On the number of nonzero digits in the beta-expansions of algebraic numbers *

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Abstract

Many mathematicians have investigated the base-b expansions for integral base-$b \geq 2$, and more general β-expansions for a real number $\beta > 1$. However, little is known on the β-expansions of algebraic numbers. The main purpose of this paper is to give new lower bounds for the numbers of nonzero digits in the β-expansions of algebraic numbers under the assumption that β is a Pisot or Salem number. As a consequence of our main results, we study the arithmetical properties of power series $\sum_{n=1}^{\infty} \beta^{-\kappa(z;n)}$, where $z > 1$ is a real number and $\kappa(z;n) = \lfloor n^z \rfloor$.

1 Normality of the digits in β-expansions

In this paper, let $\mathbb{N}$ (resp. $\mathbb{Z}^+$) be the set of nonnegative integers (resp. positive integers). We denote the integral and fractional parts of a real number x by $\lfloor x \rfloor$ and $\{x\}$, respectively. Moreover, we write the minimal integer n not less than x by $\lceil x \rceil$. We denote the length of a nonempty finite word $W = w_1 w_2 \ldots w_k$ on a certain alphabet $\mathcal{A}$ by $|W| = k$. We use the Landau symbol O and the Vinogradov symbols $\gg, \ll$ with their usual meaning.

For a real number β greater than 1, let $T_\beta : [0,1] \to [0,1)$ be the β-transformation defined by $T_\beta(x) := \{\beta x\}$. Using the β-transformation, Rényi [22] generalized the notion of the base-b expansions of real numbers for an integral base b as follows: Let x be a real number with $0 \leq x \leq 1$. Putting $t_n(\beta, x) := \lfloor \beta T_\beta^{n-1}(x) \rfloor$ for any positive integer n, we have

$$x = \sum_{n=1}^{\infty} t_n(\beta, x) \beta^{-n}. \tag{1.1}$$

The right-hand side of (1.1) is called the β-expansion of x. In what follows, we assume that $0 \leq x \leq 1$ when we consider the β-expansion of x. We have that $t_n(\beta, x) \leq \lfloor \beta \rfloor$. In particular, if $\beta = b$ is a rational integer, then we see $t_n(b, x) \leq b - 1$ except the only case of $t_1(b; 1) = b$.

Parry [21] showed that the digits $t_n(\beta, x)$ for $x < 1$ are characterized by the expansion of 1. Put

$$t_n(\beta, 1-) := \lim_{x \to 1-0} t_n(\beta, x)$$

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for any positive integer n. Then we have

$$1 = \sum_{n=1}^{\infty} t_n(\beta, 1-)\beta^{-n}.$$

For any real number $x \le 1$, let $\boldsymbol{t}(\beta, x)$ be the right-infinite sequence defined by

$$\boldsymbol{t}(\beta, x) := t_1(\beta, x)t_2(\beta, x)\ldots.$$

We also define $\boldsymbol{t}(\beta, 1-)$ in the same way. Consider the case where the sequence $\boldsymbol{t}(\beta, 1)$ is finite, namely, there exists a finite word $a_1 \ldots a_M$ on the alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ with $a_M \neq 0$ such that

$$\boldsymbol{t}(\beta, 1) = a_1 \ldots a_M 00 \ldots.$$

Then it is known that

$$\boldsymbol{t}(\beta, 1-) = a_1 \ldots a_{M-1}(a_M - 1)a_1 \ldots a_{M-1}(a_M - 1)a_1 \ldots.$$

Suppose that the sequence $\boldsymbol{t}(\beta, 1)$ is not finite, that is, there exist infinitely many n's with $t_n(\beta, 1) \neq 0$. Then

$$t_n(\beta, 1-) = t_n(\beta, 1)$$

for any positive integer n. We denote by $\prec_{lex}$ the lexicographical order on the sets of the infinite sequences of nonnegative integers. Let σ be the one-sided shift operator defined by $\sigma((s_n)_{n=1}^{\infty}) = (s_{n+1})_{n=1}^{\infty}$. Parry [21] showed for any sequence $(s_n)_{n=1}^{\infty}$ of nonnegative integers that there exists a real number $x < 1$ satisfying $s_n = t_n(\beta, x)$ for any positive integer n if and only if

$$\sigma^k((s_n)_{n=1}^{\infty}) \prec_{lex} \boldsymbol{t}(\beta, 1-)$$

holds for any nonnegative integer k.

We review metrical results on the normality in the digits of β-expansions. We now recall the notion of β-admissibility. For any positive integers n and k, we define the finite word $\boldsymbol{t}_{n,k}(\beta, x)$ by

$$\boldsymbol{t}_{n,k}(\beta, x) := t_n(\beta, x)t_{n+1}(\beta, x)\ldots t_{n+k-1}(\beta, x).$$

We call that a nonempty finite word W on the alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ is β-admissible if there exists a real number $x < 1$ such that

$$W = \boldsymbol{t}_{1,|W|}(\beta, x).$$

If $\beta = b$ is a rational integer, then any nonempty finite word W on the alphabet $\{0, 1, \ldots, b\}$ is b-admissible.

