Distribution of geometric sequences modulo 1

Hajime Kaneko

Abstract. Let $||\xi\alpha^n||$ denote the distance from $\xi\alpha^n$ to the nearest integer. In this paper we obtain a new lower bound for $\limsup_{n\to\infty} ||\xi\alpha^n||$ if α is an algebraic irrational number whose conjugates have moduli greater than 1.

1. Introduction

Weyl [13] proved that an arithmetic progression is uniformly distributed modulo 1 if and only if its common difference is irrational. Moreover, it is known that a sequence $(P(n))_{n=0}^{\infty}$, where $P(X) \in \mathbb{R}[X]$, is uniformly distributed modulo 1 if and only if $P(X) - P(0) \notin \mathbb{Q}[X]$. On the other hand, for geometric progressions no criteria of uniform distribution modulo 1 have been known so far.

In this paper we estimate the maximal limit points of the sequence $(||\xi\alpha^n||)_{n=0}^{\infty}$, where ξ is a nonzero real number, α is an algebraic number with $\alpha > 1$, and $||x||$ denotes the distance from the real number x to the nearest integer.

Maximal limit points are known if α is an algebraic integer whose conjugates different from α have absolute values not greater than 1. For such an α, if its conjugates different from α have absolute values strictly less than 1, α is called a Pisot number. Otherwise α is called a Salem number. Hardy [9] proved for algebraic $\alpha > 1$ that $\lim_{n\to\infty} ||\xi\alpha^n|| = 0$ for some nonzero ξ if and only if α is a Pisot number. Dubickas [6] proved for algebraic $\alpha > 1$ that

$$\inf_{\xi\neq 0} \limsup_{n\to\infty} ||\xi\alpha^n|| = 0$$

if and only if α is a Pisot or Salem number. It is a natural problem to determine the value

$$\mathcal{E}(\alpha) = \inf_{\xi\neq 0} \limsup_{n\to\infty} ||\xi\alpha^n||$$

in the case where α is neither a Pisot nor Salem number. We denote the length of a polynomial $C(X) = \sum_{i=0}^{m} c_i X^i \in \mathbb{R}[X]$ by $L(C(X)) = \sum_{i=0}^{m} |c_i|$. Let $P(X)$ be the minimal polynomial of α. We denote the reduced length of α by

$$l(\alpha)=\inf_{B(X)\in\Gamma} L(P(X)B(X)),$$

where

$$\Gamma=\{B(X)=b_0+b_1X+\cdots+b_mX^m\in\mathbb{R}[X]\mid b_0=1 \text{ or } b_m=1\}.$$

Dubickas [6] proved that

$$\mathcal{E}(\alpha)\geq\max\left\{\frac{1}{L(P(X))},\frac{1}{2l(\alpha)}\right\} \tag{1.1}$$

if α is neither a Pisot nor Salem number. If $\alpha=p/q$, where p and q are integers with $p>q\geq 2$ and $\gcd(p,q)=1$, the inequality (1.1) implies

$$\mathcal{E}\left(\frac{p}{q}\right)\geq\frac{1}{p+q}. \tag{1.2}$$

Dubickas [6] further obtained

$$\mathcal{E}\left(\frac{p}{q}\right)\geq\frac{1}{p}E_1\left(\frac{q}{p}\right), \tag{1.3}$$

where E_1 is defined by Mahler function as follows:

$$E_1(X)=\frac{1-(1-X)\prod_{i=0}^{\infty}\left(1-X^{2^i}\right)}{2X}. \tag{1.4}$$

The inequality (1.3) is sharper than (1.2) since

$$\begin{aligned}\frac{1}{p}E_1\left(\frac{q}{p}\right)\ &-\ \frac{1}{p+q}\\ &=\ \frac{p-q}{2q(p+q)}\left\{1-\left(1-\frac{q^2}{p^2}\right)\prod_{i=1}^{\infty}\left(1-\frac{q^{2^i}}{p^{2^i}}\right)\right\}>0.\end{aligned}$$

The main purpose of this paper is to generalize the inequality (1.3) to the case of irrational α whose conjugates have absolute values greater than 1.

2. Main results

First we give a lower bound for $\mathcal{E}(\alpha)$ in the case where α is a quadratic irrational number. The theorems in this section give improvements of the inequality (1.1).

THEOREM 2.1. *Let $\alpha>1$ be a quadratic irrational number with the minimal polynomial $a_2X^2+a_1X+a_0\in\mathbb{Z}[X]$, where $a_2>0$ and $\gcd(a_2,a_1,a_0)=1$. Let α_2 be the conjugate of α. Assume that $\alpha_2>1$ and that*

$$\alpha^{-1}+\alpha_2^{-1}\leq\frac{\sqrt{5}-1}{2}. \tag{2.1}$$

Then

$$\mathcal{E}(\alpha) \geq \frac{1}{a_0} E_2(\alpha^{-1}, \alpha_2^{-1}),$$

where

$$E_2(X, Y) = \frac{X E_1(X) - Y E_1(Y)}{X - Y}.$$

REMARK 2.2. Let α be a quadratic irrational number satisfying the assumptions of Theorem 2.1. By Dubickas's formula of reduced length in [5], we get

$$\begin{aligned} \mathcal{E}(\alpha) &\geq \max\left\{\frac{1}{L(P(X))}, \frac{1}{2|a_0|}\right\} = \frac{1}{L(P(X))} \\ &= \frac{1}{|a_0|} \frac{1}{1 + \alpha^{-1} + \alpha_2^{-1} + \alpha^{-1}\alpha_2^{-1}}. \end{aligned}$$

By using Lemma 2.1 at the end of this section, we can rewrite $E_2(\alpha^{-1}, \alpha_2^{-1})$ as an alternative series. Thus we get

$$E_2(\alpha^{-1}, \alpha_2^{-1}) > 1 - \alpha^{-2} - \alpha^{-1}\alpha_2^{-1} - \alpha_2^{-2},$$

and so

$$\frac{1}{|a_0|} E_2(\alpha^{-1}, \alpha_2^{-1}) > \max\left\{\frac{1}{L(P(X))}, \frac{1}{2|a_0|}\right\}.$$

Now we give an example. Put $\alpha = 4 + \sqrt{2}$ and $\alpha_2 = 4 - \sqrt{2}$. By the inequality (1.1) we get

$$\mathcal{E}(4 + \sqrt{2}) \geq 0.0434\ldots.$$

Since α and α_2 satisfy the conditions in Theorem 2.1, we have

$$\mathcal{E}(4 + \sqrt{2}) \geq 0.0581\ldots.$$

We note that we calculate the value $E_2(\alpha^{-1}, \alpha_2^{-1})$ by using Lemma 2.1.

