On normal numbers and powers of algebraic numbers *

Hajime Kaneko

Abstract

Let $\alpha > 1$ be an algebraic number and $\xi > 0$. Denote the fractional parts of $\xi\alpha^n$ by $\{\xi\alpha^n\}$. In this paper, we estimate a lower bound of the occurrence $\lambda_N(\alpha, \xi)$ of integers n with $0 \le n < N$ and

$$\{\xi\alpha^n\} \ge \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\}. \text{ (see (1.2))}$$

Our results show, for example, the following; Let α be an algebraic integer with Mahler measure $M(\alpha)$ and $\xi > 0$ an algebraic number with $\xi \notin \mathbf{Q}(\alpha)$. Put $[\mathbf{Q}(\alpha, \xi) : \mathbf{Q}(\alpha)] = D$. Then there exists an absolute constant c satisfying

$$\lambda_N(\alpha, \xi) \ge c \frac{(\log \alpha)^2}{(\log M(\alpha))^2 (\log(6D))^{1/2}} \frac{(\log N)^{3/2}}{(\log\log N)^{1/2}}$$

for all large N.

1 Introduction

A normal number in an integer base α is a positive number for which all finite words with letters from the alphabet $\{0, 1, \ldots, \alpha - 1\}$ occur with the proper frequency. It is easily checked that a positive number ξ is a normal number in base α if and only if the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1. Borel [6] proved that almost all positive ξ are normal numbers in every integer base. Moreover, Koksma [16] showed that if any real number $\alpha > 1$ is given, then the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1 for almost all positive ξ, which is a generalization of Borel's result. However, it is generally difficult to check a given geometric sequence is uniformly distributed modulo 1 or not. For instance, we even do not know whether the numbers $\sqrt{2}$, $\sqrt[3]{5}$ and π are normal in base 10.

Borel [7] conjectured that each algebraic irrational number is normal in every integer base. However, we know no such number whose normality was proved. We now introduce some partial results.

Let α be a natural number greater than 1 and ξ a positive algebraic irrational number. For simplicity, assume that $\xi < 1$. Write its α-ary expansion by

$$\xi = \sum_{i=-\infty}^{-1} s_i(\xi)\alpha^i = .s_{-1}(\xi)s_{-2}(\xi)\cdots$$

*2000 Mathematics Subject Classification : 11J71, 11J13, 11J61, 11K16 with $s_i(\xi) \in \{0, 1, \ldots, \alpha - 1\}$. First, we measure the complexity of the infinite word $\boldsymbol{s} = s_{-1}(\xi)s_{-2}(\xi)\cdots$ by the number $p(N)$ of distinct blocks of length N appearing in the word $\boldsymbol{s}$. If ξ is normal in base α, then $p(N) = \alpha^N$ for any positive N. Ferenczi and Mauduit [13] showed that

$$\lim_{N\to\infty} (p(N) - N) = \infty.$$

Adamczewski and Bugeaud [1] improved their results as follows:

$$\lim_{N\to\infty} \frac{p(N)}{N} = \infty.$$

Moreover, Bugeaud and Evertse [10] showed for any positive ξ with $\eta < 1/11$ that

$$\limsup_{N\to\infty} \frac{p(N)}{N(\log N)^\eta} = \infty.$$

Bugeaud and Evertse [10] gave a lower bound of the number $\mathrm{ch}(N)$ of digit changes among the first $(N+1)$ digits of the α-ary expansion of ξ. Namely,

$$\mathrm{ch}(N) = \mathrm{Card}\{i \in \mathbb{N} | 1 \le i \le N, s_{-i}(\xi) \ne s_{-i-1}(\xi)\},$$

where Card denotes the cardinality. They showed for an algebraic irrational $\xi > 0$ of degree $D(\ge 2)$ that there exist an effectively computable absolute constant c_1 and an effectively computable constant $c_2(\alpha, \xi)$, depending only on α and ξ, satisfying

$$\mathrm{ch}(N) \ge c_1 \frac{(\log N)^{3/2}}{(\log 6D)^{1/2}(\log\log N)^{1/2}}$$

for any N with $N \ge c_2(\alpha, \xi)$.

Next, we count the number $\lambda_N(\alpha, \xi)$ of nonzero digits among the first N digits of the α-ary expansion of ξ, where

$$\lambda_N(\alpha, \xi) = \mathrm{Card}\{i \in \mathbb{N} | 1 \le i \le N, s_{-i}(\xi) \ne 0\}. \tag{1.1}$$

Let ξ be an algebraic irrational number of degree D with $1 < \xi < 2$. In the case of $\alpha = 2$, Bailey, Borwein, Crandall, and Pomerance [4] showed that an arbitrary positive ε is given, then

$$\lambda_N(\alpha, \xi) > (1 - \varepsilon)(2A_D)^{-1/D} N^{1/D}$$

for all sufficiently large N, where $A_D(>0)$ is the leading coefficient of the minimal polynomial of ξ. Moreover, in the same way as the proof of the inequality above, we can show for any natural number $\alpha \ge 2$ that there exist a positive constant $c_3(\alpha, \xi)$ depending only on α and ξ satisfying

$$\lambda_N(\alpha, \xi) \ge c_3(\alpha, \xi) N^{1/D}$$

for every sufficiently large N.

In what follows, we consider the fractional parts of geometric progressions whose common ratios are algebraic numbers. Let $\alpha > 1$ be an algebraic number with minimal polynomial $a_d X^d + a_{d-1} X^{d-1} + \ldots + a_0 \in \mathbb{Z}[X]$, where $a_d > 0$ and $\gcd(a_d, a_{d-1}, \ldots, a_0) = 1$. Put

$$L_+(\alpha) = \sum_{a_i > 0} a_i, \ L_-(\alpha) = \sum_{a_i \le 0} |a_i|. \tag{1.2}$$

Moreover, write the Mahler measure of α by

$$M(\alpha) = a_d \prod_{k=1}^{d} \max\{1, |\alpha_k|\},$$

where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the conjugates of α. We now recall the definition of a Pisot and Salem number. A Pisot number is an algebraic integer greater than 1 whose conjugates different from itself have absolute values strictly less than 1. A Salem number is an algebraic integer greater than 1 which has at least one conjugate with modulus 1 and exactly one conjugate outside the unit circle. Take a positive number ξ. If α is a Pisot or Salem number, then assume $\xi \notin \mathbb{Q}(\alpha)$. Dubickas [11] showed for infinitely many $n \ge 1$ that

$$\{\xi \alpha^n\} \ge \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\},$$

where $\{\xi\alpha^n\}$ means the fractional part of $\xi\alpha^n$. In what follows we estimate the number of such n, namely, we give a lower bound of the number

$$\lambda_N(\alpha, \xi) = \mathrm{Card}\left\{n \in \mathbb{Z} \,\middle|\, 0 \le n < N, \{\xi\alpha^n\} \ge \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\}\right\}. \tag{1.3}$$

(1.3) is generalization of (1.1). In fact, assume that α is a natural number greater than 1 and that ξ is a positive number with $\xi < 1$. Then, for $n \ge 0$,

$$\{\xi \alpha^n\} \ge \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\} = \frac{1}{\alpha}$$

if and only if the $(n+1)$-th digit of α-ary expansion of ξ is nonzero.

Dubickas's result above implies

$$\lim_{N \to \infty} \lambda_N(\alpha, \xi) = \infty.$$

He verified this by showing that, for infinitely many $n \ge 0$,

$$s_{-n}(\xi) \ne 0,$$

where $s_{-n}(\xi)$ will be defined in Section 2. Moreover, in the same way as that of Theorem 3 of [11], we can show the following: Assume that α has at least one conjugate different from itself outside the unit circle. Then

$$\liminf_{N \to \infty} \frac{\lambda_N(\alpha, \xi)}{\log N} \ge \left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha) - \log(a_d \alpha)}\right)\right)^{-1}. \tag{1.4}$$

At the beginning of Section 5, we give another proof of (1.4). In this paper we improve this estimation in the case where $\alpha > 1$ and $\xi > 0$ are algebraic numbers with $\xi \notin \mathbb{Q}(\alpha)$ by using a version of the quantitative parametric subspace theorem of Bugeaud and Evertse [10]. First, we consider the case where $\alpha > 1$ is an algebraic integer.

THEOREM 1.1. *Let $\alpha > 1$ be an algebraic integer with Mahler measure $M(\alpha)$. Let ξ be a positive number with $\xi \notin \mathbb{Q}(\alpha)$. Put*

$$D = [\mathbb{Q}(\alpha, \xi) : \mathbb{Q}(\alpha)].$$

Then there exists an effectively computable absolute constant $c > 0$ such that

$$\lambda_N(\alpha, \xi) \geq c \frac{(\log \alpha)^2}{(\log M(\alpha))^2 (\log(6D))^{1/2}} \frac{(\log N)^{3/2}}{(\log \log N)^{1/2}}$$

for every sufficiently large N.

Next we give a lower bound of $\lambda_N(\alpha, \xi)$ in the case where $\alpha > 1$ is not an algebraic integer.

THEOREM 1.2. *Let $\alpha > 1$ be an algebraic number of degree d with Mahler measure $M(\alpha)$. We denote the leading coefficient of the minimal polynomial of α by $a_d (\geq 1)$. Let ξ be a positive algebraic number with $\xi \notin \mathbb{Q}(\alpha)$. Assume that α is not an algebraic integer. Then*

$$\liminf_{N \to \infty} \frac{\lambda_N(\alpha, \xi)}{\log N} \geq \left(\log \left(\frac{\log M(\alpha)}{\log M(\alpha) - \log a_d} \right) \right)^{-1}.$$

Theorem 1.2 gives an improvement of (1.4) since

$$\left(\log \left(\frac{\log M(\alpha)}{\log M(\alpha) - \log a_d} \right) \right)^{-1} > \left(\log \left(\frac{\log M(\alpha)}{\log M(\alpha) - \log(a_d \alpha)} \right) \right)^{-1}.$$

We introduce a numerical example in the case of $\alpha = 4 + 1/\sqrt{2}$. The minimal polynomial of α is $2X^2 - 16X + 31$, so we have $a_d = 2$, $M(\alpha) = 31$, and

$$\min\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\} = \min\{\frac{1}{33}, \frac{1}{16}\} = \frac{1}{33}.$$

Note that the conjugate of α is greater than 1. Thus by (1.4), for any positive ξ,

$$\liminf_{N \to \infty} \frac{\lambda_N(4 + 1/\sqrt{2}, \xi)}{\log N} \geq \left(\log \left(\frac{\log(31)}{\log(31) - \log(8 + \sqrt{2})} \right) \right)^{-1} = 0.944 \ldots.$$

On the other hand, the second statement of Theorem 1.2 implies that if $\xi > 0$ is an algebraic number with $\xi \notin \mathbb{Q}(\sqrt{2})$, then

$$\liminf_{N \to \infty} \frac{\lambda_N(4 + 1/\sqrt{2}, \xi)}{\log N} \geq \left(\log \left(\frac{\log(31)}{\log(31) - \log(2)} \right) \right)^{-1} = 4.43 \ldots.$$

REMARK 1.1. By using the same method for the proof of Theorem 1.2 and 1.1, Bugeaud [9] gave a lower bound for the number of digit changes in the β-expansion of algebraic numbers.