Borel [7] introduced the notion of normal numbers in base-b for any integer $b \ge 2$. Recall that a real number $\xi < 1$ is a normal number if, for any nonempty finite word W on the alphabet $\{0, 1, \ldots, b-1\}$, we have

$$\lim_{N\to\infty} \frac{\mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \le N, \boldsymbol{t}_{n,|W|}(b, \xi) = W\}}{N} = b^{-|W|},$$

where Card denotes the cardinality.

Rényi [22] proved for any real number $\beta > 1$ that there exists a unique T_β-invariant probability measure μ_β on $[0,1)$ which is absolutely continuous with respect to the Lebesgue measure on $[0,1)$. Moreover, he also verified that μ_β is ergodic. Consequently, almost all real numbers $\xi < 1$ are normal with respect to the β-expansion, that is, for any (nonempty finite) β-admissible word W, we have

$$\lim_{N\to\infty} \frac{\mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \le N, \boldsymbol{t}_{n,|W|}(\beta,\xi) = W\}}{N} \\ = \mu_\beta(\{x \in [0,1) \mid \boldsymbol{t}_{1,|W|}(\beta,x) = W\}).$$

On the other hand, it is difficult to determine whether a given real number $\xi < 1$ is normal with respect to the β-expansion. For instance, if $\beta = b$ is a rational integer, then Borel [8] conjectured that every algebraic irrational number is normal in base-b. However, neither proof nor counterexample is known for Borel's conjecture. The main purpose of this paper is to study the properties of digits in the β-expansions of algebraic numbers in the case where β is a Pisot or Salem number.

We recall the definition of Pisot and Salem numbers. Let β be an algebraic integer greater than 1. Then β is called a Pisot number if the conjugates of β except itself have moduli less than 1. Moreover, β is a Salem number if the conjugates of β except itself have absolute values not greater than 1, and there exists a conjugate of β with absolute value 1.

In Section 2, we study the complexity of the sequence $\boldsymbol{t}(\beta,\xi)$ in the case where β is a Pisot or Salem number and $0 < \xi \le 1$ is an algebraic number. In particular, we give new lower bounds for the numbers of nonzero digits in $\boldsymbol{t}(\beta,\xi)$. The lower bounds are deduced from Theorem 2.2, which is proved in Section 3.

2 Main results

Let $\beta > 1$ and $0 < \xi \le 1$ be algebraic numbers. Lower bounds for the numbers $\gamma(\beta,\xi;N)$ of digit changes, defined by

$$\gamma(\beta,\xi;N) := \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \le N, t_n(\beta,\xi) \ne t_{n+1}(\beta,\xi)\},$$

for positive integer N were studied in [9, 11, 13, 18, 19], which gives partial results on the normality of ξ with respect to the β-expansion. In particular, Bugeaud [11] proved the follwoing: Suppose that β is a Pisot or Salem number and that $t_n(\beta,\xi) \ne t_{n+1}(\beta,\xi)$ for infinitely many n. Then there exist effectively computable positive constants $C_1(\beta,\xi), C_2(\beta,\xi)$, depending only on β and ξ, satisfying

$$\gamma(\beta,\xi;N) \ge C_1(\beta,\xi)\frac{(\log N)^{3/2}}{(\log\log N)^{1/2}} \tag{2.1}$$

for any N with $N \ge C_2(\beta,\xi)$. Lower bounds for the block complexity $p(\beta,\xi;N)$, defined by

$$p(\beta,\xi;N) := \mathrm{Card}\{\boldsymbol{t}_{n,N}(\beta,\xi) \mid n \in \mathbb{Z}^+\}$$

for positive integer N, were also obtained in [2, 3, 10, 13, 17]. Moreover, the diophantine exponents of the sequence $\boldsymbol{t}(\beta,\xi)$ were studied in [2, 15].

Bailey, Borwein, Crandall, and Pomerance [5] studied the numbers of nonzero digits in the binary expansions of algebraic irrational numbers. More generally, we estimate lower bounds for the nonzero digits in the β-expansions of algebraic numbers. Let $\beta > 1$ and $\xi \leq 1$ be real numbers. Put

$$\nu(\beta, \xi; N) := \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \leq N, t_n(\beta, \xi) \neq 0\}$$

for any positive integer N. It is easily seen that

$$\nu(\beta, \xi; N) \geq \frac{1}{2}\gamma(\beta, \xi; N) + O(1).$$

Let β be a Pisot or Salem number and ξ an algebraic number. Assume that the digits of $\boldsymbol{t}(\beta, \xi)$ change infinitely many times. Then (2.1) implies that

$$\nu(\beta, \xi; N) \geq \frac{C_1(\beta, \xi)}{3} \cdot \frac{(\log N)^{3/2}}{(\log\log N)^{1/2}} \tag{2.2}$$

for any sufficiently large N.