On the other hand, for any $\alpha > 1$ we have

$$\mathcal{E}(\alpha) \leq \frac{1}{2\alpha - 2} \tag{2.2}$$

(see Section 5). By (2.2) we have

$$\mathcal{E}(4 + \sqrt{2}) \leq 0.113\ldots.$$

Next we consider the case where α is an algebraic number with arbitrary degree. In what follows, we write the symmetric homogeneous polynomial of degree m as

$$\rho_m(X_1, X_2, \ldots, X_r) = \sum_{\substack{i_1, i_2, \ldots, i_r \geq 0 \\ i_1 + i_2 + \cdots + i_r = m}} X_1^{i_1} X_2^{i_2} \cdots X_r^{i_r}. \tag{2.3}$$

We note that (2.1) is equivalent to

$$0 < \rho_{m+1}(\alpha^{-1}, \alpha_2^{-1}) \le \frac{\sqrt{5}-1}{2}\rho_m(\alpha^{-1}, \alpha_2^{-1}) \tag{2.4}$$

for all $m \ge 0$.

In order to estimate $\mathcal{E}(\alpha)$ in this case we need a stronger assumption than that of Theorem 2.1.

THEOREM 2.3. *Let α be an algebraic irrational number with $|\alpha| > 1$ and let $\alpha_1(=\alpha), \alpha_2, \ldots, \alpha_d$ be the conjugates of α. Denote the minimal polynomial of α by $P(X) = a_d X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$, where $a_d > 0$ and* $\gcd(a_d, a_{d-1}, \ldots, a_0) = 1$. *Assume*

$$|\alpha_i| > 1 \ (i = 1, 2, \ldots, d)$$

and

$$0 < \rho_{m+1}(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) \le \frac{1}{2}\rho_m(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) \ (m = 0, 1, \ldots). \tag{2.5}$$

Then

$$\mathcal{E}(\alpha) \ge \frac{1}{|a_0|} E_d(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}),$$

where

$$E_d(X_1, X_2, \ldots, X_d) = \sum_{i=1}^{d} \left(\prod_{\substack{1 \le j \le d \\ j \ne i}} \frac{1}{X_i - X_j} \right) X_i^{d-1} E_1(X_i)$$

and E_1 is defined by (1.4).

REMARK 2.4. Let α be an algebraic number satisfying the assumptions of Theorem 2.3. By the same way as in Remark 2.3, we have

$$\begin{aligned} \max\left\{ \frac{1}{L(P(X))}, \frac{1}{2l(\alpha)} \right\} &= \max\left\{ \frac{1}{L(P(X))}, \frac{1}{2|a_0|} \right\} \\ &\le \frac{1}{|a_0|} \frac{1}{1 + \rho_1(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1})} \end{aligned}$$

and

$$E_d(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) > 1 - \rho_2(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}).$$

Thus we get

$$\frac{1}{|a_0|} E_d(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) > \max\left\{ \frac{1}{L(P(X))}, \frac{1}{2l(\alpha)} \right\}.$$

We now set

$$\rho_m(X_1, X_2, \ldots, X_d) = 0 \ (-d+1 \le m \le -1). \tag{2.6}$$

By the equality

$$\sum_{n=-d+1}^{\infty} \rho_n(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) X^n \sum_{i=0}^{d} a_i X^i = a_0,$$

we get

$$\begin{aligned} a_0\rho_m(\alpha^{-1},\, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) + a_1\rho_{m-1}(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) \\ + \ \cdots \ + a_d\rho_{m-d}(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) = 0 \ (m = 1, 2, \ldots). \end{aligned} \tag{2.7}$$

By using (2.7) to check the inequality (2.5), we obtain the following:

COROLLARY 2.5. *Let α be an algebraic irrational number with $|\alpha| > 1$. Denote the minimal polynomial of α by $P(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$, where $a_d > 0$ and $\gcd(a_d, a_{d-1}, \ldots, a_0) = 1$. Assume*

$$a_i \geq 0 \ (1 \leq i \leq d)$$

and

$$a_0 \leq -2a_1 - 2a_2 - \cdots - 2a_d.$$

Then

$$\mathcal{E}(\alpha) \geq \frac{1}{|a_0|} E_d(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}).$$

We estimate $\mathcal{E}(\alpha)$ in the case where $\alpha = 7\sqrt[3]{2} - 7$. The minimal polynomial of α is $X^3 + 21X^2 + 147X - 343$. The inequality (1.1) implies

$$\mathcal{E}(7\sqrt[3]{2} - 7) \geq 0.00195\ldots.$$

Since α satisfies the conditions in Corollary 2.5, we have

$$\mathcal{E}(7\sqrt[3]{2} - 7) \geq 0.00242\ldots.$$

We note that we calculate the value $E_d(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1})$ by using Lemma 2.1.

We can apply Theorem 2.3 to the case where $\alpha > 1$ is a quadratic irrational number whose Galois conjugate is less than -1.

