned}
l_1 &:= |I_1| = \max_{1 \leq h \leq \omega} \{i_{1+h} - i_h\} \\
&\geq \frac{|J_2|}{\omega} \geq \frac{N}{2(1+C_{16})\lambda(N)^{2D-1}}.
\end{aligned}$$

In particular, (4.15) implies that

$$l_1 \geq 2(D + 2C_{18}) \log_\beta N \tag{4.24}$$

for any $N \in \Xi$. Recall that C_{17} and C_{21} are defined in Lemmas 4.1 and 4.2, respectively. Put $I_2 := [i_q, i_q + l_1/2)$.

LEMMA 4.3. *There exists* C_{23} *with*

$$C_{23} > \max\{C_{21}, \beta^{C_{17}}\} \tag{4.25}$$

satisfying the following: Let N *be an element of* Ξ *with* $N \geq C_{23}$*. Then, for any* $R \in I_2 \cap \mathbb{Z}$*, we have*

$$0 < Y_R < R^{-C_{18}}.$$

Proof. Let $N \in \Xi$ and $R \in I_2 \cap \mathbb{Z}$. For the proof of Lemma 4.3, it suffices to show that

$$|Y_R| < R^{-C_{18}}$$

by Lemma 4.2 and $I_2 \subset I_1 \subset J_2$. We estimate upper bounds for

$$|Y_R| = \left| \sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m+R) \right|.$$

Let $1 \leq k \leq D$. Take an integer m with $1 \leq m < i_{1+q} - R$. Then we have $i_q < m + R < i_{1+q}$. Thus, (4.23) implies that $\rho(k; m+R) = 0$ for such an m. Using (4.7) and $R + 1 \leq N$, we get

$$\begin{aligned} |Y_R| &\leq \sum_{k=1}^{D} |A_k| \sum_{m=i_{1+q}-R}^{\infty} \beta^{-m} \rho(k; m+R) \\ &\leq \sum_{k=1}^{D} |A_k| \sum_{m=i_{1+q}-R}^{\infty} \beta^{-m} B^D (m+R+1)^D \\ &\ll \sum_{m=i_{1+q}-R}^{\infty} \beta^{-m} (m+N)^D. \end{aligned}$$

Observe for any $R \in I_2 \cap \mathbb{Z}$ that

$$i_{1+q} - R > i_{1+q} - i_q - \frac{1}{2} l_1 = \frac{1}{2} l_1 \geq (D + 2C_{18}) \log_\beta N$$

by (4.24). Thus,

$$|Y_R| \ll \sum_{m=\lceil (D+2C_{18}) \log_\beta N \rceil}^{\infty} \beta^{-m} (m+N)^D.$$

Hence, using (4.20) and $N \geq C_{21} \geq C_{22}$, we obtain

$$\begin{aligned} |Y_R| &\ll \beta^{-\lceil (D+2C_{18}) \log_\beta N \rceil} (\lceil (D+2C_{18}) \log_\beta N \rceil + N)^D \sum_{m=0}^{\infty} \beta^{-m} \left(\frac{\beta+1}{2} \right)^m \\ &\ll N^{-(D+2C_{18})} \cdot N^D = N^{-2C_{18}} \leq N^{-C_{18}} R^{-C_{18}}. \end{aligned}$$

Namely, there exists C_{24} such that, for any $R \in I_2 \cap \mathbb{Z}$,

$$|Y_R| \leq C_{24} N^{-C_{18}} R^{-C_{18}}.$$

Taking a suitable constant C_{23} satisfying (4.25), we obtain that if $N \geq C_{23}$, then $|Y_R| < R^{-C_{18}}$ for any $R \in I_2 \cap \mathbb{Z}$. Therefore, we verified Lemma 4.3. $\square$

Suppose that there exists an $N \in \Xi$ with $N \geq C_{23}$. Recall that $i_q = i_q(N)$ is defined by N. Put

$$R(0) := i_q + C_{17} \in \mathbb{Z}^+.$$

Since $R \geq C_{17}$, Lemma 4.1 implies that

$$Y_{R(0)} = 0 \text{ or } |Y_{R(0)}| \geq R(0)^{-C_{18}}. \tag{4.26}$$

On the other hand, by (4.24) and (4.25), we get

$$\begin{aligned} C_{17} &< (D + 2C_{18}) \log_\beta C_{23} \\ &\leq (D + 2C_{18}) \log_\beta N \leq \frac{1}{2} l_1, \end{aligned}$$

which implies that $R(0) \in I_2$. Thus, using Lemma 4.3, we obtain

$$0 < Y_{R(0)} < R(0)^{-C_{18}},$$

which contradicts (4.26). Finally, we deduce that $\Xi \subset [2, C_{23})$. Namely, we proved Theorem 2.1. □

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