2 Preliminaries

Let $\alpha > 1$ be an algebraic number of degree d and ξ a positive number. Write the minimal polynomial of α by $P_\alpha(X) = a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$ $(a_d > 0)$. In this section, we study the sequence $(s_m(\xi))_{m=-\infty}^{\infty}$ defined by

$$s_m(\xi) = -\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{-m-i}\}.$$

Let $[x]$ be the integral part of a real number x. Since

$$0 = \sum_{i=0}^{d} a_{d-i}\xi\alpha^{-m-i} = \sum_{i=0}^{d} a_{d-i}\Big([\xi\alpha^{-m-i}] + \{\xi\alpha^{-m-i}\}\Big),$$

we have

$$\begin{aligned} s_m(\xi) &= \sum_{i=0}^{d} a_{d-i}\Big([\xi\alpha^{-m-i}] - \xi\alpha^{-m-i}\Big) \\ &= \sum_{i=0}^{d} a_{d-i}[\xi\alpha^{-m-i}]. \qquad (2.1) \end{aligned}$$

In particular, $s_m(\xi)$ is a rational integer. Thus we get the following:

LEMMA 2.1. *Let ξ be a positive number.*
(1) *If $s_m(\xi) \neq 0$,then*

$$\max_{-m-d \le n \le -m} \{\xi\alpha^n\} \ge \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\}.$$

(2) $s_m(\xi) = 0$ *for all sufficiently large m.*

Proof. We first show the first statement. Since $s_m(\xi)$ is a nonzero integer, we have

$$1 \le |s_m(\xi)| = \left| -\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{-m-i}\} \right|.$$

By using $0 \le \{\xi\alpha^{-m-i}\} < 1$, we obtained the first statement. The second statement follows from (2.1) and $[\xi\alpha^{-m}] = 0$ for each sufficiently large m. $\square$

PROPOSITION 2.2. *Write the conjugates of α with moduli greater than 1 by $\alpha_1 (= \alpha), \ldots, \alpha_p$. Let ξ be a positive number. Then*
(1) *For $2 \le k \le p$,*

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\xi) = 0.$$

(2)

$$\sum_{i=-\infty}^{\infty} \alpha^i s_i(\xi) = -\frac{\xi}{\alpha}(P_\alpha^*)'\left(\frac{1}{\alpha}\right) \neq 0,$$

where $P_\alpha^*(X) = a_d + a_{d-1}X + \cdots + a_0 X^d$ denotes the reciprocal polynomial of $P_\alpha(X)$ and $(P_\alpha^*)'(X)$ its derivative.

REMARK 2.1. By the second statement of Lemma 2.1, the series

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\xi)$$

converges for any k with $1 \leq k \leq p$.

Proof. We first consider the case of $0 < \xi < 1$. Then, for any $m \leq 0$, $[\xi\alpha^m] = 0$, and so $s_{-m}(\xi) = 0$ by (2.1). Put

$$f(z) = \sum_{n=0}^{\infty} [\xi\alpha^n] z^n, \; g(z) = \sum_{n=0}^{\infty} \{\xi\alpha^n\} z^n.$$

Then we have

$$\begin{aligned} \left(\frac{\xi}{1-\alpha z} - g(z)\right) P_\alpha^*(z) &= f(z) P_\alpha^*(z) \\ &= \sum_{h=0}^{\infty} \sum_{\substack{i,j \geq 0 \\ i+j=h}} [\xi\alpha^i] a_{d-j} z^h \\ &= \sum_{h=0}^{\infty} \sum_{i=h-d}^{h} [\xi\alpha^i] a_{d-h+i} z^h = \sum_{h=0}^{\infty} s_{-h}(\xi) z^h. \end{aligned}$$

Consider the region of $z \in \mathbb{C}$ satisfying

$$\left(\frac{\xi}{1-\alpha z} - g(z)\right) P_\alpha^*(z) = \sum_{h=0}^{\infty} s_{-h}(\xi) z^h. \tag{2.2}$$

Since $0 \leq \{\xi\alpha^n\} < 1$ for any n, the left-hand side of (2.2) is a meromorphic function on $\{z \mid |z| < 1\}$. Moreover, because the sequence $s_{-m}(\xi)$ $(m = 0, 1, \ldots)$ is bounded, the right-hand side of (2.2) converges for $|z| < 1$. Hence (2.2) holds for $|z| < 1$. In particular, since the left-hand side of (2.2) has zero at $z = \alpha_k^{-1}$ with $2 \leq k \leq p$, we obtain

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\xi) = \sum_{i=0}^{\infty} \alpha_k^{-i} s_{-i}(\xi) = 0.$$

Let $\alpha_1 = \alpha, \ldots, \alpha_p, \alpha_{p+1}, \ldots, \alpha_d$ be the conjugates of α. $P_\alpha^*(z)$ has a simple zero at $z = 1/\alpha$ since

$$P_\alpha^*(z) = z^d P_\alpha\left(\frac{1}{z}\right) = a_d(1-\alpha z)(1-\alpha_2 z)\cdots(1-\alpha_d z).$$

Note that $g(z)$ is holomorphic for $|z| < 1$. Hence

$$\begin{aligned} \sum_{i=-\infty}^{\infty} \alpha^i s_i(\xi) &= \sum_{i=0}^{\infty} \alpha^{-i} s_{-i}(\xi) \\ &= \lim_{z \to 1/\alpha} \frac{\xi P_\alpha^*(z)}{1-\alpha z} = -\frac{\xi}{\alpha} (P_\alpha^*)'\left(\frac{1}{\alpha}\right) \neq 0. \end{aligned}$$

Next, we check the case of $\xi \geq 1$. Take a positive integer R satisfying $\xi\alpha^{-R} < 1$. Then we obtain

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\xi) = \alpha_k^R \sum_{i=-\infty}^{\infty} \alpha_k^{i-R} s_{i-R}(\xi\alpha^{-R}) = 0$$

for $2 \leq k \leq p$, and

$$\sum_{i=-\infty}^{\infty} \alpha^i s_i(\xi) = \alpha^R \sum_{i=-\infty}^{\infty} \alpha^{i-R} s_{i-R}(\xi\alpha^{-R}) = -\frac{\xi}{\alpha}(P_\alpha^*)'\left(\frac{1}{\alpha}\right).$$

□

3 The quantitative subspace theorem

First, we consider approximations of given algebraic numbers by algebraic numbers which lies in a fixed number field. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. In what follows, assume that all algebraic number fields are subfields of $\overline{\mathbb{Q}}$. Let us begin with some notation about the absolute values on $\mathbf{K}$, where $\mathbf{K}$ is a number field of degree d. Let $\mathcal{M}_{arc}(\mathbf{K})$ be the set of archimedean places of $\mathbf{K}$ and $\mathcal{M}_{non}(\mathbf{K})$ the set of non-archimedean places of $\mathbf{K}$, respectively. Moreover, put $\mathcal{M}(\mathbf{K}) = \mathcal{M}_{arc}(\mathbf{K}) \cup \mathcal{M}_{non}(\mathbf{K})$. We define the absolute values $|\cdot|_v$ and $||\cdot||_v$ associated to a place $v \in \mathbf{K}$. In the case of $\mathbf{K} = \mathbb{Q}$, we have

$$\mathcal{M}(\mathbb{Q}) = \{\infty\} \cup \{\text{primes}\}.$$

In the case of $v = \infty$, let $|\cdot|_\infty$ be the ordinary archimedean absolute value on $\mathbb{Q}$. If $v = p$ is a prime number, then denote $|\cdot|_p$ the p-adic absolute value, normalized such that $|p|_p = p^{-1}$.

Next, we consider the case where $\mathbf{K}$ is an arbitrary number field. Suppose a place $v \in \mathcal{M}(\mathbf{K})$ lies above the place $p_v \in \mathcal{M}(\mathbb{Q})$. We choose the normalized absolute value $|\cdot|_v$ in such a way that the restriction of $|\cdot|_v$ to $\mathbb{Q}$ is $|\cdot|_{p_v}$. Let $\mathbf{K}_v$ (resp. $\mathbb{Q}_{p_v}$) be the completion of $(\mathbf{K}, |\cdot|_v)$ (resp. $(\mathbb{Q}, |\cdot|_{p_v})$). Put

$$d(v) = \frac{[\mathbf{K}_v : \mathbb{Q}_{p_v}]}{[\mathbf{K} : \mathbb{Q}]}.$$

and

$$||\cdot||_v = |\cdot|_v^{d(v)}.$$

Define the height of x by

$$H(x) = \prod_{v \in \mathcal{M}(\mathbf{K})} \max\{1, ||x||_v\}.$$

By Lemma 3.10 of [20], we have

$$H(x)^{\deg x} = M(x) \tag{3.1}$$

Moreover, the product formula (for instance see [20], p. 74) implies for any nonzero $x \in \mathbf{K}$ that

$$H(x^{-1}) = H(x). \tag{3.2}$$

Now we introduce Theorem 2 of [17] in the case of $d = 1$, which we use to prove Theorem 1.2. Suppose every valuation of $\mathbf{K}$ to be extended to $\overline{\mathbb{Q}}$.