COROLLARY 2.6. *Let $\alpha > 1$ be a quadratic irrational number with the minimal polynomial $a_2X^2 + a_1X + a_0 \in \mathbb{Z}[X]$, where $a_2 > 0$ and $\gcd(a_2, a_1, a_0) = 1$. Let α_2 be the conjugate of α. Put $\zeta = 1$ if $\alpha < |\alpha_2|$, otherwise put $\zeta = -1$.*

Assume $\alpha_2 < -1$ and

$$0 < \rho_{m+1}(\zeta\alpha^{-1}, \zeta\alpha_2^{-1}) \leq \frac{1}{2}\rho_m(\zeta\alpha^{-1}, \zeta\alpha_2^{-1}) \ (i = 0, 1).$$

Then

$$\mathcal{E}(\alpha) \geq \frac{1}{|a_0|} E_2(\zeta\alpha^{-1}, \zeta\alpha_2^{-1}).$$

In the rest of this section we determine the Taylor expansion of the function $E_d(X_1, X_2, \ldots, X_d)$ at the origin. The calculation was obtained by Dubickas [6] if $d = 1$. Let A_n $(n = 0, 1, \ldots)$ be finite words given by $A_0 = 2,\ A_1 = 2\ 1\ 1$ and

$$A_n = A_{n-1}A_{n-2}A_{n-2}.$$

Let $\mathbf{w} = (w_n)_{n=0}^{\infty}$ be a sequence defined by

$$\begin{aligned} \mathbf{w} &= A_1 A_0 A_0 A_1 A_1 A_2 A_2 \ldots A_n A_n \ldots \\ &= 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, \ldots . \end{aligned}$$

Then we define the finite words γ_n $(n = 0, 1, \ldots)$ and the sequence $\mathbf{e} = (e_n)_{n=0}^{\infty}$ as follows:

$$\gamma_n = \begin{cases} \emptyset & \text{if } w_n = 1, \\ 0 & \text{if } w_n = 2; \end{cases}$$

$$\begin{aligned} \mathbf{e} &= 1, \gamma_0, -1, \gamma_1, 1, \gamma_2, -1, \gamma_3, 1, \gamma_4, -1, \ldots \\ &= 1, 0, -1, 1, -1, 0, 1, 0, -1, \ldots . \end{aligned}$$

LEMMA 2.1. *The function $E_d(X_1, X_2, \ldots, X_d)$ is represented as*

$$E_d(X_1, X_2, \ldots, X_d) = \sum_{i=0}^{\infty} \rho_i(X_1, X_2, \ldots, X_d) e_i. \tag{2.8}$$

Proof. Since

$$E_1(X) = \sum_{i=0}^{\infty} (-1)^i X^{w_0 + w_1 + \cdots + w_{i-1}},$$

we have

$$\begin{aligned} E_d(X_1, X_2, \ldots, X_d) &= \sum_{i=0}^{\infty} (-1)^i \sum_{j=1}^{d} \prod_{\substack{1 \le k \le d \\ k \ne j}} \frac{1}{X_j - X_k} X_j^{w_0 + w_1 + \cdots + w_{i-1} + d - 1} \\ &= \sum_{i=0}^{\infty} (-1)^i \rho_{w_0 + w_1 + \cdots + w_{i-1}}(X_1, X_2, \ldots, X_d) \\ &= \sum_{i=0}^{\infty} \rho_i(X_1, X_2, \ldots, X_d) e_i. \end{aligned}$$

Note that we use Lemma 3.1 to check the second equation. □

3. Preliminaries

We define the integral part and the fractional part of a real number.

DEFINITION 3.1. *Let x be a real number. Then the integer $u(x)$ and the real number $\varepsilon(x)$ are uniquely determined by $x = u(x) + \varepsilon(x)$ with $-1/2 \le \varepsilon(x) < 1/2$. We call $u(x)$ the integral part of x and $\varepsilon(x)$ the fractional part of x.*

We note that $u(x)$ is one of the nearest integers to x and that $||x||$ is given by $||x|| = |\varepsilon(x)|$.

If $\alpha \geq 2$ is a rational integer, the fractional part $\varepsilon(\xi\alpha^n)$ can be denoted by the α-ary expansion of ξ. Mahler [11] considered the "3/2-ary" expansion of real numbers to study the distribution of the geometric progressions whose common ratios are 3/2. In this section we construct the "α-ary" expansion for an algebraic number α.

At first, we check the following:

LEMMA 3.1. *Let the symmetric homogeneous polynomial $\rho_m(X_1, X_2, \ldots, X_r)$ be defined by (2.3). Then*

$$\rho_m(X_1, X_2, \ldots, X_r) = \sum_{i=1}^{r} \left(\prod_{\substack{1 \leq j \leq r \\ j \neq i}} \frac{1}{X_i - X_j} \right) X_i^{m+r-1}. \tag{3.1}$$

Proof. We denote the right-hand side of (3.1) by ρ'_m. We consider the polynomial of Y defined by

$$g(Y) = \prod_{i=1}^{r} (1 - X_i Y) \sum_{m=0}^{\infty} \rho'_m Y^m = \sum_{i=1}^{r} \prod_{\substack{1 \leq j \leq r \\ j \neq i}} \frac{1 - X_j Y}{1 - X_i^{-1} X_j}.$$

Since

$$g(X_l^{-1}) = 1 (1 \leq l \leq r)$$

and the degree of $g(Y)$ is at most $r-1$, we have $g(Y) = 1$. Thus we conclude that

$$\sum_{i=0}^{\infty} \rho'_i Y^i = \prod_{i=1}^{r} (1 - X_i Y)^{-1} = \sum_{i=0}^{\infty} \rho_i(X_1, X_2, \ldots, X_r) Y^i.$$

$\square$

We define $\rho_m(X_1, X_2, \ldots, X_r)$ also for a negative integer m by using (3.1). We note this definition coincides with (2.6) for $-r+1 \leq m \leq -1$.

Let α be an algebraic number and let $P(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$ be its minimal polynomial. We denote the conjugates of α by $\alpha_1 (= \alpha), \alpha_2, \ldots, \alpha_d$. In the rest of this section we assume that $|\alpha_i| > 1$ $(i = 1, 2, \ldots, d)$. Let ξ be a nonzero real number. We define the sequence $(s_n(\xi))_{n=-\infty}^{\infty}$ by

$$s_n(\xi) = a_d u(\xi\alpha^{-n}) + a_{d-1} u(\xi\alpha^{-n-1}) + \cdots + a_0 u(\xi\alpha^{-n-d}).$$

It is easily checked that

$$|s_n(\xi)| < \frac{1}{2} L(P(X)) = \frac{1}{2} \sum_{i=0}^{d} |a_i|.$$

In fact, $s_n(\xi)$ can be rewritten as

$$s_n(\xi) = -a_d \varepsilon(\xi\alpha^{-n}) - a_{d-1} \varepsilon(\xi\alpha^{-n-1}) - \cdots - a_0 \varepsilon(\xi\alpha^{-n-d}).$$

Moreover, Dubickas [6] proved that the sequence $(s_n(\xi))_{n=-\infty}^{M}$ is not periodic for any integer M.