The main purpose of this paper is to improve lower bound (2.2). Bailey, Borwein, Crandall, and Pomerance [5] proved for any algebraic irrational number $\xi \leq 1$ of degree D that there exist positive constants $C_3(\xi)$ and $C_4(\xi)$, depending only on ξ, satisfying

$$\nu(2, \xi; N) \geq C_3(\xi) N^{1/D} \tag{2.3}$$

for any integer N with $N \geq C_4(\xi)$. Note that $C_3(\xi)$ is effectively computable but $C_4(\xi)$ is not. Rivoal [23] improved the constant $C_3(\xi)$ for certain classes of algebraic irrational numbers.

Adamczewski, Faverjon [4] and Bugeaud [12] independently verified for each integral base $b \geq 2$ and any algebraic irrational number ξ of degree D that there exist effectively computable positive constants $C_5(b, \xi)$ and $C_6(b, \xi)$, depending only on b and ξ, satisfying

$$\nu(b, \xi; N) \geq C_5(b, \xi) N^{1/D}$$

for any integer N with $N \geq C_6(b, \xi)$.

Let again β be a Pisot or Salem number and $\xi \leq 1$ an algebraic number. Put $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$, where $[L : K]$ denotes the degree of the field extension L/K. Suppose that there exist infinitely many nonzero digits in the sequence $\boldsymbol{t}(\beta, \xi)$. Then we have [20]

$$\nu(\beta, \xi; N) \geq C_7(\beta, \xi) \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}} \tag{2.4}$$

for any integer N with $N \geq C_8(\beta, \xi)$, where $C_7(\beta, \xi)$ and $C_8(\beta, \xi)$ are effectively computable positive constants depending only on β and ξ. The inequality (2.4) follows from Theorem 2.1 in [20], which we introduce as follows: For any sequence $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ of integers, we set

$$\Gamma(\boldsymbol{s}) = \{n \in \mathbb{N} \mid s_n \neq 0\}$$

and

$$f(\boldsymbol{s}; X) := \sum_{n=0}^{\infty} s_n X^n.$$

Moreover, for any nonnegative integer N and any nonempty set $\mathcal{A}$ of nonnegative integers, we put

$$\lambda(\mathcal{A};N) := \mathrm{Card}([0,N]\cap\mathcal{A}).$$

THEOREM 2.1 ([20, Theorem 2.1])**.** *Let β be a Pisot or Salem number and ξ an algebraic number with $[\mathbb{Q}(\beta,\xi):\mathbb{Q}(\beta)] = D$. Suppose that there exists a sequence $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ of integers satisfying the following two assumptions:*

1. *There exists a positive integer B such that, for any $n \in \mathbb{N}$, we have $0 \le s_n \le B$. Moreover, there exist infinitely many n such that $s_n > 0$.*
2. *$\xi = f(\boldsymbol{s};\beta^{-1})$.*

Then there exist effectively computable positive constants $C_9 = C_9(\beta,\xi,B)$ and $C_{10} = C_{10}(\beta,\xi,B)$, depending only on β,ξ and B, such that, for any integer N with $N \ge C_{10}$, we have

$$\lambda(\Gamma(\boldsymbol{s});N) \ge C_9 \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}}. \tag{2.5}$$

In what follows, we improve Theorem 2.1 under the same assumptions.

THEOREM 2.2. *Let β be a Pisot or Salem number and ξ an algebraic number with $[\mathbb{Q}(\beta,\xi):\mathbb{Q}(\beta)] = D$. Suppose that there exists a sequence $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ of integers satisfying the following two assumptions:*

1. *There exists a positive integer B such that, for any $n \in \mathbb{N}$, we have $0 \le s_n \le B$. Moreover, there exist infinitely many n such that $s_n > 0$.*
2.
$$\xi = f(\boldsymbol{s};\beta^{-1}). \tag{2.6}$$

Then there exist effectively computable positive constants $C_{11} = C_{11}(\beta,\xi,B)$ and $C_{12} = C_{12}(\beta,\xi,B)$, depending only on β,ξ and B, such that, for any integer N with $N \ge C_{12}$, we have

$$\lambda(\Gamma(\boldsymbol{s});N) \ge C_{11} \frac{N^{1/D}}{(\log N)^{1/D}}. \tag{2.7}$$

We note that Theorems 2.1 and 2.2 are applicable not only to the β-expansion but also to a general β-representation

$$\xi = \sum_{n=0}^{\infty} t_n \beta^{-n},$$

where $(t_n)_{n=0}^{\infty}$ is a bounded sequence of nonnegative integers.

As a consequence of Theorem 2.2, we improve (2.4) as follows:

COROLLARY 2.3. *Let β be a Pisot or Salem number and $\xi \leq 1$ an algebraic number with $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$. Suppose that there exist infinitely many nonzero digits in $\boldsymbol{t}(\beta, \xi)$. Then there exist effectively computable positive constants $C_{13}(\beta, \xi)$ and $C_{14}(\beta, \xi)$, depending only on β and ξ, satisfying*

$$\nu(\beta, \xi; N) \geq C_{13}(\beta, \xi) \frac{N^{1/D}}{(\log N)^{1/D}}$$

for any integer N with $N \geq C_{14}(\beta, \xi)$.