THEOREM 3.1 (Locher [17]). *Let $0 < \varepsilon \leq 1$ and $\mathbf{F}/\mathbf{K}$ be an extension of number fields of degree D. Let S be a finite set of places of $\mathbf{K}$ with cardinality s. Suppose that for each $v \in S$, a fixed element $\theta_v \in \mathbf{F}$ is given. Let H be a real number with $H \geq H(\theta_v)$ for all $v \in S$. Consider the inequality*

$$\prod_{v \in S} \min\{1, ||\theta_v - \gamma||_v\} < H(\gamma)^{-2-\varepsilon} \tag{3.3}$$

to be solved in elements $\gamma \in \mathbf{K}$. Then there are at most

$$e^{7s+19}\varepsilon^{-s-4} \log(6D) \log\left(\varepsilon^{-1} \log(6D)\right)$$

solutions $\gamma \in \mathbf{K}$ of (3.3) with

$$H(\gamma) \geq \max\left\{H, 4^{4/\varepsilon}\right\}.$$

Next, we consider approximations of given algebraic numbers by algebraic numbers with arbitrary degree. Let us introduce the quantitative subspace theorem proved by Bugeaud and Evertse [10]. Let $\mathcal{L} = (L_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ be a tuple of linear forms with the following properties:

$$\begin{cases} L_{iv} \in \mathbf{K}[X, Y] \text{ for } v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2, \\ L_{1v} = X,\ L_{2v} = Y \text{ for all but finitely many } v \in \mathcal{M}(\mathbf{K}), \\ \det(L_{1v}, L_{2v}) = 1 \text{ for } v \in \mathcal{M}(\mathrm{K}), \\ \mathrm{Card}\left(\bigcup_{v \in \mathcal{M}(\mathbf{K})} \{L_{1v}, L_{2v}\}\right) \leq r. \end{cases} \tag{3.4}$$

Put

$$\bigcup_{v \in \mathcal{M}(\mathbf{K})} \{L_{1v}, L_{2v}\} = \{L_1, \ldots, L_s\}$$

and

$$\mathcal{H} = \mathcal{H}(\mathcal{L}) = \prod_{v \in \mathcal{M}(\mathbf{K})} \max_{1 \leq i < j \leq s} || \det(L_i, L_j)||_v. \tag{3.5}$$

Moreover, let $\mathbf{c} = (c_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ be a tuple of reals with the following properties:

$$\begin{cases} c_{1v} = c_{2v} = 0 \text{ for all but finitely many } v \in \mathcal{M}(\mathbf{K}), \\ \sum_{v \in \mathcal{M}(\mathbf{K})} \sum_{i=1}^{2} c_{iv} = 0, \\ \sum_{v \in \mathcal{M}(\mathbf{K})} \max\{c_{1v}, c_{2v}\} \leq 1. \end{cases} \tag{3.6}$$

Next, take any finite extension $\mathbf{E}$ of $\mathbf{K}$ and any place $w \in \mathcal{M}(\mathbf{E})$. Let $v \in \mathcal{M}(\mathbf{K})$ be the place lying below w. Write the completion of $(\mathbf{E}, |\cdot|_w)$ (resp. $(\mathbf{K}, |\cdot|_v)$) by $\mathbf{E}_w$ (resp. $\mathbf{K}_v$). For $i = 1, 2$, define the linear forms L_{1w}, L_{2w} and the real numbers c_{1w}, c_{2w} by

$$L_{iw} = L_{iv} \text{ and } c_{iw} = d(w|v)c_{iv}, \tag{3.7}$$

where

$$d(w|v) = \frac{[\mathbf{E}_w : \mathbf{K}_v]}{[\mathbf{E} : \mathbf{K}]}.$$

Note that

$$||x||_w = ||x||_v^{d(w|v)} \text{ for } x \in \mathbf{K} \tag{3.8}$$

and that

$$\sum_{\substack{w \in \mathcal{M}(\mathbf{E}) \\ w|v}} d(w|v) = 1 \text{ for } v \in \mathcal{M}(\mathbf{K}). \tag{3.9}$$

Take a positive number Q and $\mathbf{x} = (x, y) \in \overline{\mathbb{Q}}^2$. We define the twisted height $H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x})$. There exists a number field $\mathbf{E}$ including the field $\mathbf{K}(x, y)$. Then put

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) = \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}},$$

which is a finite product by the assumption of $\mathcal{L}$ and $\mathbf{c}$. We show that $H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x})$ does not depend on the choice of $\mathbf{E}$. Let $\mathbf{E}'$ be another number field including $\mathbf{K}(x, y)$. Take a number field $\mathbf{F}$ with $\mathbf{F} \supset \mathbf{E} \cup \mathbf{E}'$. By (3.7), (3.8), and (3.9)

$$\begin{aligned}
&\prod_{u \in \mathcal{M}(\mathbf{F})} \max_{1 \le i \le 2} ||L_{iu}(\mathbf{x})||_u Q^{-c_{iu}} \\
&= \prod_{w \in \mathcal{M}(\mathbf{E})} \prod_{\substack{u \in \mathcal{M}(\mathbf{F}) \\ u|w}} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w^{d(u|w)} Q^{-d(u|w)c_{iw}} \\
&= \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\prod_{w' \in \mathcal{M}(\mathbf{E}')} \max_{1 \le i \le 2} ||L_{iw'}(\mathbf{x})||_{w'} Q^{-c_{iw'}} &= \prod_{u \in \mathcal{M}(\mathbf{F})} \max_{1 \le i \le 2} ||L_{iu}(\mathbf{x})||_u Q^{-c_{iu}} \\
&= \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}}.
\end{aligned}$$

Now we consider the inequality

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \le Q^{-\delta}, \tag{3.10}$$

where $\mathbf{x} \in \overline{\mathbf{Q}}^2$ and $Q, \delta > 0$.

THEOREM 3.2 (Bugeaud and Evertse [10]). *Let* $\mathcal{L} = (L_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ *be a tuple of linear forms satisfying* (3.4) *and* $\mathbf{c} = (c_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ *a tuple of reals fulfilling* (3.6). *Moreover, let* $0 < \delta \le 1$.
Then there are proper linear subspaces $T_1, \ldots, T_{t_1}$ *of* $\overline{\mathbb{Q}}^2$*, all defined over* $\mathbf{K}$*, with*

$$t_1 = t_1(r, \delta) = 2^{25}\delta^{-3}\log(2r)\log\left(\delta^{-1}\log(2r)\right) \tag{3.11}$$

such that the following holds: for every real Q with

$$Q > \max\left(\mathcal{H}^{1/\binom{r}{2}}, 2^{2/\delta}\right) \tag{3.12}$$

there is a subspace $T_i \in \{T_1, \dots, T_{t_1}\}$ which contains all solutions $\mathbf{x} \in \overline{\mathbf{Q}}^2$ of (3.10).

This is Proposition 4.1 of [10] in the case of $n = 2$.

4 Systems of inequalities

In this section we apply Theorem 3.2 to certain systems of inequalities, which are generalization of Theorem 5.1 in [10]. Let $\mathbf{K} \subset \overline{\mathbb{Q}}$ be a number field of degree d. We define some notation about linear forms with algebraic coefficients. Take a linear form $L(X,Y) = \alpha X + \beta Y \in \overline{\mathbb{Q}}[X,Y]$ and put

$$\mathbf{K}(L) = \mathbf{K}(\alpha, \beta).$$

Define the inhomogeneous height $H^*(L)$ of L by

$$H^*(L) = \prod_{v \in \mathcal{M}(\mathbf{K}(L))} \max\{1, ||\alpha||_v, ||\beta||_v\}.$$

Note that, for a number field $\mathbf{E}$ including $\mathbf{K}(L)$,

$$\begin{aligned} &\prod_{w \in \mathcal{M}(\mathbf{E})} \max\{1, ||\alpha||_v, ||\beta||_v\} \\ &= \prod_{v \in \mathcal{M}(\mathbf{K}(L))} \prod_{\substack{w \in \mathcal{M}(\mathbf{E}) \\ w|v}} \max\{1, ||\alpha||_v, ||\beta||_v\}^{d(w|v)} = H^*(L) \end{aligned} \tag{4.1}$$

by (3.9). In what follows we put, for $w \in \mathbf{E}$,

$$||L||_w = \max\{||\alpha||_w, ||\beta||_w\}.$$

Moreover, if an automorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ is given, let

$$\sigma(L) = \sigma(\alpha)X + \sigma(\beta)Y.$$

Write the archimedean place associated to the inclusion map $\mathbf{K} \hookrightarrow \mathbb{C}$ by ∞, namely,

$$||x||_\infty = |x|^{1/d} \text{ for } x \in \mathbf{K}. \tag{4.2}$$

Let ε be a real with $0 < \varepsilon \leq 1/2$ and S a finite subset of $\mathcal{M}(\mathbf{K})$ including all archimedean places of $\mathbf{K}$. Moreover, let L_{iv} ($v \in S$, $i = 1, 2$) be linear forms in X, Y with coefficients in $\overline{\mathbb{Q}}$ such that

$$\begin{cases} \det(L_{1v}, L_{2v}) = 1 \text{ for } v \in S, \\ \text{Card}\left(\bigcup_{v \in S}\{L_{1v}, L_{2v}\}\right) \leq R, \\ [K(L_{iv}) : K] \leq D \text{ for } v \in S,\ i = 1, 2, \\ H^*(L_{iv}) \leq H \text{ for } v \in S,\ i = 1, 2, \end{cases} \tag{4.3}$$

and e_{iv} $(v \in S,\ i = 1, 2)$ be reals satisfying

$$\sum_{v \in S} \sum_{i=1}^{2} e_{iv} = -\varepsilon. \tag{4.4}$$

Put

$$A = 1 + \sum_{v \in S} \max\{0, e_{1v}, e_{2v}\}\ (\geq 1).$$

Finally, let Ψ be a function from $O_{\mathbf{K}}^2$ to $\mathbb{R}_{\geq 0}$, where $O_{\mathbf{K}}$ is the ring of integers of $\mathbf{K}$. Suppose every valuation v of $\mathbf{K}$ to be extended to $\overline{\mathbb{Q}}$. Consider the system of inequalities

$$||L_{iv}(\mathbf{x})||_v \leq \Psi(\mathbf{x})^{e_{iv}}\ \ (v \in S, i = 1, 2), \tag{4.5}$$

where $\mathbf{x} \in O_{\mathbf{K}}^2$ with $\Psi(\mathbf{x}) \neq 0$.