PROPOSITION 3.1. *The integral part $u(\xi\alpha^n)$ and the fractional part $\varepsilon(\xi\alpha^n)$ are given by*

$$u(\xi\alpha^n) = \frac{1}{a_d}\sum_{i=0}^{\infty}\rho_i(\alpha,\alpha_2,\ldots,\alpha_d)s_{i-n}(\xi)$$

and

$$\varepsilon(\xi\alpha^n) = \frac{1}{a_d}\sum_{i=-\infty}^{-1}\rho_i(\alpha,\alpha_2,\ldots,\alpha_d)s_{i-n}(\xi),$$

respectively. In particular

$$\xi\alpha^n = \frac{1}{a_d}\sum_{i=-\infty}^{\infty}\rho_i(\alpha,\alpha_2,\ldots,\alpha_d)s_{i-n}(\xi).$$

Proof. We get

$$\begin{aligned}
&\frac{1}{a_d}\sum_{i=-\infty}^{-1}\rho_i(\alpha,\alpha_2,\ldots,\alpha_d)s_{i-n}(\xi)\\
= &-\frac{1}{a_d}\sum_{i=-\infty}^{-d}\rho_i(\alpha,\alpha_2,\ldots,\alpha_d)\sum_{j=0}^{d}a_{d-j}\varepsilon(\xi\alpha^{n-i-j})\\
= &-\frac{1}{a_d}\sum_{i=-\infty}^{0}\varepsilon(\xi\alpha^{n-i})\sum_{j=0}^{\min\{d,-i\}}\rho_{i+j-d}(\alpha,\alpha_2,\ldots,\alpha_d)a_j.
\end{aligned}$$

Since

$$\rho_{-d}(\alpha,\alpha_2,\ldots,\alpha_d) = (-1)^{1+d}\alpha^{-1}\alpha_2^{-1}\ldots\alpha_d^{-1}$$

and since

$$\begin{aligned}
&\sum_{j=0}^{\min\{d,-i\}}\rho_{i+j-d}(\alpha,\alpha_2,\ldots,\alpha_d)a_j\\
= &\sum_{j=-d+\min\{d,-i\}}^{\min\{d,-i\}}\rho_{i+j-d}(\alpha,\alpha_2,\ldots,\alpha_d)a_j = 0
\end{aligned}$$

for $i \le -1$, we conclude that

$$\frac{1}{a_d}\sum_{i=-\infty}^{-1}\rho_i(\alpha,\alpha_2,\ldots,\alpha_d)s_{i-n}(\xi) = \varepsilon(\xi\alpha^n).$$

By the same way as above we can check the representation of $u(\xi\alpha^n)$. □

COROLLARY 3.2. *Let ξ be arbitrary real number. Then*

$$\varepsilon(\xi\alpha^n) = -\frac{1}{a_0}\sum_{i=0}^{\infty}\rho_i(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1})s_{-i-n-d}(\xi).$$

Proof. Since

$$\begin{aligned}\frac{1}{a_d}\rho_i(\alpha, \alpha_2, \ldots, \alpha_d) &= \frac{1}{a_d}\prod_{l=1}^{d}\left(\prod_{\substack{1\le h\le d\\ h\ne l}}\frac{-\alpha_l^{-1}\alpha_h^{-1}}{\alpha_l^{-1}-\alpha_h^{-1}}\right)\alpha_l^{i+d-1}\\ &= -\frac{1}{a_0}\rho_{-i-d}(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}),\end{aligned}$$

we obtain

$$\begin{aligned}\varepsilon(\xi\alpha^n) &= \frac{1}{a_d}\sum_{i=-\infty}^{-d}\rho_i(\alpha, \alpha_2, \ldots, \alpha_d)s_{i-n}(\xi)\\ &= -\frac{1}{a_0}\sum_{i=-\infty}^{-d}\rho_{-i-d}(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1})s_{i-n}(\xi)\\ &= -\frac{1}{a_0}\sum_{i=0}^{\infty}\rho_i(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1})s_{-i-n-d}(\xi).\end{aligned}$$

$\square$

We end this section by introducing a property of the sequence $\mathbf{w} = (w_n)_{n=0}^{\infty}$ defined in Section 2.

PROPOSITION 3.3 (Dubickas [6]). *Let $\mathbf{b} = (b_n)_{n=0}^{\infty}$ be a sequence with $b_n \in \{1, 2\}$ which is not ultimately periodic. Then $\mathbf{b}$ satisfies at least one of the following:*

1. *For any $N \ge 0$, there exists an $m \ge 0$ such that*

$$b_{m+i} = w_i \ (i = 0, 1, \ldots, N);$$

2. *There exist $N \ge 2$ and infinitely many $m \ge 0$ such that*

$$b_{m+i} = w_i \ (i = 0, 1, \ldots, N-1)$$

and

$$\begin{cases} b_{m+N} = 2, w_N = 1 & \text{if } N \text{ is even,}\\ b_{m+N} = 1, w_N = 2 & \text{if } N \text{ is odd.}\end{cases}$$

4. Proof of the main results

Proof of Theorem 2.3. For simplicity we set

$$\mu_m = \frac{1}{|a_0|}\rho_m(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}) \ (m = 0, 1, \ldots),$$

and

$$\psi = \frac{1}{|a_0|}E_d(\alpha^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}).$$

Take an arbitrary nonzero real number ξ. We put

$$t_n = s_{-n-d}(\xi) \ (n = 0, 1, \ldots).$$

By Lemma 2.1 and Corollary 3.2

$$||\xi\alpha^n|| = \left|\sum_{i=0}^{\infty} \mu_i t_{i+n}\right|, \ \psi = \sum_{i=0}^{\infty} \mu_i e_i.$$

Let

$$t = \limsup_{n\to\infty} |t_n|.$$

We first consider the case where $t \geq 2$. Since the sequence $\mathbf{t} = (t_n)_{n=0}^{\infty}$ is not ultimately periodic, at least one of the words $W_m = t, m \ (-t+1 \leq m \leq t)$ or $\overline{W_m} = -t, m \ (-t \leq m \leq t-1)$ appears infinitely many times in $\mathbf{t}$. If $\mathbf{t}$ contains infinitely many W_m for some m with $-t+2 \leq m \leq t$ or infinitely many $\overline{W_m}$ for some m with $-t \leq m \leq t-2$, then by (2.5)

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \geq t\mu_0 + (2-t)\mu_1 - \sum_{i=2}^{\infty} t\mu_i \geq \mu_0 \geq \psi.$$

Thus we may assume that W_m with $-t+2 \leq m \leq t$ and $\overline{W_m}$ with $-t \leq m \leq t-2$ appear in $\mathbf{t}$ only finitely many times.