We apply Theorem 2.2 to the arithmetical properties on certain values of power series at algebraic points. Let $(v_n)_{n=1}^{\infty}$ be a sequence of nonnegative integers such that $v_{n+1} > v_n$ for sufficiently large n. Bugeaud [9, 11] posed a problem on the transcendence of $\sum_{n=1}^{\infty} \alpha^{v_n}$, where α is an algebraic number with $0 < |\alpha| < 1$, under the assumption that $(v_n)_{n=1}^{\infty}$ increases sufficiently rapidly. Corvaja and Zannier [14] proved for any algebraic number α with $0 < |\alpha| < 1$ that if

$$\liminf_{n\to\infty} \frac{v_{n+1}}{v_n} > 1$$

holds, then $\sum_{n=1}^{\infty} \alpha^{v_n}$ is transcendental. In particular, consider the case of $\alpha = \beta^{-1}$, where β is a Pisot or Salem number. Adamczewski [1] proved that if

$$\limsup_{n\to\infty} \frac{v_{n+1}}{v_n} > 1,$$

then $\sum_{n=1}^{\infty} \beta^{-v_n}$ is transcendental. However, if

$$\lim_{n\to\infty} \frac{v_{n+1}}{v_n} = 1, \tag{2.8}$$

then it is generally difficult to determine whether $\sum_{n=1}^{\infty} \alpha^{v_n}$ is transcendental. For instance, put, for any real number $z > 1$ and any positive integer n, $\kappa(z; n) := \lfloor n^z \rfloor$. Moreover, set $\psi(z; X) := \sum_{n=1}^{\infty} X^{\kappa(z;n)}$. Then the transcendence of $\psi(z; \alpha)$ is unknown except the case where $\psi(2; \alpha)$ is transcendental for any algebraic number α with $0 < |\alpha| < 1$, which was proved by Duverney, Nishioka, Nishioka, Shiokawa [16], and Bertrand [6] independently.

Using Theorem 2.1 or Theorem 2.2, we obtain that if

$$\limsup_{n\to\infty} \frac{v_n}{n^R} = \infty \tag{2.9}$$

for any positive real number R, then, for any Pisot or Salem number β, we have $\sum_{n=1}^{\infty} \beta^{-v_n}$ is transcendental. This criterion for transcendence is applicable to certain sequences $(v_n)_{n=1}^{\infty}$ satisfying (2.8). For instance, let, for any positive integer n,

$$w_n := \left\lfloor n^{\log n} \right\rfloor = \left\lfloor \exp\left((\log n)^2\right) \right\rfloor .$$

Then $(w_n)_{n=1}^{\infty}$ fulfills (2.8). Since $(w_n)_{n=1}^{\infty}$ satisfies (2.9), we see that $\sum_{n=1}^{\infty} \beta^{-w_n}$ is transcendental.

Moreover, Using Theorem 2.1, we get for real number $z > 1$ and any Pisot or Salem number β that $\psi(z; \beta^{-1})$ cannot be algebraic of small degree over $\mathbb{Q}(\beta)$, precisely

$$\left[\mathbb{Q}\big(\psi(z; \beta^{-1}), \beta\big) : \mathbb{Q}\,(\beta)\right] \geq \left\lceil \frac{z+1}{2} \right\rceil . \tag{2.10}$$

In fact, we put

$$\psi(z;X) =: \sum_{n=0}^{\infty} s_n X^n.$$

Then a bounded sequence $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ of nonnegative integers satisfies

$$\lim_{N\to\infty} \frac{\lambda(\Gamma(\boldsymbol{s}); N)}{N^{1/z}} = 1.$$

If $\psi(z;\beta^{-1})$ is transcendental, then (2.10) is clear because the left-hand side is equal to infinity. Assume that $\psi(z;\beta^{-1})$ is an algebraic number satisfying $\left[\mathbb{Q}\left(\psi(z;\beta^{-1}),\beta\right):\mathbb{Q}(\beta)\right] = D$. Then (2.5) holds only in the case of $z \le 2D-1$. Similarly, using Theorem 2.2, we deduce that

$$\left[\mathbb{Q}\big(\psi(z;\beta^{-1}),\beta\big):\mathbb{Q}(\beta)\right] \ge \lceil z \rceil,$$

which improves (2.10).

3 Proof of Theorem 2.2

For the proof of Theorem 2.2, we recall the following Liouville type inequality deduced from Theorem 11 in [24, p. 34].