PROPOSITION 4.1. *The set of solutions* $\mathbf{x} \in O_{\mathbf{K}}^2$ *of* (4.5) *with*

$$\Psi(\mathbf{x}) > \max\{2H, 2^{4/\varepsilon}\} \tag{4.6}$$

is contained in the union of at most

$$2^{31} A^4 \varepsilon^{-3} \log(2RD) \log\left(\varepsilon^{-1} \log(2RD)\right)$$

proper linear subspaces of $\mathbf{K}^2$.

Proof. We can prove this proposition in the same way as Theorem 5.1 in [10]. Let $\mathbf{E}$ be a finite normal extension of $\mathbf{K}$, containing the coefficients of L_{iv} as well as the conjugates over $\mathbf{K}$ of these coefficients, for $v \in S$, $i = 1, 2$. Let $\widetilde{S}$ denote the set of places of $\mathbf{E}$ lying above the places in S. Note that $\widetilde{S} \supset \mathcal{M}_{arc}(\mathbf{E})$. Take a place $w \in \mathcal{M}(\mathbf{E})$ above the place $v \in \mathcal{M}(\mathbf{K})$. For simplicity, put

$$d_w = d(w|v).$$

If $w \in \widetilde{S}$, then there exists an automorphism σ_w of $\mathbf{E}$ satisfying

$$||x||_w = ||\sigma_w(x)||_v^{d_w} \text{ for } x \in \mathbf{E}.$$

For $i = 1, 2$, we define the linear forms L_{iw} and the real numbers e_{iw} by

$$L_{iw} = \begin{cases} \sigma_w^{-1}(L_{iv}) & (w \in \widetilde{S}), \\ X & (i = 1,\ w \notin \widetilde{S}) \\ Y & (i = 2,\ w \notin \widetilde{S}) \end{cases}$$

and

$$e_{iw} = \begin{cases} d_w e_{iv} & (w \in \widetilde{S}), \\ 0 & (w \notin \widetilde{S}), \end{cases}$$

respectively. Take an $\mathbf{x} \in O_{\mathbf{K}}^2$ with (4.5). If $w \notin \widetilde{S}$, then w is non-archimedean, so

$$||L_{iw}(\mathbf{x})||_w \leq 1.$$

Moreover, since

$$||L_{iv}(\mathbf{x})||_v^{d_w} = ||\sigma_w\big(L_{iw}(\mathbf{x})\big)||_v^{d_w} = ||L_{iw}(\mathbf{x})||_w,$$

$\mathbf{x}$ satisfies the system of inequalities

$$||L_{iw}(\mathbf{x})||_w \leq \Psi(\mathbf{x})^{e_{iw}} \ (w \in \mathcal{M}(\mathbf{E}),\, i = 1, 2). \tag{4.7}$$

By using (3.9) and (4.4) we get

$$\begin{aligned} \sum_{w\in\mathcal{M}(\mathbf{E})} \sum_{i=1}^{2} e_{iw} &= \sum_{v\in S} \sum_{\substack{v\in\mathcal{M}(\mathbf{E})\\ w|v}} \sum_{i=1}^{2} d_w e_{iv} \\ &= \sum_{v\in S} \sum_{i=1}^{2} e_{iv} = -\varepsilon. \end{aligned} \tag{4.8}$$

By the definition of L_{iw} with $1 \leq i \leq 2$ and $w \in \mathcal{M}(\mathbf{E})$

$$\text{Card}\left(\bigcup_{w\in\mathcal{M}(\mathbf{E})} \{L_{1w}, L_{2w}\}\right) \leq 2 + DR.$$

Let $\mathcal{L} = (L_{iw} : w \in \mathcal{M}(\mathbf{E}),\, i = 1, 2)$. Define the tuple of reals $\mathbf{c} = (c_{iw} : w \in \mathcal{M}(\mathbf{E}),\, i = 1, 2)$ by

$$c_{iw} = A^{-1}\left(e_{iw} - \frac{1}{2}\sum_{j=1}^{2} e_{jw}\right).$$

We apply Theorem 3.2 with $\mathcal{L}$, $\mathbf{c}$, $r = 2 + DR(\geq 4)$, and

$$\delta = \frac{\varepsilon}{2A}.$$

It is easy to check the condition (3.4). We verify the condition (3.6). The first statement is clear by the definition of c_{iw} and e_{iw}. The second statement follows from $c_{1w} + c_{2w} = 0$ for each $w \in \mathcal{M}(\mathbf{E})$. Moreover, by using (4.4) and (4.8), we obtain

$$\begin{aligned} A\sum_{w\in\mathcal{M}(\mathbf{E})} \max\{c_{1w}, c_{2w}\} &= \sum_{w\in\mathcal{M}(\mathbf{E})} \max\{e_{1w}, e_{2w}\} - \frac{1}{2}\sum_{w\in\mathcal{M}(\mathbf{E})} \sum_{j=1}^{2} e_{jw} \\ &= \sum_{v\in S} \sum_{\substack{w\in\mathcal{M}(\mathbf{E})\\ w|v}} \max\{d_w e_{1v}, d_w e_{2v}\} + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \sum_{v\in S} \max\{0, e_{1v}, e_{2v}\} \sum_{\substack{w\in\mathcal{M}(\mathbf{E})\\ w|v}} d_w \\ &\leq 1 + \sum_{v\in S} \max\{0, e_{1v}, e_{2v}\} = A. \end{aligned}$$

Therefore we proved the last inequality of (3.6).

Let $\mathbf{x} \in O_{\mathbf{K}}^2$ be a solution of (4.5) with (4.6). Then $\mathbf{x}$ also fulfills (4.7). Put

$$Q = \Psi(\mathbf{x})^A.$$

Finally, we show that such an $\mathbf{x}$ satisfies (3.10) and (3.12). By (4.7) and the definition of c_{iw},

$$\begin{aligned} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}} &= ||L_{iw}(\mathbf{x})||_w \Psi(\mathbf{x})^{-e_{iw}} \Psi(\mathbf{x})^{(e_{1w}+e_{2w})/2} \\ &\leq \Psi(\mathbf{x})^{(e_{1w}+e_{2w})/2} \end{aligned}$$

for $w \in \mathcal{M}(\mathbf{E})$, $i = 1, 2$. By taking product over $w \in \mathcal{M}(\mathbf{E})$ and using (4.8), we get

$$\begin{aligned} H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) &= \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \leq i \leq 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}} \\ &\leq \prod_{w \in \mathcal{M}(\mathbf{E})} \Psi(\mathbf{x})^{(e_{1w}+e_{2w})/2} \\ &= \Psi(\mathbf{x})^{-\varepsilon/2} = Q^{-\delta}. \end{aligned}$$

Thus (3.10) is verified. Put

$$\bigcup_{w \in \mathcal{M}(\mathbf{E})} \{L_{1w}, L_{2w}\} = \{L_1, \ldots, L_s\},$$

where $s \leq r$. We check $H^*(L_{iw}) \leq H$ for $w \in \mathcal{M}(\mathbf{E})$ and $i = 1, 2$. We may assume that $w \in \widetilde{S}$. There exists an automorphism σ_w of $\mathbf{E}$ such that $L_{iw} = \sigma_w^{-1}(L_{iv})$, where $v \in S$ is the place below w. By (4.1) and (4.3)

$$\begin{aligned} H^*(L_{iw}) &= \prod_{u \in \mathcal{M}(\mathbf{E})} \max\{1, ||\sigma_w^{-1}(L_{iv})||_u\} \\ &= \prod_{u \in \mathcal{M}(\mathbf{E})} \max\{1, ||L_{iv}||_u\} = H^*(L_{iv}) \leq H. \end{aligned}$$

Let $\widetilde{D} = [\mathbf{E} : \mathbb{Q}]$ and $1 \leq i < j \leq s$. If w is an Archimedean place, then

$$\begin{aligned} ||\det(L_i, L_j)||_w &\leq 2^{[\mathbf{E}_w:\mathbb{R}]/\widetilde{D}} ||L_i||_w ||L_j||_w \\ &\leq 2^{[\mathbf{E}_w:\mathbb{R}]/\widetilde{D}} \prod_{l=1}^{s} \max\{1, ||L_l||_w\}. \end{aligned}$$

Similarly, if w is non-Archimedean, then by the ultrametric inequality

$$||\det(L_i, L_j)||_w \leq ||L_i||_w ||L_j||_w \leq \prod_{l=1}^{s} \max\{1, ||L_l||_w\}.$$

Since $\sum_{w \in \mathcal{M}_{arc}(\mathbf{E})} [\mathbf{E}_w : \mathbb{R}] = \widetilde{D}$, we conclude that

$$\begin{aligned} \mathcal{H}(\mathcal{L}) &= \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \leq i < j \leq s} ||\det(L_i, L_j)||_w \\ &\leq \prod_{w \in \mathcal{M}_{arc}(\mathbf{E})} 2^{[\mathbf{E}_w:\mathbb{R}]/\widetilde{D}} \prod_{l=1}^{s} H^*(L_l) \leq 2H^r, \end{aligned}$$

hence

$$\begin{aligned}\max\left\{\mathcal{H}(\mathcal{L})^{1/\binom{r}{2}}, 2^{2/\delta}\right\} &\leq \max\left\{2^{1/\binom{r}{2}} H^{r/\binom{r}{2}}, 2^{4A/\varepsilon}\right\} \\ &\leq \max\left\{2H, 2^{4/\varepsilon}\right\}^A < \Psi(\mathbf{x})^A = Q.\end{aligned}$$

Let $t_1 = t_1(r, \delta)$ be defined as (3.11). Theorem 3.2 implies the following: there are proper subspaces $T_1, \ldots, T_{t_1}$ of $\overline{\mathbb{Q}}$ all defined over $\mathbf{E}$ such that any solution $\mathbf{x} \in O_{\mathbf{K}}^2$ of (4.5) with (4.6) satisfies

$$\mathbf{x} \in \bigcup_{i=1}^{t_1} (T_i \cap \mathbf{K}^2).$$

Therefore, for the proof of the proposition it suffices to check

$$t_1 \leq 2^{31} A^4 \varepsilon^{-3} \log(2RD) \log\left(\varepsilon^{-1} \log(2RD)\right) \tag{4.9}$$

Since $DR \geq 2$, we have

$$\log(2r) \leq 2 \log(2DR).$$

Moreover, by $0 < \varepsilon \leq 1/2$

$$\begin{aligned}\log\left(\delta^{-1} \log(2r)\right) &\leq \log\left(4A\varepsilon^{-1} \log(2DR)\right) \\ &\leq 4A \log\left(\varepsilon^{-1} \log(2DR)\right).\end{aligned}$$

Thus (4.9) follows. □

5 Proof of main results

We give another proof of the in equality (1.4). Without loss of generality, we may assume that $1/\alpha \leq \xi < 1$. In fact, there is an integer R with $1/\alpha \leq \xi\alpha^R < 1$. Then since $\xi\alpha^n = (\xi\alpha^R)\alpha^{n-R}$, we have

$$|\lambda_N(\alpha, \xi) - \lambda_N(\alpha, \xi\alpha^R)| \leq |R|.$$

In particular, for any $n \leq 0$, $[\xi\alpha^n] = 0$, and so $s_{-n}(\xi) = 0$ by (2.1).

