On the number of digit changes in base-b expansions of algebraic numbers *

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Abstract

Bugeaud [6] introduced the number of digit changes to measure the complexity of the base-b expansions of algebraic irrational numbers. We give lower bounds of the number of digit changes which are generalizations of results in [12]. We also study the number of occurrences of words in the binary expansions of algebraic irrational numbers.

1 Introduction

Let b be an integer greater than 1. Then, as is well known, the base-b expansions of all rational numbers are ultimately periodic. However, very little is known on the base-b expansions of algebraic irrational numbers. For instance, it is still not proven that the digit 0 appears infinitely often in the decimal expansion of $\sqrt{2}$.

We recall the definition of normal numbers. We call a positive real number ξ to be normal in base b if and only if, for any word w on the alphabet $\{0, 1, \ldots, b-1\}$, w occurs in the b-ary expansion of ξ with average frequency tending to $b^{-|w|}$, where $|w|$ denotes the length of w. Borel [4] proved that almost all positive numbers are normal in any integral base b. He [5] conjectured that each algebraic irrational number is normal in any base b. However, it is generally difficult to check whether a given positive number is normal in base b. For instance, there is no algebraic irrational number proven to be normal in base 10.

We introduce known results on the digits of the base-b expansions of algebraic irrational numbers. In what follows, Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{Z}^+$ the set of positive integers. We write the integral and fractional parts of a real number x by $\lfloor x \rfloor$ and $\{x\}$, respectively. Moreover, let $\lceil x \rceil$ be the smallest integer not less than x. Let ξ be an algebraic irrational number and b an integer greater than 1. In what follows, denote the base-b expansion of ξ by

$$\xi = \sum_{h=-\infty}^{R} s_h^{(b)}(\xi) b^h, \tag{1.1}$$

where $s_h^{(b)}(\xi)$ is the h-th digit in the base-b expansion of ξ written as

$$s_h^{(b)}(\xi) = \lfloor \xi b^{-h} \rfloor - b \lfloor \xi b^{-h-1} \rfloor \in \{0, 1, \ldots, b-1\}$$

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and $R = R(b;\xi)$ is the maximal integer with $s_R(\xi) \neq 0$. The base-b expansion of ξ is often written as $\xi = \sum_{h=-M}^{\infty} a_h b^{-h}$, where M is an integer and a_h is the $-h$-th digit of ξ in the base-b expansion. However, we use the representation (1.1) because it is convenient for introducing symmetric signed expansions of integers and real numbers.

We first consider the block complexity. Namely, we count the number $\beta_b(\xi;N)$ of distinct blocks of length N occurring in the base-b expansions of ξ. The number $\beta_b(\xi;N)$ is written as

$$\beta_b(\xi;N) = \mathrm{Card}\{s_h^{(b)}(\xi)s_{h-1}^{(b)}(\xi)\ldots s_{h-N+1}^{(b)}(\xi) \mid h \in \mathbb{Z},\, h \leq R\},$$

where Card denotes the cardinality. If Borel's conjecture is true, then any finite word w on the alphabet $\{0,1,\ldots,b-1\}$ appears in the b-ary expansion of ξ. That is, we have

$$\beta_b(\xi;N) = b^N$$

for any positive integer N. Ferenczi and Mauduit [10] showed that

$$\lim_{N\to\infty} (\beta_b(\xi;N) - N) = \infty,$$

applying a reformulation of Ridout's theorem [14]. Adamczewski and Bugeaud [1] verified that

$$\lim_{N\to\infty} \frac{\beta_b(\xi;N)}{N} = \infty,$$

using the Schmidt subspace theorem by Adamczewski, Bugeaud, and Luca [2]. Moreover, Bugeaud and Evertse [7] proved for any positive real number δ less than 1/11 that

$$\limsup_{N\to\infty} \frac{\beta_b(\xi;N)}{N(\log N)^{\delta}} = \infty,$$

improving the quantitative parametric subspace theorem from [9].