LEMMA 3.1 ([20, Proposition 3.1])**.** *Let z and ξ be algebraic numbers. Suppose that there exists a sequence $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ of integers satisfying the following three assumptions:*

1. *There exists a positive integer B such that, for any $n \in \mathbb{N}$, we have $0 \le s_n \le B$.*
2. *$\xi = f(\boldsymbol{s}; z)$.*
3. *For any $M \in \mathbb{N}$, we have*

$$\sum_{n=0}^{M} s_n z^n \ne \xi.$$

Let $(w(m))_{m=0}^{\infty}$ be a strictly increasing sequence of nonnegative integers defined by

$$\{n \in \mathbb{N} \mid s_n \ne 0\} =: \{w(0) < w(1) < \cdots\}.$$

Then there exist effectively computable positive constants $C_{15} = C_{15}(z,\xi,B)$ and $C_{16} = C_{16}(z,\xi,B)$, depending only on z,ξ and B, such that, for any integer m with $m \ge C_{16}$, we have

$$\frac{w(m+1)}{w(m)} < C_{15}.$$

If $D = 1$, then (2.7) is deduced from (2.5). Thus, we may assume that $D \geq 2$. For simplicity, put

$$\Gamma := \Gamma(\boldsymbol{s}), \lambda(N) := \lambda(\Gamma; N).$$

We may assume that $s_0 \neq 0$, that is ,

$$0 \in \Gamma. \tag{3.1}$$

In what follows, the implied constants in the symbol $\ll$ and the constants $C_{17}, C_{18}, \ldots$ are effectively computable positive ones depending only on β, ξ and B. We see for any $M \in \mathbb{N}$ that $\sum_{n=0}^{M} s_n \beta^{-n} \neq \xi$ by (2.6) and the first assumption of Theorem 2.2. Thus, using Lemma 3.1, we get that there exist C_{17} and C_{18} satisfying

$$\Gamma \cap [x, C_{17}x) \neq \emptyset \tag{3.2}$$

for any real number x with $x \geq C_{18}$. By $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$, there exists an polynomial $P(X) = A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0 \in \mathbb{Z}[\beta][X]$ with $A_D > 0$ such that $P(\xi) = 0$. In the same way as the proof of Theorem 2.1 in [20], we see for any k with $1 \leq k \leq D$ that

$$\xi^k = \left(\sum_{m \in \Gamma} s_m \beta^{-m} \right)^k = \sum_{m=0}^{\infty} \beta^{-m} \rho(k; m), \tag{3.3}$$

where

$$\rho(k; m) = \sum_{\substack{m_1, \ldots, m_k \in \Gamma \\ m_1 + \cdots + m_k = m}} s_{m_1} \cdots s_{m_k}.$$

Note for any nonnegative integer m that $\rho(k; m)$ is a nonnegative integer. Moreover, putting

$$k\Gamma := \{m_1 + \cdots + m_k \mid m_1, \ldots, m_k \in \Gamma\},$$

we get that $\rho(k; m)$ is positive if and only if $m \in k\Gamma$. By (3.1), we have

$$(0 \in)\ \Gamma \subset 2\Gamma \subset \cdots \subset (D-1)\Gamma \subset D\Gamma. \tag{3.4}$$

Observe that

$$\lambda(k\Gamma; N) = \mathrm{Card}([0, N] \cap k\Gamma) \leq \mathrm{Card}([0, N] \cap \Gamma)^k = \lambda(N)^k \tag{3.5}$$

and that

$$\rho(k; m) \leq B^k \sum_{\substack{m_1, \ldots, m_k \in \Gamma \\ m_1 + \cdots + m_k = m}} 1 \leq B^k (m+1)^k. \tag{3.6}$$

We see that

$$\begin{aligned} 0 &= P(\xi) = A_0 + \sum_{k=1}^{D} A_k \xi^k \\ &= A_0 + \sum_{k=1}^{D} A_k \sum_{m=0}^{\infty} \beta^{-m} \rho(k; m) \end{aligned} \tag{3.7}$$

by (3.3). Let R be a nonnegative integer. Then, multiplying (3.7) by β^R, we get

$$0 = A_0\beta^R + \sum_{k=1}^{D} A_k \sum_{m=-R}^{\infty} \beta^{-m}\rho(k; m+R).$$

In particular, putting

$$Y_R := \sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m}\rho(k; m+R),$$

we obtain

$$Y_R = -A_0\beta^R - \sum_{k=1}^{D} A_k \sum_{m=-R}^{0} \beta^{-m}\rho(k; m+R). \tag{3.8}$$