Recall that $s_{-n}(\xi) \neq 0$ for infinitely many positive n, which we introduced in Section 1. Define the increasing sequence of positive integers $(n_j)_{j=1}^{\infty}$ by $s_{-n}(\xi) \neq 0$ if and only if $n = n_j$ for some $j \geq 1$. By the first statement of Lemma 2.1, it suffices to show that

$$\liminf_{j \to \infty} \frac{j}{\log n_j} \geq \left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha) - \log(a_d\alpha)}\right)\right)^{-1}.$$

Write the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Without loss of generality, we may assume that

$$|\alpha_k| > 1 \ (1 \leq k \leq p),$$

where p is the number of the conjugates of α whose absolute values are greater than 1. In what follows, $C_1(\alpha), C_2(\alpha), \ldots$ denote positive constants depending only on α. We first check

$$C_1(\alpha)\alpha^{n_j} \le \left|\sum_{i=0}^{n_j} \alpha^i s_{i-n_j}(\xi)\right| \le C_2(\alpha)\alpha^{n_j} \tag{5.1}$$

for any sufficiently large $j \ge 1$. By using $s_m(\xi) = 0$ for any $m > 0$ and the second statement of Proposition 2.2, we get

$$\begin{aligned}
\sum_{i=0}^{n_j} \alpha^i s_{i-n_j}(\xi) &= \sum_{i=0}^{\infty} \alpha^i s_{i-n_j}(\xi) \\
&= \alpha^{n_j} \sum_{i=-\infty}^{\infty} \alpha^{i-n_j} s_{i-n_j}(\xi) - \sum_{i=-\infty}^{-1} \alpha^i s_{i-n_j}(\xi) \\
&= -\xi\alpha^{-1+n_j}(P_\alpha^*)'\left(\frac{1}{\alpha}\right) - \sum_{i=-\infty}^{-1} \alpha^i s_{i-n_j}(\xi),
\end{aligned}$$

where $(P_\alpha^*)'$ is defined in Proposition 2.2. Thus

$$\begin{aligned}
\left|\sum_{i=0}^{n_j} \alpha^i s_{i-n_j}(\xi) + \xi\alpha^{-1+n_j}(P_\alpha^*)'\left(\frac{1}{\alpha}\right)\right| &\le \max\{L_+(\alpha), L_-(\alpha)\} \sum_{i=-\infty}^{-1} \alpha^i \\
&\le C_3(\alpha).
\end{aligned}$$

By considering $(P_\alpha^*)'(1/\alpha) \neq 0$, we obtain (5.1). Recall that, for any nonempty subset I of $\{1, 2, \ldots, d\}$, the number

$$a_d \prod_{k \in I} \alpha_k$$

is an algebraic integer (for example, see pages 71 and 72 of [20]). So (5.1) implies that

$$1 \le \left| a_d^{n_j} \prod_{k=1}^{d} \left(\sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi) \right) \right| \tag{5.2}$$

since the right-hand side of this inequality is the absolute value of a nonzero rational integer. By the first statement of Proposition 2.2, for $2 \le k \le p$,

$$\left|\sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi)\right| = \left| - \sum_{i=1+n_j}^{\infty} \alpha_k^i s_{i-n_j}(\xi) - \sum_{i=-\infty}^{-1} \alpha_k^i s_{i-n_j}(\xi) \right|.$$

Because $s_{i-n_j}(\xi) = 0$ for each i with $i \ge 1 + n_j$, we have

$$\begin{aligned}
\left|\sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi)\right| &\le \max\{L_+(\alpha), L_-(\alpha)\} \sum_{i=-\infty}^{n_j - n_{1+j}} |\alpha_k^i| \\
&\le C_4(\alpha)|\alpha_k|^{n_j - n_{1+j}}.
\end{aligned} \tag{5.3}$$

Similarly, if $p+1 \le k \le d$, then

$$\left|\sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi)\right| \le C_5(\alpha) n_j. \tag{5.4}$$

Take an arbitrary positive ε. By combining (5.1), (5.2), (5.3), and (5.4), we conclude for sufficiently large j that

$$\begin{aligned} 1 &\le C_6(\alpha) a_d^{n_j} \alpha^{n_j} \left(\prod_{k=2}^{p} |\alpha_k|^{n_j - n_{1+j}}\right) \left(\prod_{k=1+p}^{d} n_j\right) \\ &\le |\alpha_2 \cdots \alpha_p|^{-n_{1+j}} \left((1+\varepsilon) M(\alpha)\right)^{n_j}. \end{aligned}$$

Hence, for $j \ge j_0$,

$$\frac{n_{1+j}}{n_j} \le \frac{\log((1+\varepsilon)M(\alpha))}{\log M(\alpha) - \log(a_d \alpha)} =: F_1(\varepsilon)$$

and

$$n_j \le n_{j_0} F_1(\varepsilon)^{j-j_0}.$$

Therefore we conclude that

$$\liminf_{j\to\infty} \frac{j}{\log n_j} \ge \frac{1}{\log F_1(\varepsilon)}.$$

Since ε is an arbitrary positive number, (1.4) is proved.

Proof of Theorem 1.2. Theorem 3 of [11] shows for infinitely many $n \ge 0$ that $s_{-n}(\xi) \ne 0$. There exists the unique increasing sequence of positive integers $(n_j)_{j=1}^{\infty}$ such that $s_{-n}(\xi) \ne 0$ if and only if $n = n_j$ for some $j \ge 1$. Put

$$\xi' = \sum_{i=-\infty}^{-1} \alpha^i s_i(\xi) \text{ and } \xi_j = \sum_{i=-n_j}^{-1} \alpha^i s_i(\xi).$$

We may assume $\xi \in [1/\alpha, 1)$. Then $s_n(\xi) = 0$ for any $n \ge 0$. By Proposition 2.2, $\xi' \notin \mathbb{Q}(\alpha)$. Thus we get $\xi' \ne \xi_j$ for any $j \ge 1$. Recall that ∞ is the archimedean place defined by (4.2). In what follows, let $C_1(\alpha)$, $C_2(\alpha), \ldots$ be positive constants depending only on α. Then

$$\begin{cases} 0 < ||\xi' - \xi_j||_\infty \le C_1(\alpha) \alpha^{-n_{1+j}/d}, \\ ||\xi_j||_\infty \le C_1(\alpha). \end{cases} \tag{5.5}$$

Take an arbitrary positive number ε. Apply Theorem 3.1 with

$$\mathbf{K} = \mathbb{Q}(\alpha),\ S = \mathcal{M}_{arc}(\mathbf{K}) \cup \left\{v \in \mathcal{M}_{non}(\mathbf{K}) \middle| ||\alpha||_v < 1\right\},$$

and

$$\theta_v = \begin{cases} 1/\xi' & (\text{if } v = \infty), \\ 0 & (\text{otherwise}). \end{cases}$$

Consider solutions γ of (3.3) satisfying $\gamma = 1/\xi_j$ for some j. Let us take any $j_0 \geq 0$. By (5.5) there exists at most finitely many $j \geq 1$ with $\xi_j = \xi_{j_0}$. Thus by Theorem 3.1 there exist at most finitely many j such that $\gamma = 1/\xi_j$ fulfills (3.3). Namely, for all sufficiently large j,

$$\prod_{v \in S} \min\left\{1, \left|\left|\theta_v - \frac{1}{\xi_j}\right|\right|_v\right\} \geq \ H\left(\frac{1}{\xi_j}\right)^{-2-\varepsilon} = H(\xi_j)^{-2-\varepsilon}. \tag{5.6}$$

We have

$$\prod_{v \in S} \min\left\{1, \left|\left|\theta_v - \frac{1}{\xi_j}\right|\right|_v\right\} = \left|\left|\frac{1}{\xi'} - \frac{1}{\xi_j}\right|\right|_\infty \prod_{v \in S \setminus \{\infty\}} \min\left\{1, \left|\left|\frac{1}{\xi_j}\right|\right|_v\right\}$$
$$= \left|\left|\frac{1}{\xi'} - \frac{1}{\xi_j}\right|\right|_\infty \max\{1, ||\xi_j||_\infty\} \left(\prod_{v \in S} \max\{1, ||\xi_j||_v\}\right)^{-1}$$
$$\leq C_1(\alpha)^2 \alpha^{-n_{1+j}/d} \left(\prod_{v \in S} \max\{1, ||\xi_j||_v\}\right)^{-1}.$$

Note that if $v \in \mathcal{M}(\mathbf{K}) \backslash S$, then $||\xi_j||_v \leq 1$ by the ultrametric inequality. Hence

$$\prod_{v \in S} \min\left\{1, \left|\left|\theta_v - \frac{1}{\xi_j}\right|\right|_v\right\}$$
$$\leq C_1(\alpha)^2 \alpha^{-n_{1+j}/d} H(\xi_j)^{-1} \prod_{v \in \mathcal{M}(\mathbf{K}) \backslash S} \max\{1, ||\xi_j||\}$$
$$\leq C_1(\alpha)^2 \alpha^{-n_{1+j}/d} H(\xi_j)^{-1}.$$

By combining the inequality above and (5.6), we obtain, for any sufficiently large j,

$$\alpha^{n_{1+j}} \leq C_1(\alpha)^{2d} H(\xi_j)^{(1+\varepsilon)d}.$$

Write the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Let p (resp. q) be the number of the conjugates of α whose absolute values are greater (resp. smaller) than 1. Without loss of generality we may assume $|\alpha_k| > 1$ if $1 \leq k \leq p$, $|\alpha_k| < 1$ if $p + 1 \leq k \leq p + q$, and $|\alpha_k| = 1$ otherwise. By the ultrametric inequality

$$\prod_{v \in \mathcal{M}_{non}(\mathbf{K})} \max\{1, ||\xi_j||_v\} \quad \leq \prod_{v \in \mathcal{M}_{non}(\mathbf{K})} \max\{1, ||\alpha^{-1}||_v\}^{n_j}.$$