Next we estimate the number of nonzero digits in the base-b expansions of algebraic irrational numbers ξ. For any integer N, put

$$\lambda_b(\xi;N) = \mathrm{Card}\{h \in \mathbb{Z} \mid -N \leq h \leq R,\, s_h^{(b)}(\xi) \neq 0\}.$$

In this paper O denotes the Landau symbol and $\gg$ the Vinogradov symbol. Namely, $f = O(g)$ and $g \gg f$ imply that $|f| \leq Cg$ for some constant C. $f \sim g$ means that the ratio of f and g tends to 1. If ξ is normal in base b, then we have

$$\lambda_b(\xi,N) \sim \frac{b-1}{b}N$$

as N tends to infinity. Let D be the degree of ξ. Bailey, Borwein, Crandall, and Pomerance [3] showed that if $b=2$, then

$$\lambda_2(\xi;N) \geq C_1(\xi)N^{1/D} \tag{1.2}$$

for all sufficiently large N, where $C_1(\xi)$ is an effectively computable positive constant depending only on ξ. Moreover, Rivoal [15] improved the constant $C_1(\xi)$ for certain classes of algebraic irrational numbers ξ.

Bugeaud [6] introduced the number of digit changes to measure the complexity of the base-b expansions of algebraic irrational numbers ξ. Put

$$\gamma_b(\xi; N) = \mathrm{Card}\{h \in \mathbb{Z} \mid -N \leq h \leq R-1,\ s_h^{(b)}(\xi) \neq s_{h+1}^{(b)}(\xi)\}.$$

If ξ is normal in base b, then we have

$$\gamma_b(\xi; N) \sim \frac{b^2-b}{b^2} N = \frac{b-1}{b} N$$

as N tends to infinity because

$$\mathrm{Card}\{(i,j) \in \mathbb{N}^2 \mid i, j \leq b-1,\ i \neq j\} = b^2 - b.$$

Bugeaud [6] proved, using a quantitative Ridout's theorem [13], that

$$\gamma_b(\xi; N) \geq 3(\log N)^{1+1/(w(b)+4)} \cdot (\log\log N)^{-1/4}$$

for any sufficiently large N, where $w(b)$ means the number of the distinct prime factors of b. Bugeaud and Evertse [7] improved the quantitative parametric subspace theorem by Evertse and Schlickewei [9]. Consequently, they showed that

$$\gamma_b(\xi; N) \geq C_2 \frac{(\log N)^{3/2}}{(\log 6D)^{1/2}(\log\log N)^{1/2}} \tag{1.3}$$

for all sufficiently large N, where C_2 is an effectively computable positive absolute constant. In the case of $b = 2$, the author [12] improved (1.3) as follows: Let ξ be an algebraic irrational number with minimal polynomial $A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0 \in \mathbb{Z}[X]$, where $A_D > 0$. Suppose that there exists an odd prime number p which divides the coefficients $A_D, A_{D-1}, \ldots, A_1$, but not the constant term A_0, which we call the prime divisor assumption. Then there exists an effectively computable positive constant $C_3(\xi)$ depending only on ξ such that

$$\gamma_2(\xi; N) \geq C_3(\xi) N^{1/D} \tag{1.4}$$

for any sufficiently large N. We give a numerical example of (1.4). We consider the case of $\xi = 1/\sqrt{3}$. Then the minimal polynomial of ξ is $A_2X^2 + A_1X + A_0 = 3X^2 - 1$. Thus, p=3 satisfies the prime divisor assumption because 3 divides A_2 and A_1, but not A_0. Let ε be an arbitrary positive real number less than 1. Then there exists an effectively computable positive constant $C_4(\varepsilon)$ depending only on ε such that

$$\gamma_2\left(\frac{1}{\sqrt{3}}, N\right) \geq \frac{1-\varepsilon}{\sqrt{2}} \sqrt{N}$$

for each integer N greater than $C_4(\varepsilon)$.

The main purpose of this paper is to generalize the inequality (1.4) to any integral base b. We also estimate the number of occurrences of words in the binary expansions of algebraic irrational numbers. In Section 2 we state the main results, introducing the symmetric signed expansions of integers and real numbers. In Section 3 we study more details of the symmetric signed expansions. In Section 5 we prove the main results, using the preliminaries in Section 4.