Note that Y_R is an algebraic integer by (3.8) because β is a Pisot or Salem number. In the same way as the proof of Lemma 4.1 in [20], we deduce the following: There exists positive integers C_{19} and C_{20} such that if R is an integer with $R \ge C_{20}$, then we have

$$Y_R = 0 \text{ or } |Y_R| \ge R^{-C_{19}}. \tag{3.9}$$

In the case of $\beta = 2$, Bailey, Borwein, Crandall, and Pomerance [5] investigated the numbers of positive Y_R to prove (2.3). More precisely, they estimated upper and lower bounds for the value

$$\mathrm{Card}\{R \in \mathbb{N} \mid R \le N,\ Y_R > 0\}$$

for a nonnegative integer N. However, if β is a general Pisot or Salem number, then it is difficult to obtain upper bounds. So we modify their definition, that is, we consider the value

$$y_N := \mathrm{Card}\{R \in \mathbb{N} \mid R \le N,\ Y_R \ge C_{21}\}$$

for a integer N with $N \gg 1$, where $C_{21} = \min\{1/\beta, A_D/\beta\}$. We give upper bounds for y_N in Lemma 3.2, using the function $\lambda(N)$. Note that we modify the definition of y_N to get (3.11), which is the key inequality for the proof of Lemma 3.2. On the other hand, we estimate upper bounds for y_N in Lemma 3.5. The main tool for the proof of Lemma 3.5 is Lemma 3.4, which is deduced from Liouville type inequality (3.9).

In what follows, we assume that N is a sufficiently large integer satisfying

$$\left(1 + \frac{1}{N}\right)^D < \frac{1+\beta}{2}. \tag{3.10}$$

LEMMA 3.2.

$$y_N \ll \log N + \lambda(N)^D.$$

for any integer N with $N \gg 1$.

Proof. Putting $K := \lceil (D+1)\log_\beta N \rceil$, we get

$$y_N \le K + y_{N-K} = K + \sum_{\substack{0\le R\le N-K \\ Y_R \ge C_{21}}} 1 \le K + \frac{1}{C_{21}} \sum_{R=0}^{N-K} |Y_R|. \qquad (3.11)$$

Observe that

$$\begin{aligned} \sum_{R=0}^{N-K} |Y_R| &\le \sum_{R=0}^{N-K}\sum_{k=1}^{D}\sum_{m=1}^{\infty} |A_k|\beta^{-m}\rho(k;m+R) \\ &= \sum_{k=1}^{D}|A_k| \sum_{R=0}^{N-K}\sum_{m=1}^{\infty}\beta^{-m}\rho(k;m+R) \\ &=: \sum_{k=1}^{D}|A_k| z_N^{(k)}, \end{aligned} \qquad (3.12)$$

where

$$z_N^{(k)} = \sum_{R=0}^{N-K}\sum_{m=1}^{\infty}\beta^{-m}\rho(k;m+R)$$

for any N and k with $N \ge 0$ and $1 \le k \le D$. By (3.11) and (3.12), it suffices to show

$$z_N^{(k)} \ll \lambda(N)^D \qquad (3.13)$$

for any N and k with $N \gg 1$ and $1 \le k \le D$. We see that

$$\begin{aligned} z_N^{(k)} &= \sum_{m=1}^{K}\beta^{-m}\sum_{R=0}^{N-K}\rho(k;m+R) \\ &\qquad + \sum_{m=K+1}^{\infty}\beta^{-m}\sum_{R=0}^{N-K}\rho(k;m+R) \\ &=: S_1(k) + S_2(k). \end{aligned} \qquad (3.14)$$

Using the first assumption of Theorem 2.2 and the definition of $\rho(k;R), \lambda(N)$, we obtain

$$\begin{aligned} S_1(k) &\le \sum_{m=1}^{K}\beta^{-m}\sum_{R=0}^{N}\rho(k;R) \le \sum_{m=1}^{\infty}\beta^{-m}\sum_{R=0}^{N}\rho(k;R) \\ &\ll \sum_{R=0}^{N}\rho(k;R) = \sum_{R=0}^{N}\sum_{\substack{m_1,\ldots,m_k\in\Gamma \\ m_1+\cdots+m_k=R}} s_{m_1}\cdots s_{m_k} \\ &= \sum_{\substack{m_1,\ldots,m_k\in\Gamma \\ m_1+\cdots+m_k\le N}} s_{m_1}\cdots s_{m_k} \le B^k \sum_{\substack{m_1,\ldots,m_k\in\Gamma \\ m_1+\cdots+m_k\le N}} 1 \\ &\le B^D\lambda(N)^D \ll \lambda(N)^D. \end{aligned} \qquad (3.15)$$

On the other hand, (3.6) implies by $k \le D$ that

$$S_2(k) \ll \sum_{m=K+1}^{\infty}\beta^{-m}\sum_{R=0}^{N-K}(m+R+1)^D \le N\sum_{m=K+1}^{\infty}\beta^{-m}(m+N)^D.$$

Thus, using (3.10), we obtain for any integer N with $N \gg 1$ that

$$\begin{aligned} S_2(k) &\ll N\beta^{-1-K}(1+K+N)^D \sum_{m=0}^{\infty} \beta^{-m} \left(\frac{1+\beta}{2}\right)^m \\ &\ll \beta^{-K} N^{D+1} \le 1. \end{aligned} \tag{3.16}$$

Therefore, combining (3.14), (3.15), and (3.16), we deduce (3.13). □

Recalling that $0 \in (D-1)\Gamma$ by (3.4), set

$$[0, N) \cap (D-1)\Gamma =: \{0 = i(1) < i(2) < \cdots < i(\tau)\},$$

where

$$\tau = \tau(N) \le \lambda(N)^{D-1} \tag{3.17}$$

by (3.5). Put $i(1+\tau) := N$.