Suppose that W_{-t+1} appears infinitely many times in $\mathbf{t}$. Then for every sufficiently large n and for each $m \geq 0$,

$$\mu_i t_{n+i} + \mu_{i+1} t_{n+i+1} \geq -t\mu_i - (t-2)\mu_{i+1},$$

consequently

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \geq t\mu_0 - (t-1)\mu_1 - \sum_{i=1}^{\infty} \big(t\mu_{2i} + (t-2)\mu_{2i+1}\big). \tag{4.1}$$

The inequality (4.1) is true also in the case where $\overline{W_{t-1}}$ appears infinitely many times in $\mathbf{t}$. Using (2.5), (4.1) and

$$\psi \leq \mu_0 - \mu_2 + \mu_3 - \mu_4 + \mu_6,$$

we obtain

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \geq \psi.$$

In what follows, we assume that $t = 1$. By the same way as above $\limsup_{n\to\infty} ||\xi\alpha^n|| \geq \psi$ if at least one of the following words appears infinitely many times in $\mathbf{t}$:

$$U_1 = 1, 1;\ U_2 = -1, -1;\ U_3 = 1, 0, 1;\ U_4 = -1, 0, -1,\ V = 0, 0.$$

It suffices to prove the theorem in the case where these words appear in $\mathbf{t}$ only finitely many times. As a result, there exists an $M \geq 0$ such that

$$t_M, t_{M+1}, t_{M+2}, \ldots = 1, \underbrace{0, \ldots, 0}_{x_0}, -1, \underbrace{0, \ldots, 0}_{x_1}, 1, \underbrace{0, \ldots, 0}_{x_2}, -1, \ldots,$$

where

$$x_n \in \{0, 1\}\ (n = 0, 1, \ldots).$$

Put $y_n = 1 + x_n$. Note that the sequence $\mathbf{y} = (y_n)_{n=0}^{\infty}$ is not ultimately periodic and

$$\begin{aligned}
\limsup_{n\to\infty} ||\xi\alpha^n|| &\geq \sum_{i=0}^{\infty} (-1)^i \mu_{y_m + y_{m+1} + \cdots + y_{m+i-1}}\ (m = 0, 1, \ldots), \\
\psi &= \sum_{i=0}^{\infty} (-1)^i \mu_{w_0 + w_1 + \cdots + w_{i-1}}.
\end{aligned}$$

We apply Proposition 3.3 to $\mathbf{y}$. If $\mathbf{y}$ satisfies the statement 1, then $\limsup_{n\to\infty} ||\xi\alpha^n|| \geq \psi$. Otherwise, we choose N and m satisfying the statement 2 of Proposition 3.3. Put

$$w = w_0 + w_1 + \cdots + w_{N-1}.$$

Then by (2.5) we have

$$\limsup_{n\to\infty} ||\xi\alpha^n|| - \psi \geq \mu_{1+w} - \mu_{2+w} - \mu_{1+w+v} \geq 0,$$

where

$$v = \begin{cases} w_{N+1} & \text{if } N \text{ is even,} \\ y_{m+N+1} & \text{if } N \text{ is odd.} \end{cases}$$

$\square$

Proof of Corollary 2.6. Since $\zeta\alpha$ and its conjugate $\zeta\alpha_2$ satisfy the assumptions of Theorem 2.3, we have

$$\mathcal{E}(\alpha) = \mathcal{E}(\zeta\alpha) \geq \frac{1}{|a_0|} E_2(\zeta\alpha^{-1}, \zeta\alpha_2^{-1}).$$

$\square$

Proof of Theorem 2.1. We use the same notation as in the proof of Theorem 2.3.

First we assume $t \geq 2$. Since

$$|t_n| = |a_2\varepsilon(\xi\alpha^{-n}) + a_1\varepsilon(\xi\alpha^{-n-1}) + a_0\varepsilon(\xi\alpha^{-n-2})| \geq 2$$

for infinitely many n, we have

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \geq \frac{2}{|a_0|+|a_1|+|a_2|} > \frac{1}{|a_0|} > \psi.$$

In what follows, we suppose that $t=1$. By the same way as in the proof of Theorem 2.3, it suffices to check the case where the following words appear in $\mathbf{t}=(t_n)_{n=0}^{\infty}$ only finitely many times:

$$U_1=1,1\,;\quad U_2=-1,-1;\ U_3=1,0,1;\ U_4=-1,0,-1;$$
$$U_5\ =\ 1,0,0,1;\ U_6=-1,0,0,-1;\ U_7=0,0,0.$$

Consequently, there exists an $M\geq 0$ such that

$$t_M,t_{M+1},t_{M+2},\ldots=1,\underbrace{0,\ldots,0}_{x_0},-1,\underbrace{0,\ldots,0}_{x_1},1,\underbrace{0,\ldots,0}_{x_2},-1,\ldots,$$

where

$$x_n\in\{0,1,2\}\ (n=0,1,2,\ldots).$$

Suppose that $x_n=2$ for infinitely many n. Then we may assume that

$$\mu_2<2\mu_3$$

since by (2.4)

$$\limsup_{n\to\infty}||\xi\alpha^n||-\psi\geq(\mu_0-\mu_3)-(\mu_0-\mu_2+\mu_3)=\mu_2-2\mu_3.$$

Let $(\tau_n)_{n=0}^{\infty}$ be defined by the recurrence

$$\tau_0=1,\ \tau_{n+1}=(\sqrt{5}-1)\frac{1+\tau_n+{\tau_n}^2+{\tau_n}^3}{1+\tau_n+{\tau_n}^2}-1.$$