Since $s_n(\xi) \leq \max\{L_+(\alpha), L_-(\alpha)\}$ for every integer n

$$\prod_{k=p+1}^{d} \left|\sum_{i=-n_j}^{-1} \alpha_k^i s_i(\xi)\right|^{1/d} \leq \prod_{k=p+1}^{p+q} \left|C_2(\alpha)\alpha_k^{-n_j}\right|^{1/d} \prod_{k=p+q+1}^{d} (C_2(\alpha)n_j)^{1/d}$$
$$= C_2(\alpha)^{(d-p)/d} n_j^{(d-p-q)/d} \prod_{k=p+1}^{p+q} |\alpha_k|^{n_j/d}.$$

By using the first statement of Proposition 2.2 and $s_n(\xi) = 0$ for any $n \geq 0$,

$$\begin{aligned}
\prod_{k=1}^{p} \left| \sum_{i=-n_j}^{-1} \alpha_k^i s_i(\xi) \right|^{1/d} &= \left| \sum_{i=-n_j}^{-1} \alpha^i s_i(\xi) \right|^{1/d} \prod_{k=2}^{p} \left| \sum_{i=-\infty}^{-n_{1+j}} \alpha_k^i s_i(\xi) \right|^{1/d} \\
&\leq C_3(\alpha)^{1/d} \prod_{k=2}^{p} \left| C_3(\alpha) \alpha_k^{-n_{1+j}} \right|^{1/d} \\
&= C_3(\alpha)^{p/d} \prod_{k=2}^{p} \left| \alpha_k^{-n_{1+j}} \right|^{1/d}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\prod_{v \in \mathcal{M}_{arc}(\mathbf{K})} \max\{1, ||\xi_j||_v\} = \prod_{k=1}^{d} \left| \sum_{i=-n_j}^{-1} \alpha_k^i s_i(\xi) \right|^{1/d} \\
&\leq C_4(\alpha) n_j^{(d-p-q)/d} \prod_{v \in \mathcal{M}_{arc}(\mathbf{K})} \max\{1, ||\alpha^{-1}||_v\}^{n_j} \prod_{k=2}^{p} \left| \alpha_k^{-n_{1+j}} \right|^{1/d}
\end{aligned}$$

and so

$$H(\xi_j) \leq C_4(\alpha) n_j^{(d-p-q)/d} H(\alpha^{-1})^{n_j} \prod_{k=2}^{p} \left| \alpha_k^{-n_{1+j}} \right|^{1/d}.$$

Finally, we conclude for sufficiently large j that

$$\begin{aligned}
\alpha^{n_{1+j}} &\leq C_5(\alpha) n_j^{(1+\varepsilon)(d-p-q)} H(\alpha^{-1})^{(1+\varepsilon)dn_j} \prod_{k=2}^{p} \left| \alpha_k^{-n_{1+j}} \right|^{(1+\varepsilon)} \\
&\leq H(\alpha^{-1})^{(1+2\varepsilon)dn_j} \prod_{k=2}^{p} \left| \alpha_k^{-n_{1+j}} \right|^{(1+\varepsilon)} \\
&= M(\alpha)^{(1+2\varepsilon)n_j} \prod_{k=2}^{p} \left| \alpha_k^{-n_{1+j}} \right|^{(1+\varepsilon)},
\end{aligned}$$

where for the last equality we use (3.1). Taking logarithms of both sides of the inequality above, we get

$$\frac{n_{1+j}}{n_j} \leq \frac{(1+2\varepsilon) \log M(\alpha)}{\log \alpha + (1+\varepsilon) \log |\alpha_2 \cdots \alpha_p|} =: F_2(\varepsilon), \tag{5.7}$$

consequently

$$\liminf_{j \to \infty} \frac{j}{\log n_j} \geq \frac{1}{\log F_2(\varepsilon)}.$$

Therefore by the first statement of Lemma 2.1

$$\liminf_{N \to \infty} \frac{\lambda_N(\alpha, \xi)}{\log N} \geq \frac{1}{\log F_2(\varepsilon)}.$$

Since ε is an arbitrary positive number, we proved the theorem. In fact,

$$\lim_{\varepsilon \to +0} \frac{1}{\log F_2(\varepsilon)} = \left(\log \left(\frac{\log M(\alpha)}{\log M(\alpha) - \log a_d} \right) \right)^{-1}.$$

□

Proof of Theorem 1.1. We may assume $\xi \in [1/\alpha, 1)$, and so $s_n(\xi) = 0$ for any $n \geq 0$. Put

$$\xi' = \sum_{i=-\infty}^{-1} \alpha^i s_i(\xi). \tag{5.8}$$

By the second statement of Proposition 2.2, we have $\xi' \notin \mathbb{Q}(\alpha)$. Let p be the number of the conjugates of α whose absolute values are greater than 1. Write the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$, where d is the degree of α. Without loss of generality, we may assume that $|\alpha_k| > 1$ for $k = 1, 2, \ldots, p$.

First we show the following:

LEMMA 5.1. *There is a sequence of integers $\mathbf{y} = (y_n)_{n=1}^{\infty}$ satisfying the following:*

1. $y_n = 0$ *or* $y_n = s_{-n}(\xi)$*;*
2. $\sum_{i=1}^{\infty} y_i \alpha^{-i} = \xi'$*;*
3. $\sum_{i=1}^{\infty} y_i \alpha_k^{-i} = 0$ *for any* k *with* $2 \leq k \leq p$*;*
4. *Put*

$$\{n \geq 1 | y_n \neq 0\} =: \{n_1 < n_2 < \cdots\}$$

and

$$\xi_j = \sum_{i=1}^{n_j} y_i \alpha^{-i}.$$

Then, for any h and l with $h < l$, $\xi_h \neq \xi_l$.

Proof. We construct the bounded sequences of integers $\mathbf{y}_m = (y(m,n))_{n=1}^{\infty}$ $(m = 1, 2, \ldots)$ by induction on m fulfilling the following:

1. For any $n \geq 1$,

$$y(m,n) = 0 \text{ or } y(m,n) = s_{-n}(\xi); \tag{5.9}$$

2.

$$\sum_{i=1}^{\infty} \alpha^{-i} y(m,i) = \xi'; \tag{5.10}$$

3. For any k with $2 \leq k \leq p$,

$$\sum_{i=1}^{\infty} \alpha_k^{-i} y(m,i) = 0. \tag{5.11}$$

In particular, we have, for any $m, n \geq 1$,

$$|y(m,n)| \leq |s_{-n}(\xi)| \leq \max\{L_+(\alpha), L_-(\alpha)\}.$$

Define $\mathbf{y}_1 = (y(1,n))_{n=1}^{\infty}$ by

$$y(1,n) = s_{-n}(\xi) \ (n \geq 1).$$

For $m = 1$, (5.9) and (5.10) hold. Moreover, (5.11) follows from the first statement of Proposition 2.2.

Next, assume that we have a sequence of integers $\mathbf{y}_m$ with (5.9), (5.10), and (5.11) for $m \geq 1$. Let

$$\begin{aligned} \Xi_m &= \{n \geq 1 | y(m,n) \neq 0\} \\ &=: \{n(m,1) < n(m,2) < \cdots\} \end{aligned}$$

and

$$\xi(m,j) = \sum_{i=1}^{n(m,j)} \alpha^{-i} y(m,i).$$

By (5.10) and $\xi' \notin \mathbb{Q}(\alpha)$, Ξ_m is an infinite set. If $\xi(m,h) \neq \xi(m,l)$ for any $h \neq l$, then then $\mathbf{y} = \mathbf{y}_m$ satisfies the last condition of Lemma 5.1. Moreover, first, second, and third conditions of Lemma 5.1 follow immediately from (5.9), (5.10), and (5.11). Otherwise, we define $\mathbf{y}_{m+1}$ by using $\mathbf{y}_m$. There exists an $h \geq 1$ such that $\xi(m,h) = \xi(m,l)$ for some $l > h$. For such an h, write the minimal value by h_m. Put

$$\Lambda_m = \{l > h_m | \xi(m,l) = \xi(m,h_m)\}.$$

Then Λ_m is a finite set. In fact, if Λ_m is an infinite set, then

$$\xi' = \lim_{\substack{n \to \infty \\ n \in \Lambda_m}} \xi(m,n) = \xi(m,h_m) \in \mathbb{Q}(\alpha),$$

which contradicts to $\xi \notin \mathbb{Q}(\alpha)$. So let

$$l_m = \max \Lambda_m.$$

We define $\mathbf{y}_{m+1} = (y(m+1,n))_{n=1}^{\infty}$ by

$$y(m+1,n) = \begin{cases} 0 & (\text{if } 1 + n(m,h_m) \leq n \leq n(m,l_m)), \\ y(m,n) & (\text{otherwise}). \end{cases}$$

Note that

$$\xi(m+1,j) = \begin{cases} \xi(m,j) & (\text{if } j \leq h_m) \\ \xi(m, j + l_m - h_m) & (\text{if } j > h_m). \end{cases} \tag{5.12}$$

Now we verify that $\mathbf{y}_{m+1}$ fulfills (5.9), (5.10), and (5.11). (5.9) is obvious by the definition of $\mathbf{y}_{m+1}$. By the inductive hypothesis and

$$0 = \xi(m, l_m) - \xi(m, h_m) = \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha^{-i} y(m, i), \tag{5.13}$$

we get

$$\sum_{i=1}^{\infty} \alpha^{-i} y(m+1, i) = \sum_{i=1}^{\infty} \alpha^{-i} y(m, i) - \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha^{-i} y(m, i) = \xi'.$$

By taking the conjugate of (5.13), we deduce for any k with $2 \le k \le p$ that

$$0 = \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha_k^{-i} y(m, i).$$

Thus

$$\begin{aligned} \sum_{i=1}^{\infty} \alpha_k^{-i} y(m+1, i) &= \sum_{i=1}^{\infty} \alpha_k^{-i} y(m, i) - \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha_k^{-i} y(m, i) \\ &= \sum_{i=1}^{\infty} \alpha_k^{-i} y(m, i) = 0. \end{aligned}$$

For the proof of Lemma 5.1 we may assume that, for any $m \ge 1$, $\mathbf{y} = \mathbf{y}_m$ is defined and does not satisfy the conditions of Lemma 5.1. We verify that $h_{m+1} > h_m$ for each $m \ge 1$. It suffices to check for $1 \le h < l$ with $h \le h_m$ that

$$\xi(m+1, l) \ne \xi(m+1, h).$$

In the case of $h < h_m$, this follows from (5.12) and the definition of h_m. So consider the case of $h = h_m$. Since $l + l_m - h_m > l_m$ we get

$$\xi(m+1, l) = \xi(m, l + l_m - h_m) \ne \xi(m, h_m) = \xi(m+1, h_m)$$

by the definition of l_m. Hence the sequence h_m $(m = 1, 2, \ldots)$ is strictly increasing. In particular, $h_m \ge m$.