Let $1 \le h \le \tau$. We define the interval I_h by

$$I_h := \begin{cases} [i(h), i(1+h)) & (1 \le h \le -1+\tau), \\ [i(\tau), i(1+\tau)] & (h = \tau). \end{cases}$$

Moreover, let $|I_h| := i(1+h) - i(h)$ and

$$y_N(h) := \mathrm{Card}\,\{R \in I_h \mid Y_R \ge C_{21}\}.$$

Then we have

$$\sum_{h=1}^{\tau} |I_h| = N \tag{3.18}$$

and

$$\sum_{h=1}^{\tau} y_N(h) = y_N. \tag{3.19}$$

Consider the case where I_h satisfies

$$|I_h| > 8D(1+C_{17})C_{19} \log_\beta N =: C_{22} \log_\beta N. \tag{3.20}$$

If $N \gg 1$, then applying (3.2) with $x = |I_h|/(1+C_{17})$, we see by (3.20) that there exists $\theta(h) \in \Gamma$ with

$$\frac{1}{1+C_{17}}|I_h| \le \theta(h) < \frac{C_{17}}{1+C_{17}}|I_h|.$$

Putting $M(h) := i(h) + \theta(h)$, we get

$$i(h) + \frac{1}{1+C_{17}}|I_h| \le M(h) < i(h) + \frac{C_{17}}{1+C_{17}}|I_h|. \tag{3.21}$$

Moreover, we obtain $M(h) \in D\Gamma$, by $i(h) \in (D-1)\Gamma$ and $\theta(h) \in \Gamma$.

LEMMA 3.3. *Let N, h be integers with $N \gg 1$ and $1 \le h \le \tau$. Assume that (3.20) holds. Then $Y_R > 0$ for any integer R with $i(h) \le R < M(h)$.*

Proof. We prove the lemma by induction on R. We first consider the case where $R = -1 + M(h)$. Observe that

$$\begin{aligned} Y_{-1+M(h)} &= A_D \sum_{m=1}^{\infty} \beta^{-m} \rho(D; m + M(h) - 1) \\ &\qquad + \sum_{k=1}^{D-1} A_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + M(h) - 1) \\ &=: S_3 + S_4. \end{aligned} \tag{3.22}$$

By $M(h) \in D\Gamma$, we get

$$S_3 \geq \frac{A_D}{\beta} \rho(D; M(h)) \geq \frac{A_D}{\beta}. \tag{3.23}$$

We estimate upper bounds for $|S_4|$. Let k, m be integers with $1 \leq k \leq D - 1$ and $1 \leq m \leq -1 + \lceil 2D \log_\beta N \rceil$. Observe that, by (3.21), (3.20), and $C_{19} \geq 1$,

$$\begin{aligned} i(1 + h) - M(h) &\geq i(1 + h) - i(h) - \frac{C_{17}}{1 + C_{17}} |I_h| \\ &= \frac{1}{1 + C_{17}} |I_h| > 8D \log_\beta N > m \end{aligned}$$

Hence, we see

$$i(h) < m + M(h) - 1 < i(1 + h),$$

by $i(h) < M(h) \leq m + M(h) - 1$. Consequently, $m + M(h) - 1 \notin (D - 1)\Gamma$. In particular, by (3.4) we obtain $m + M(h) - 1 \notin k\Gamma$. Therefore, we deduce that

$$\rho(k; m + M(h) - 1) = 0$$

for any k, m with $1 \leq k \leq D - 1$ and $1 \leq m \leq -1 + \lceil 2D \log_\beta N \rceil$.

Using (3.6), we obtain

$$\begin{aligned} |S_4| &\leq \sum_{k=1}^{D-1} |A_k| \sum_{m \geq \lceil 2D \log_\beta N \rceil} \beta^{-m} \rho(k; m + M(h) - 1) \\ &\leq \sum_{k=1}^{D-1} |A_k| \sum_{m \geq \lceil 2D \log_\beta N \rceil} \beta^{-m} B^D (m + N)^D \\ &\ll \sum_{m \geq \lceil 2D \log_\beta N \rceil} \beta^{-m} (m + N)^D. \end{aligned}$$

Consequently, (3.10) implies that

$$\begin{aligned} |S_4| &\ll \beta^{-\lceil 2D \log_\beta N \rceil} (\lceil 2D \log_\beta N \rceil + N)^D \sum_{m=0}^{\infty} \beta^{-m} \left(\frac{1 + \beta}{2} \right)^m \\ &\ll N^{-D}. \end{aligned}$$

If $N \gg 1$, then

$$|S_4| < \frac{A_D}{2\beta}. \tag{3.24}$$

Combining (3.22), (3.23), and (3.24), we deduce $Y_{-1+M(h)} > 0$.