Then $(\tau_n)_{n=0}^{\infty}$ is a positive decreasing sequence, which converges to $\tau=0.25093\ldots$ as n tends to infinity. Put

$$p=\max\{\alpha^{-1},\alpha_2^{-1}\},\ q=\min\{\alpha^{-1},\alpha_2^{-1}\}.$$

Now we verify that

$$q\leq\tau p, \tag{4.2}$$

checking that

$$q\leq\tau_n p\ (n=0,1,\ldots) \tag{4.3}$$

by induction on n. The inequality (4.3) is clear if $n=0$. Suppose $n\geq 1$. By the induction hypothesis we have

$$\begin{aligned}0<2\mu_3-\mu_2 &= \mu_2(2p-1)+\frac{2}{|a_0|}q^3\\ &\leq \mu_2\left(2p\frac{1+\tau_n+\tau_n^2+\tau_n^3}{1+\tau_n+\tau_n^2}-1\right)\end{aligned}$$

and so

$$\begin{aligned} p &\geq \frac{1+\tau_n+\tau_n^2}{2(1+\tau_n+\tau_n^2+\tau_n^3)}, \\ q &\leq \frac{\sqrt{5}-1}{2} - \frac{1+\tau_n+\tau_n^2}{2(1+\tau_n+\tau_n^2+\tau_n^3)}. \end{aligned}$$

Hence we obtain $q \leq \tau_{n+1} p$. Moreover, we have the following:

$$\frac{\sqrt{5}-1}{2} \geq p \geq \frac{1+\tau+\tau^2}{2(1+\tau+\tau^2+\tau^3)} = 0.49405\ldots; \tag{4.4}$$

$$q \leq 0.12397\ldots. \tag{4.5}$$

Since $(x_n)_{n=0}^{\infty}$ is not ultimately periodic, there exist infinitely many $n \geq 0$ such that $x_n = 2$ and $x_{n+1} \leq 1$. Thus by (2.4)

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \geq \mu_0 - \mu_3 + \min\{\mu_4 - \mu_5, \mu_5 - \mu_6\}.$$

Since

$$\begin{aligned} \mu_4 - 2\mu_5 + \mu_6 &\geq (1-2p+p^2)\mu_4 - \frac{2}{|a_0|} q^5 \\ &\geq \left(\frac{7-3\sqrt{5}}{2} - \frac{2q}{1+\tau^{-1}+\tau^{-2}+\tau^{-3}+\tau^{-4}} \right) \mu_4 \\ &> 0, \end{aligned}$$

we get

$$\begin{aligned} \limsup_{n\to\infty} ||\xi\alpha^n|| - \psi &\geq \mu_2 - 2\mu_3 + \mu_4 + \mu_5 - 2\mu_6 \\ &\geq (1-2p+p^2+p^3-2p^4)\mu_2 - \frac{3}{|a_0|} q^3 \\ &> 0. \end{aligned}$$

In what follows, we suppose that $x_n \in \{0, 1\}$ for all n. We may assume that the sequence $\mathbf{y} = (y_n)_{n=0}^{\infty}$, where $y_n = 1 + x_n$, satisfies the second statement of Proposition 3.3. Let N, m, w and v be as in the proof of Theorem 2.3. Then by (2.4)

$$\limsup_{n\to\infty} ||\xi\alpha^n|| - \psi \geq \mu_{1+w} - \mu_{2+w} - \mu_{1+w+v}$$

and so we may assume that $v = 1$ and that $2\mu_{2+w} \geq \mu_{1+w}$. Since $(\mu_{n+1}/\mu_n)_{n=0}^{\infty}$ is a decreasing sequence, we see that $2\mu_3 > \mu_2$. Therefore the inequalities (4.2), (4.4) and (4.5) hold.

Assume that N is odd. Then by (2.4)

$$\limsup_{n\to\infty} ||\xi\alpha^n|| - \psi \geq \mu(1+w) - 2\mu(2+w) + \sum_{i=0}^{3}(-1)^i\mu\left(2+w+\sum_{k=0}^{i} w_{N+1+k}\right) + \sum_{i=0}^{3}(-1)^i\mu\left(2+w+\sum_{k=0}^{i} y_{m+N+2+k}\right),$$

where

$$\mu(h) = \mu_h$$

for $h \geq 0$. Since $y_n \in \{1,2\}$ for each n, it is easy to check

$$\sum_{i=0}^{3}(-1)^i \mu\left(2+w+\sum_{k=0}^{i} y_{m+N+2+k}\right) \geq \mu(4+w) - \mu(5+w) + \mu(7+w) - \mu(8+w).$$

Since $w_N = 2$ and since $\mathbf{w}$ is a concatenation of the words $A_2 = 2,1,1$ and $A_3 = 2,1,1,2,2$, we have

$$(w_{N+1}, w_{N+2}, w_{N+3}, w_{N+4}) \in \{(1,1,2,1), (1,1,2,2), (2,1,1,2), (2,2,1,1)\}.$$

Thus we get

$$\sum_{i=0}^{3}(-1)^i \mu\left(2+w+\sum_{k=0}^{i} w_{N+1+k}\right) \geq \mu(4+w) - \mu(5+w) + \mu(6+w) - \mu(8+w).$$

Using (4.4), (4.5), and $w \geq 3$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} &||\ \xi\alpha^n|| - \psi \\ &\geq (1-2p+2p^3-2p^4+p^5+p^6-2p^7)\mu(1+w) - \frac{4q^{2+w}}{|a_0|} \\ &\geq \left(\frac{\sqrt{5}-1}{2}\right)^8 \mu(1+w) - \frac{4q^{2+w}}{|a_0|} > 0.\end{aligned}$$

Next, we consider the case where N is even. Then by (2.4) we have

$$\limsup_{n\to\infty} ||\xi\alpha^n|| - \psi \geq \mu(1+w) - 2\mu(2+w) + \sum_{i=0}^{3}(-1)^i\mu\left(2+w+\sum_{k=0}^{i} w_{N+2+k}\right) + \sum_{i=0}^{3}(-1)^i\mu\left(2+w+\sum_{k=0}^{i} y_{m+N+1+k}\right)$$

and

$$\sum_{i=0}^{3}(-1)^i \ \mu\left(2+w+\sum_{k=0}^{i} y_{m+N+1+k}\right)$$
$$\geq \ \mu(4+w)-\mu(5+w)+\mu(7+w)-\mu(8+w).$$