Let $n \ge 1$. Take an integer m with $m \ge n$. Note that

$$n \le m \le h_m \le n(m, h_m).$$

So, by the definition of $\mathbf{y}_{m+1}$, we have $y(m+1, n) = y(m, n)$. Thus

$$y(m, n) = y(n, n) \text{ for any } m \ge n. \tag{5.14}$$

We define the sequence $\mathbf{y} = (y_n)_{n=1}^{\infty}$ by

$$y_n = y(n, n).$$

In what follows we check the conditions of Lemma 5.1. The first condition is clear. Let $m \geq 1$ be any integer. Then by (5.14)

$$\begin{aligned}
\left|\xi' - \sum_{i=1}^{\infty} \alpha^{-i} y_i\right| &= \left|\sum_{i=1}^{\infty} \alpha^{-i}(y(m,i) - y(i,i))\right| \\
&= \left|\sum_{i=m+1}^{\infty} \alpha^{-i}(y(m,i) - y(i,i))\right| \\
&\leq 2\max\{L_+(\alpha), L_-(\alpha)\}\frac{1}{(\alpha-1)\alpha^m}.
\end{aligned}$$

Similarly, for $2 \leq k \leq p$,

$$\begin{aligned}
\left|\sum_{i=1}^{\infty} \alpha_k^{-i} y_i\right| &= \left|\sum_{i=1}^{\infty} \alpha_k^{-i}(y(m,i) - y(i,i))\right| \\
&= \left|\sum_{i=m+1}^{\infty} \alpha_k^{-i}(y(m,i) - y(i,i))\right| \\
&\leq 2\max\{L_+(\alpha), L_-(\alpha)\}\frac{1}{(\alpha_k-1)\alpha_k^m},
\end{aligned}$$

where for the first equality we use (5.11). Since m is arbitrary, we obtain the second and third conditions.

Finally, assume that $\xi_h = \xi_l$ for some $h < l$. Take an integer m with $m > l$. Then by (5.14)

$$\xi(m,h) = \xi_h = \xi_l = \xi(m,l).$$

By the definition of h_m, we get $h_m \leq h < m$, which contradicts to $h_m \geq m$. Therefore, the last condition follows. □

For $N \geq 1$ put

$$\tau_N = \mathrm{Card}\{n \in \mathbb{Z} | 1 \leq n \leq N, y_n \neq 0\}.$$

By the first condition of Lemma 5.1

$$\tau_N \leq \mathrm{Card}\{n \in \mathbb{Z} | 1 \leq n \leq N, s_{-n}(\xi) \neq 0\}. \tag{5.15}$$

In what follows, we verify for all sufficiently large N that

$$\tau_N \geq c\frac{(\log \alpha)^2}{(\log M(\alpha))^2 (\log(6D))^{1/2}} \frac{(\log N)^{3/2}}{(\log\log N)^{1/2}}. \tag{5.16}$$

Theorem 1.1 follows from (5.15), (5.16), and the first statement of Lemma 2.1.

Put $\mathbf{K} = \mathbb{Q}(\alpha)$. Let ∞ be the Archimedean place in $\mathbf{K}$ which is defined by (4.2). In what follows, let $C_1(\alpha,\xi), C_2(\alpha,\xi), \ldots$ be positive constants depending only on α and ξ. Put

$$C_1(\alpha,\xi) = \max\{L_+(\alpha), L_-(\alpha)\}^{1/d} \max_{1\leq k\leq p}\left(\frac{1}{1-|\alpha_k^{-1}|}\right)^{1/d}.$$

Then we have

$$\begin{aligned} 0 < ||\xi' - \xi_j||_\infty &= \left| \sum_{i=n_{1+j}}^{\infty} y_i \alpha^{-i} \right|^{1/d} \\ &\le C_1(\alpha,\xi) ||\alpha||_\infty^{-n_{1+j}}. \end{aligned} \tag{5.17}$$

Let ε be an arbitrary positive number with $\varepsilon \le 1/2$ and $F_2(\varepsilon)$ be defined by (5.7). In the same way as the proof of Theorem 1.2, we can verify that

$$\frac{n_{1+j}}{n_j} \le F_2(\varepsilon)$$

for sufficiently large j. In particular, since

$$\lim_{\varepsilon \to +0} F_2(\varepsilon) = 1,$$

we get, for $j \ge C_2(\alpha, \xi)$,

$$n_{1+j} \le 2n_j. \tag{5.18}$$

We count the numbers of j fulfilling

$$n_{1+j} \ge (1+2\varepsilon) n_j. \tag{5.19}$$

Assume (5.19) and

$$n_j \ge C_3(\alpha,\xi)\varepsilon^{-9/8}. \tag{5.20}$$

We determine $C_3(\alpha, \xi)$ later. Let

$$S = \mathcal{M}_{arc}(\mathbf{K}) \cup \left\{ v \in \mathcal{M}_{non}(\mathbf{K}) \middle| ||\alpha||_v < 1 \right\}.$$

Define the linear forms $L_{i,v}$ ($v \in S$, $i = 1, 2$) by

$$\begin{aligned} L_{1v} &= \begin{cases} X - \xi' Y & \text{for } v = \infty, \\ X & \text{for } v \in S \backslash \{\infty\}, \end{cases} \\ L_{2v} &= Y \text{ for } v \in S. \end{aligned}$$

Then (4.3) is satisfied with $R = 3$, $D = [\mathbb{Q}(\alpha, \xi) : \mathbb{Q}(\alpha)]$, and $H = H(\xi)$. Consider the system of inequalities (4.5) with

$$\begin{aligned} e_{1v} &= \begin{cases} -(5\varepsilon)/4 & \text{for } v = \infty, \\ \varepsilon/(4d') & \text{for } v \in \mathcal{M}_{arc}(\mathbf{K}) \backslash \{\infty\}, \\ 0 & \text{for } v \in \mathcal{M}_{non}(\mathbf{K}) \cap S, \end{cases} \\ e_{2v} &= (\log ||\alpha||_v)/(\log ||\alpha||_\infty), \text{ for } v \in S \\ \Psi(x,y) &= ||y||_\infty, \end{aligned}$$

where $d' = \mathrm{Card}(\mathcal{M}_{arc}(\mathbf{K}) \backslash \{\infty\})$. Then (4.4) follows from the product formula. Apply Proposition 4.1 with

$$\mathbf{x}_j = \left(\sum_{i=1}^{n_j} y_i \alpha^{-i+n_j}, \alpha^{n_j} \right) \in O_{\mathbf{K}}^2.$$

If $C_3(\alpha,\xi)$ is sufficiently large, then (4.6) follows from (5.20). In fact,

$$\log\Psi(\mathbf{x}_j)=n_j\log||\alpha||_\infty>\max\{\log(2H),\frac{4}{\varepsilon}\log 2\}.$$

We check that $\mathbf{x}_j$ satisfies the system of inequalities (4.5). If $i=2$, then

$$\begin{aligned}||L_{2v}(\mathbf{x}_j)||_v &= ||\alpha^{n_j}||_v=||\alpha^{n_j}||_\infty^{(\log||\alpha||_v)/(\log||\alpha||_\infty)}\\ &= \Psi(\mathbf{x}_j)^{e_{2v}}.\end{aligned}$$

In the case of $v\in S\cap\mathcal{M}_{non}(\mathbf{K})$, by the ultrametric inequality

$$||L_{1v}(\mathbf{x}_j)||_v=\left\|\sum_{i=1}^{n_j}y_i\alpha^{-i+n_j}\right\|_v\le 1=\Psi(\mathbf{x}_j)^{e_{1v}}.$$

Now we show that

$$n_j^{1/d}C_1(\alpha,\xi)\le||\alpha^{n_j}||_\infty^{\varepsilon/(4d')}=\Psi(\mathbf{x}_j)^{\varepsilon/(4d')}. \tag{5.21}$$

(5.21) is equivalent to

$$\frac{4d'}{\log||\alpha||_\infty}\left(\frac{1}{d}+\frac{\log C_1(\alpha,\xi)}{\log n_j}\right)\le\frac{\varepsilon n_j}{\log n_j}.$$

In what follows, constants implied by the Vinogradov symbols $\ll$, $\gg$ are absolute. If $n_j\ge C_3(\alpha,\xi)\varepsilon^{-9/8}$, then

$$\frac{\varepsilon n_j}{\log n_j}\gg\varepsilon n_j^{8/9}\ge C_3(\alpha,\xi)^{8/9}.$$

Thus, if $C_3(\alpha,\xi)$ is sufficiently large, then (5.21) follows. By (5.17), (5.19), and (5.21), we get

$$\begin{aligned}||L_{1\infty}(\mathbf{x}_j)||_\infty &= ||\alpha^{n_j}||_\infty||\xi_j-\xi'||_\infty\\ &\le C_1(\alpha,\xi)||\alpha||_\infty^{n_j-n_{1+j}}\le C_1(\alpha,\xi)||\alpha^{n_j}||_\infty^{-2\varepsilon}\\ &\le ||\alpha^{n_j}||_\infty^{-(5\varepsilon)/4}=\Psi(\mathbf{x}_j)^{-(5\varepsilon)/4}.\end{aligned}$$