Next we assume $Y_R > 0$ for some R with $i(h) < R < M(h)(< i(1+h))$. Using $\rho(k;R) = 0$ for $k = 1, \ldots, D-1$ by (3.4), we see

$$\begin{aligned} Y_{R-1} &= \sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m}\rho(k; m+R-1) \\ &= \frac{1}{\beta} A_D \rho(D;R) + \frac{1}{\beta}\sum_{k=1}^{D} A_k \sum_{m=2}^{\infty} \beta^{-(m-1)}\rho(k; m-1+R) \\ &= \frac{1}{\beta} A_D \rho(D;R) + \frac{1}{\beta} Y_R \geq \frac{1}{\beta} Y_R > 0 \end{aligned} \tag{3.25}$$

by the inductive hypothesis. Therefore, we proved the lemma. □

LEMMA 3.4. *Let N, h be integers with $N \gg 1$ and $1 \leq h \leq \tau$. Assume that (3.20) holds. Let R be an integer with*

$$i(h) + 4C_{19} \log_\beta N \leq R < M(h).$$

Then we have

$$R - \max\{R' \in \mathbb{N} \mid R' < R, Y_{R'} \geq C_{21}\} \leq 2C_{19} \log_\beta N.$$

Proof. Let

$$R_1 := \max\{R' \in \mathbb{N} \mid R' < R, Y_{R'} \geq C_{21}\}.$$

In the same way as the proof of (3.25), we see for any integer r with $i(h) < r < i(1+h)$ that

$$Y_{r-1} = \frac{1}{\beta} A_D \rho(D;r) + \frac{1}{\beta} Y_r. \tag{3.26}$$

For the proof of the lemma, we may assume that $Y_R < 1$. In fact, if $Y_R \geq 1$, then we have $Y_{R-1} \geq 1/\beta \geq C_{21}$ by (3.26) and $R - R_1 = 1 \leq 2C_{19} \log_\beta N$.

Put $S := \lceil C_{19} \log_\beta N \rceil$. Assume for any integer m with $0 \leq m \leq S$ that

$$\rho(D; R-m) = 0.$$

Since $M(h) > R > R-1 > \cdots > R-S > i(h)$, we get by (3.26) that

$$1 > Y_R = \beta Y_{R-1} = \cdots = \beta^S Y_{R-S} = \beta^{1+S} Y_{R-S-1} > 0.$$

In fact, Lemma 3.3 implies $Y_{R-S-1} > 0$ by $R - S - 1 \geq i(h)$. Consequently, we see

$$\beta^{S+1} < Y_{R-S-1}^{-1} = |Y_{R-S-1}|^{-1}.$$

If $N \gg 1$, then we have $R - S - 1 \geq 2C_{19} \log_\beta N \geq C_{20}$. Thus, using (3.9), we obtain

$$\beta^{S+1} < |Y_{R-S-1}|^{-1} \leq (R-S-1)^{C_{19}} < N^{C_{19}}.$$

Hence, we deduce that

$$\lceil C_{19} \log_\beta N \rceil + 1 = S + 1 < C_{19} \log_\beta N,$$

a contradiction. Therefore, there exists an integer m' with $0 \le m' \le S$ such that $\rho(D; R-m') \ge 1$. Finally, using (3.26) and $Y_{R-m'} > 0$ by Lemma 3.3, we obtain

$$Y_{R-m'-1} \ge \frac{A_D}{\beta} \ge C_{21}$$

and

$$R - R_1 \le m' + 1 \le 2C_{19} \log_\beta N.$$

□

LEMMA 3.5. *There exists C_{23} satisfying the following: If $N \gg 1$, then, for any integer h with $1 \le h \le \tau$, we have*

$$y_N(h) \ge \left\lfloor \frac{|I_h|}{C_{23} \log_\beta N} \right\rfloor. \tag{3.27}$$

Proof. If (3.20) holds, then (3.27) follows from Lemma 3.4. In what follows, we suppose that $|I_h| \le C_{22} \log_\beta N$. If necessary, increasing C_{23}, we may assume that $C_{23} > C_{22}$. Thus, (3.27) holds by

$$\left\lfloor \frac{|I_h|}{C_{23} \log_\beta N} \right\rfloor = 0.$$

□

If $N \gg 1$, then, combining (3.19), Lemma 3.5, and (3.18), (3.17), we deduce that

$$\begin{aligned} y_N &= \sum_{h=1}^{\tau} y_N(h) \ge \sum_{h=1}^{\tau} \left(\frac{|I_h|}{C_{23} \log_\beta N} - 1 \right) \\ &\ge \frac{N}{C_{23} \log_\beta N} - \tau \gg \frac{N}{\log N} - \lambda(N)^{D-1}. \end{aligned}$$

On the other hand, Lemma 3.2 implies that

$$\log N + \lambda(N)^D \gg y_N \gg \frac{N}{\log N} - \lambda(N)^{D-1}.$$

Therefore, we proved Theorem 2.2.

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