Since $w_N = w_{N+1} = 1$, we get

$$(w_{N+2}, w_{N+3}, w_{N+4}, w_{N+5}) \in \{(2,1,1,2),(2,2,2,1)\},$$

and so

$$\sum_{i=0}^{3}(-1)^i \ \mu\left(2+w+\sum_{k=0}^{i} w_{N+2+k}\right)$$
$$\geq \ \mu(4+w)-\mu(5+w)+\mu(6+w)-\mu(8+w).$$

Hence we obtain

$$\limsup_{n\to\infty} ||\xi\alpha^n|| > \psi.$$

$\square$

5. Geometric sequences with small fractional parts

Koksma [10] proved that, if any real number $\alpha > 1$ is given, the sequence $(\xi\alpha^n)_{n=0}^{\infty}$ is uniformly distributed modulo 1 for almost all ξ. Similarly, if any nonzero real number ξ is given, the sequence $(\xi\alpha^n)_{n=0}^{\infty}$ is uniformly distributed modulo 1 for almost all $\alpha > 1$. In this section we study the exceptional set of Koksma's Theorem. Boyd proved the following:

THEOREM 5.1 (Boyd [3]**).** *Let δ, M be arbitrary positive numbers. Then the set of real numbers $\alpha \geq M$ such that*

$$\sup_{n\geq 0} \|\xi\alpha^n\| \leq \frac{1}{(\alpha-1)(\alpha-3)}$$

for some ξ with $|\xi-2| \leq \delta$ is uncountable.

It is known that there exist only countable pairs (ξ, α), where $\xi \neq 0$ and $\alpha > 1$, satisfying

$$\limsup_{n\to\infty} \|\xi\alpha^n\| < \frac{1}{2(1+\alpha)^2}$$

(see for instance [2, p.95]).

We now consider the exceptional set of Koksma's Theorem for a fixed $\alpha > 1$. Tijdeman [12] proved for any $\alpha > 1$ that there exists a nonzero ξ such that $\{\xi\alpha^n\} = \xi\alpha^n - [\xi\alpha^n]$ and $[\xi\alpha^n] \leq 1/(\alpha-1)(n = 0, 1, \ldots)$ is the largest integer not greater than $\xi\alpha^n$. By the same way as Tijdeman we can prove the existence of ξ such that $\|\xi\alpha^n\| \leq 1/(2\alpha-2)(n = 0, 1, \ldots)$. We note that Dubickas [4, 7] obtained sharper estimation.

Next, we consider the exceptional set for a fixed nonzero ξ.

THEOREM 5.2. (1) *Let ξ be a nonzero real number. Then for arbitrary positive numbers δ and M, the set of real numbers α with $\alpha > M$ satisfying*

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \le \frac{1+\delta}{2\alpha}$$

is at least countable.

(2) *Let ξ be a nonzero real number. Then for arbitrary positive numbers δ and M, the set of real numbers α with $\alpha > M$ satisfying*

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \le \frac{1+\delta}{\alpha}$$

is uncountable.

Proof. We may assume $\xi > 0$. First we prove the statement (1) of the theorem. Take any positive integer R with $R \ge \max\{12, \xi, \xi^{-1/2}\}$. We define the sequence $(z_n)_{n=0}^{\infty}$ by

$$z_1 = R^2, \;\; z_{n+1} = u(\xi^{-1/n} z_n^{(n+1)/n}) \; (n = 1, 2, \ldots).$$

Since $R \ge \xi$, we can easily check by induction on n that

$$z_n \ge R^{n+1}.$$

We put

$$\beta_n = \xi^{-1/n} z_n^{1/n}.$$

Since

$$|z_{n+1} - \xi^{-1/n} z_n^{(n+1)/n}| \le \frac{1}{2},$$

we have

$$|\beta_n - \beta_{n+1}| \le \frac{1}{2}\xi^{-1}\left(\sum_{i=0}^{n} \beta_{n+1}^i \beta_n^{n-i}\right)^{-1}. \tag{5.1}$$

We denote the right-hand side of (5.1) by q_n. Since

$$q_n \le \frac{1}{2}\xi^{-1}(n+1)^{-1}R^{-n}, \tag{5.2}$$

the sequence $(\beta_n)_{n=1}^{\infty}$ converges. Let $\alpha = \lim_{n\to\infty} \beta_n$. Using (5.2) and

$$\beta_{n+1} \ge \beta_n(1 - \beta_n^{-1}q_n) \ge \beta_n(1 - R^{-1}q_n),$$

we obtain

$$\begin{aligned}\beta_n &\ge \beta_1 \prod_{i=1}^{n-1}\left(1 - \frac{1}{2}\xi^{-1}(i+1)^{-1}R^{-i-1}\right) \\ &\ge \xi^{-1}R^2 \prod_{i=2}^{\infty}\left(1 - \frac{1}{2}\xi^{-1}i^{-1}R^{-i}\right).\end{aligned}$$

In what follows, C_1, C_2, and C_3 denote positive constants depending only on ξ. Then

$$\beta_n \geq \xi^{-1} R^2 (1 - C_1 R^{-2}).$$

By considering only the case where R is sufficiently large, we may assume that

$$C_1 R^{-2} \leq \frac{1}{2}.$$

Thus

$$\begin{aligned}
\frac{q_{n+1}}{q_n} &\leq \frac{\sum_{i=0}^{n} \beta_{n+1}^i \beta_n^{n-i}}{\beta_{n+1} \sum_{i=0}^{n} \beta_{n+2}^i \beta_{n+1}^{n-i}} \\
&\leq \xi R^{-2} (1 - C_1 R^{-2})^{-1} \left(1 - \frac{1}{2} \xi^{-1} R^{-2} (n+1)^{-1}\right)^{-n} \\
&\leq \xi R^{-2} (1 + 2C_1 R^{-2}) \left(1 + \xi^{-1} R^{-2} (n+1)^{-1}\right)^n \\
&\leq \xi R^{-2} (1 + 2C_1 R^{-2})(1 + 2\xi^{-1} R^{-2}) \\
&\leq 6\xi R^{-2}
\end{aligned}$$

and so

$$|\alpha - \beta_n| \leq \sum_{i=0}^{\infty} q_{n+i} \leq (1 + 12\xi R^{-2}) q_n$$