Let $v\in\mathcal{M}_{arc}(\mathbf{K})\backslash\{\infty\}$. Then there exists an embedding $\sigma:\mathbf{K}\hookrightarrow\mathbb{C}$ such that

$$||x||_v=||\sigma(x)||_\infty$$

for any $x\in\mathbf{K}$. Let $\sigma(\alpha)=\alpha_k$, where $2\le k\le d$. If $2\le k\le p$, then by (5.21) and the third condition of Lemma 5.1

$$\begin{aligned}||L_{1v}(\mathbf{x}_j)||_v &= \left\|\sum_{i=1}^{n_j}y_i\alpha_k^{-i+n_j}\right\|_\infty=\left\|\alpha_k^{n_j}\right\|_\infty\left\|\sum_{i=n_{1+j}}^{\infty}y_i\alpha_k^{-i}\right\|_\infty\\ &\le C_1(\alpha,\xi)\left\|\alpha_k^{n_j-n_{1+j}}\right\|_\infty\\ &\le C_1(\alpha,\xi)\le\Psi(\mathbf{x}_j)^{\varepsilon/(4d')}=\Psi(\mathbf{x}_j)^{e_{1v}}.\end{aligned}$$

In the case of $k \geq p+1$, by using $|\alpha_k| \leq 1$ and (5.21), we obtain

$$\begin{aligned} ||L_{1v}(\mathbf{x}_j)||_v &= \left\| \sum_{i=1}^{n_j} y_i \alpha_k^{-i+n_j} \right\|_\infty \\ &\leq |n_j \max\{L_+(\alpha), L_-(\alpha)\}|^{1/d} \leq n_j^{1/d} C_1(\alpha, \xi) \\ &\leq \Psi(\mathbf{x}_j)^{\varepsilon/(4d')} = \Psi(\mathbf{x}_j)^{e_{1v}}. \end{aligned}$$

Since

$$\begin{aligned} A &= 1 + \sum_{v \in S} \max\{0, e_{1v}, e_{2v}\} \\ &\leq 1 + \frac{\varepsilon}{4} + \sum_{k=1}^{p} \frac{\log ||\alpha_k||_\infty}{\log ||\alpha||_\infty} \ll \frac{\log M(\alpha)}{\log \alpha}. \end{aligned}$$

Proposition 4.1 indicates that the vectors $\mathbf{x}_j$ satisfying (5.19) and (5.20) lie in

$$\ll \left(\frac{\log M(\alpha)}{\log \alpha} \right)^4 \varepsilon^{-3} \log(6D) \log(\varepsilon^{-1} \log(6D))$$

one-dimensional linear subspaces of $\mathbf{K}^2$. By the last condition of Lemma 5.1, if $j \neq l$, then $\mathbf{x}_j$ and $\mathbf{x}_l$ are linearly independent over $\mathbf{K}$. Thus we obtain

$$\begin{aligned} \mathrm{Card}\{j \geq 0 | n_j \geq & C_3(\alpha, \xi) \varepsilon^{-9/8}, n_{1+j} \geq (1+2\varepsilon) n_j\} \\ & \ll \left(\frac{\log M(\alpha)}{\log \alpha} \right)^4 \varepsilon^{-3} \log(6D) \log(\varepsilon^{-1} \log(6D)). \quad (5.22) \end{aligned}$$

Let j_1 be the smallest j such that $n_j \geq C_2(\alpha, \xi)$ and J an integer with

$$J > \max\{n_{j_1}^3, 2^{12} C_3(\alpha, \xi)^{12}\}. \quad (5.23)$$

Moreover, let j_2 be the largest integer with $n_{j_2} \leq 2C_3(\alpha, \xi) J^{5/12}$. Then since

$$n_{j_1} \leq J^{1/3} \leq 2C_3(\alpha, \xi) J^{5/12},$$

we get

$$n_{j_2} \geq n_{j_1} \geq C_2(\alpha, \xi). \quad (5.24)$$

So by (5.18)

$$n_{j_2} \geq \frac{n_{1+j_2}}{2} \geq C_3(\alpha, \xi) J^{5/12}.$$

For a positive integer $u (\geq 2)$, put

$$\varepsilon_1 = \frac{(\log(6D))^{1/3} (\log M(\alpha))^{4/3}}{(\log \alpha)^{4/3}} \left(\frac{\log J}{J} \right)^{1/3}, \ \varepsilon_u = \frac{1}{2}.$$

Note that if $C_3(\alpha, \xi)$ is sufficiently large, then $\log(\varepsilon_1^{-1}) \geq \log(6D)$. Next, let $\varepsilon_2, \ldots, \varepsilon_{u-1}$ be any reals satisfying

$$\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{u-1} < \varepsilon_u.$$

Then we have

$$n_{j_2} \geq C_3(\alpha,\xi)\varepsilon_h^{-9/8} \tag{5.25}$$

for $h = 1,\ldots,u$. In fact,

$$n_{j_2} C_3(\alpha,\xi)^{-1}\varepsilon_h^{9/8} \geq J^{5/12}\varepsilon_1^{9/8} \geq J^{5/12}\cdot J^{-3/8} \geq 1.$$

Let $\mathcal{S}_0 = \{j_2, 1+j_2,\ldots,J\}$ and, for $h = 1,\ldots,u$, let $\mathcal{S}_h$ denote the set of positive integers j such that $j_2 \leq j < J$ and $n_{1+j} \geq (1+2\varepsilon_h)n_j$. Moreover, write the cardinality of $\mathcal{S}_h$ by T_h for $h = 1,\ldots,u$. Then $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_u$. If $j \in \mathcal{S}_0$, then by (5.24) and (5.25) we have

$$\frac{n_{1+j}}{n_j} \leq 2$$

and

$$n_j \geq C_3(\alpha,\xi)\varepsilon_h^{-9/8} \text{ for } h = 1,\ldots,u.$$

Thus we get

$$\begin{aligned}\frac{n_J}{n_{j_2}} &= \frac{n_J}{n_{-1+J}}\frac{n_{-1+J}}{n_{-2+J}}\cdots\frac{n_{1+j_2}}{n_{j_2}}\\ &\leq \left(\prod_{j\in\mathcal{S}_u}\frac{n_{1+j}}{n_j}\right)\prod_{h=0}^{u-1}\left(\prod_{j\in\mathcal{S}_h\setminus\mathcal{S}_{1+h}}\frac{n_{1+j}}{n_j}\right)\\ &\leq 2^{T_u}(1+2\varepsilon_1)^J\prod_{h=1}^{u-1}(1+2\varepsilon_{h+1})^{T_h-T_{1+h}}.\end{aligned}$$

Taking logarithms, we obtain

$$\begin{aligned}\log\left(\frac{n_J}{n_{j_2}}\right) &\leq T_u\log 2 + 2\varepsilon_1 J + \sum_{h=1}^{u-1}2\varepsilon_{1+h}(T_h - T_{1+h})\\ &\leq T_u\log 2 + 2\varepsilon_1 J + 2\varepsilon_2 T_1 + 2\sum_{h=2}^{u-1}(\varepsilon_{1+h}-\varepsilon_h)T_h - T_u.\end{aligned}$$

(5.22) implies

$$T_h \ll \left(\frac{\log M(\alpha)}{\log\alpha}\right)^4\log(6D)\varepsilon_h^{-3}\log(\varepsilon_h^{-1}\log(6D))$$

for $h = 1,\ldots,u$, and so

$$\begin{aligned}\log\left(\frac{n_J}{n_{j_2}}\right) &\ll \varepsilon_1 J + \left(\frac{\log M(\alpha)}{\log\alpha}\right)^4\log(6D)\\ &\quad\times\Bigg(\log(\log(6D)) + \varepsilon_2\varepsilon_1^{-3}\log(\varepsilon_1^{-1}\log(6D))\\ &\quad\quad+\sum_{h=2}^{u-1}(\varepsilon_{1+h}-\varepsilon_h)\varepsilon_h^{-3}\log(\varepsilon_h^{-1}\log(6D))\Bigg)\end{aligned}$$

If u tends to infinity and if $\max_{1\le h\le u-1}(\varepsilon_{1+h}-\varepsilon_h)$ tends to zero, then the sum converges to a Riemann integral, so

$$\begin{aligned}
&\lim_{u\to\infty}\sum_{h=2}^{u-1}(\varepsilon_{1+h}-\varepsilon_h)\varepsilon_h^{-3}\log(\varepsilon_h^{-1}\log(6D))\\
&=\int_{\varepsilon_1}^{1/2} x^{-3}\log(x^{-1}\log(6D))dx\\
&\ll \varepsilon_1^{-2}\log(\varepsilon_1^{-1})+\log(\log(6D))\varepsilon_1^{-2}\ll \varepsilon_1^{-2}\log(\varepsilon_1^{-1}).
\end{aligned}$$

Thus

$$\begin{aligned}
\log\left(\frac{n_J}{n_{j_2}}\right) &\ll \varepsilon_1 J+\left(\frac{\log M(\alpha)}{\log\alpha}\right)^4\log(6D)\varepsilon_1^{-2}\log(\varepsilon_1^{-1})\\
&\ll \left(\frac{\log M(\alpha)}{\log\alpha}\right)^{4/3}(\log(6D))^{1/3}J^{2/3}(\log J)^{1/3},
\end{aligned}$$

where for the second inequality we use $\log(\varepsilon_1^{-1})\le\log J$. By using (5.23) and the definition of j_2 we obtain

$$\frac{n_{j_2}}{n_J^{1/2}}\le\frac{n_{j_2}}{J^{1/2}}\le 2C_3(\alpha,\xi)J^{-1/12}\le 1,$$

and so

$$\log n_J\ll\left(\frac{\log M(\alpha)}{\log\alpha}\right)^{4/3}(\log(6D))^{1/3}J^{2/3}(\log J)^{1/3}=:G(J).$$

Hence

$$J\gg\frac{(\log\alpha)^2}{(\log M(\alpha))^2(\log(6D))^{1/2}}\frac{(\log n_J)^{3/2}}{(\log\log n_J)^{1/2}}.$$

In fact, since the function $x^{3/2}(\log x)^{-1/2}$ is monotone increasing for $x>e$,

$$\frac{(\log n_J)^{3/2}}{(\log\log n_J)^{1/2}}\ll\frac{(\log G(J))^{3/2}}{(\log\log G(J))^{1/2}}\ll\left(\frac{\log M(\alpha)}{\log\alpha}\right)^2(\log(6D))^{1/2}J.$$

Therefore, we proved the theorem. □

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Department of Mathematics, Kyoto University
Oiwake-tyou
Kitashirakawa
Kyoto-shi, Kyoto, Japan
e-mail: kanekoha@math.kyoto-u.ac.jp