for any $n \geq 1$. Therefore

$$\begin{aligned}
|\xi\alpha^n - z_n| &\leq \xi |\alpha - \beta_n| \sum_{i=0}^{n-1} \alpha^i \beta_n^{n-1-i} \\
&\leq \frac{1}{2}(1 + 12\xi R^{-2}) \frac{\sum_{i=0}^{n-1} \alpha^i \beta_n^{n-1-i}}{\sum_{i=0}^{n} \beta_{n+1}^i \beta_n^{n-i}} \\
&\leq \frac{1}{2}\xi R^{-2} (1 + 12\xi R^{-2})(1 - C_1 R^{-2})^{-1} \frac{\sum_{i=0}^{n-1} \alpha^i \beta_n^{n-1-i}}{\sum_{i=0}^{n-1} \beta_{n+1}^i \beta_n^{n-1-i}} \\
&\leq \frac{1}{2}\xi R^{-2} (1 + C_2 R^{-2})
\end{aligned}$$

for $n \geq 1$. We may assume that

$$\frac{1}{2}\xi R^{-2} (1 + C_2 R^{-2}) < \frac{1}{2}.$$

Hence

$$|\xi R^{-2} - \alpha^{-1}| = \frac{1}{\alpha R^2} |\xi\alpha - z_1| < \frac{1}{2\alpha R^2},$$

and so

$$|\xi\alpha^n - z_n| \leq \frac{1 + C_3 R^{-2}}{2\alpha}.$$

Since z_n is a rational integer, the statement (1) was proved.

Next we prove the statement (2). Take any sufficiently large integer R. Let $\mathbf{p} = (p_n)_{n=0}^{\infty}$ be a sequence with $p_n \in \{0,1\}$. We define the sequence $(z_n(\mathbf{p}))_{n=0}^{\infty}$ by

$$\begin{aligned} z_1(\mathbf{p}) &= R^2, \\ z_{n+1}(\mathbf{p}) &= \left[\xi^{-1/n} z_n(\mathbf{p})^{(n+1)/n}\right] + p_n \ (n = 1, 2, \ldots). \end{aligned}$$

Then we have

$$\left|\xi^{-1/n} z_n(\mathbf{p})^{(n+1)/n} - z_{n+1}(\mathbf{p})\right| \leq 1.$$

By the same way as in the proof of statement (1) we can check that the sequence $(\xi^{-1/n} z_n(\mathbf{p})^{1/n})_{n=1}^{\infty}$ has a limit $\alpha(\mathbf{p})$ and that

$$|\xi\alpha(\mathbf{p})^n - z_n(\mathbf{p})| \leq \frac{1+\delta}{\alpha} < \frac{1}{2}.$$

This implies the statement (2). In fact, if the sequences $\mathbf{p}$ and $\mathbf{p}'$ are different, then $\alpha(\mathbf{p}) \neq \alpha(\mathbf{p}')$. $\square$

In contrast with Tijdeman's result and Theorem 5.2, the following theorem implies that, if $\alpha > 1$ (resp. $\xi \neq 0$), then the set of nonzero ξ (resp. $\alpha > 1$) satisfying a stronger inequality is at most countable.

THEOREM 5.3. (1) *Let $\alpha > 1$. Then the set of real numbers ξ satisfying*

$$\limsup_{n\to\infty} ||\xi\alpha^n|| < \frac{1}{2\alpha+2} \tag{5.3}$$

is at most countable.
(2) *Let ξ be a nonzero real number. Then the set of real numbers α with $\alpha > 1$ satisfying*

$$\limsup_{n\to\infty} ||\xi\alpha^n|| < \frac{1}{2\alpha+2} \tag{5.4}$$

is at most countable.

Proof. If ξ satisfies (5.3), we have

$$|u(\xi\alpha^{n+1}) - \alpha u(\xi\alpha^n)| = |\varepsilon(\xi\alpha^{n+1}) - \alpha\varepsilon(\xi\alpha^n)| < \frac{1}{2}$$

for all large n. Thus we get

$$u(\xi\alpha^{n+1}) = u(\alpha u(\xi\alpha^n)).$$

It is clear that the set of the sequences $(y_n)_{n=0}^{\infty}$ of rational integers satisfying $y_{n+1} = u(\alpha y_n)$ for all sufficiently large n is countable. Thus the statement (1) was proved. In fact, if $\xi \neq \xi'$, then the sequences $(u(\xi\alpha^n))_{n=0}^{\infty}$ and $(u(\xi'\alpha^n))_{n=0}^{\infty}$ are different.

Similarly, for the proof of the statement (2) it is sufficient to check the following: If α satisfies (5.4), then

$$u(\xi\alpha^{n+1}) = u\left(\xi^{-1/n} u(\xi\alpha^n)^{(n+1)/n}\right)$$

for all large n. Putting

$$u_n = u(\xi\alpha^n) \text{ and } \varepsilon_n = \varepsilon(\xi\alpha^n),$$

we have

$$\begin{aligned} |u_{n+1} - \xi^{-1/n} u_n^{(n+1)/n}| &\leq |u_{n+1} - \xi\alpha^{n+1}| + |\xi^{-1/n} u_n^{(n+1)/n} - \xi\alpha^{n+1}| \\ &\leq |\varepsilon_{n+1}| + |\xi\alpha^{n+1}| \big| (1 - \varepsilon_n \xi^{-1}\alpha^{-n})^{(n+1)/n} - 1 \big|. \end{aligned}$$

Using the mean value theorem, we obtain

$$(1 - \varepsilon_n \xi^{-1}\alpha^{-n})^{(n+1)/n} = 1 - \varepsilon_n \xi^{-1}\alpha^{-n} \frac{n+1}{n} (1 - \delta\varepsilon_n \xi^{-1}\alpha^{-n})^{1/n},$$

where $0 < \delta < 1$. Hence we conclude that

$$|u_{n+1} - \xi^{-1/n} u_n^{(n+1)/n}| < \frac{1}{2}$$

for all large n. □

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Department of Mathematics, Kyoto University
Oiwake-tyou
Kitashirakawa
Kyoto-shi, Kyoto, Japan
e-mail: kanekoha@math.kyoto-u.ac.jp

Hajime Kaneko