2 Main results

We introduce signed digit expansions of integers and real numbers in integral base $b \geq 2$. Heuberger and Prodinger [11] showed that any integer n is uniquely represented as a finite sum

$$n = \sum_{h=0}^{M} \sigma_h^{(b)}(n) b^h =: \left(\sigma_M^{(b)}(n) \ldots \sigma_0^{(b)}(n)\right)_b, \tag{2.1}$$

where the word $\sigma_M^{(b)}(n) \ldots \sigma_0^{(b)}(n)$ satisfies the following conditions which we call the digit conditions in this paper:

1. $$|\sigma_h^{(b)}(n)| \leq \frac{b}{2} \text{ for any } h; \tag{2.2}$$

2. $$\text{If } \sigma_h^{(b)}(n) = \frac{b}{2}, \text{ then } 0 \leq \sigma_{h+1}^{(b)}(n) \leq \frac{b}{2} - 1; \tag{2.3}$$

3. $$\text{If } \sigma_h^{(b)}(n) = -\frac{b}{2}, \text{ then } -\frac{b}{2} + 1 \leq \sigma_{h+1}^{(b)}(n) \leq 0. \tag{2.4}$$

(2.1) is called the symmetric signed digit expansion of n. In what follows, we denote it by the SSDE of n for simplicity. We call

$$\nu_b(n) := \sum_{h=0}^{M} |\sigma_h^{(b)}(n)|$$

the cost of the expansion (2.1). In the case of $b = 2$, SSDEs coincide with non-adjacent forms or signed separated binary expansions. In a sequence of signed bits, we will denote $-a$ by $\overline{a}$ for any integer a. For instance, we have $(10\overline{1})_3 = 3^2 - 1 = 8$ and $\nu_3(8) = 2$. In Section 3 we show that any real number is represented as

$$\xi = \sum_{h=-\infty}^{M} \sigma_h^{(b)}(\xi) b^h =: \left(\sigma_M^{(b)}(\xi) \ldots \sigma_0^{(b)}(\xi).\sigma_{-1}^{(b)}(\xi) \ldots\right)_b \tag{2.5}$$

where the sequence $\sigma_h^{(b)}(\xi)$ $(h = M, M-1, \ldots)$ satisfies the digit conditions. We also call (2.5) the SSDE of ξ. Note that (2.5) converges absolutely because the sequence $\sigma_h^{(b)}(\xi)$ $(h = M, M-1, \ldots)$ is bounded. Although the SSDEs (2.1) of any integers are uniquely determined, the SSDEs (2.5) of real numbers are not generally unique. For instance, we have

$$\frac{1}{2} = (0.1^\omega)_3 = (1.\overline{1}^\omega)_3,$$

where, for any nonempty finite word v, we denote the right-infinite word $vv\ldots$ by v^ω. In Section 3 we prove that the SSDE of any rational number ξ is ultimately periodic. Namely,

$$\xi = \left(\sigma_M^{(b)}(\xi)\ldots\sigma_0^{(b)}(\xi).\sigma_{-1}^{(b)}(\xi)\ldots\sigma_{-L}^{(b)}(\xi)v^\omega\right)_b,$$

where $v = v_1\ldots v_r$ is a nonempty finite word. Assume that r is the least period of the SSDE of ξ. We call $\rho = \sum_{h=1}^r |v_h|$ the cost of the period of ξ. For instance, the cost of the period of $1/2 = (0.1^\omega)_3$ is 1. Lemma 3.2 implies that the period r and the cost ρ of the period are uniquely determined by ξ although the SSDE of ξ is not generally unique.

We state the main results on the number of digit changes of the (ordinary) base-b expansions of algebraic irrational numbers, using the SSDEs of certain rational numbers. The key observation is as follows: Let η be a rational number with $|\eta| = p/q$, where $p \geq q \geq 2$ are relatively prime integers. Assume that q and b are also relatively prime. Then the SSDE of η is not finite. Namely, the cost of the period of η is not zero. In fact, if the SSDE of η is finite, then we have $\eta = A/b^l$ with $A \in \mathbb{Z}$ and $l \in \mathbb{N}$, which is a contradiction.

THEOREM 2.1. *Let b be an integer greater than 1 and ξ a positive algebraic irrational number with minimal polynomial $A_D X^D + \cdots + A_0 \in \mathbb{Z}[X]$, where $A_D > 0$. Assume that there is a positive integer u satisfying the following three assumptions:*

1. *u and b are relatively prime;*
2. *u does not divide $A_0(b-1)^D$;*
3. *u divides $A_h(b-1)^{D-h}$ for any $h = 1,\ldots,D$.*

Let r be the least period of the SSDE of

$$\eta := -\frac{A_0(b-1)^D}{u}$$

and let ρ be the cost of the period of η. Then, for an arbitrary positive real number ε less than 1, there exists an effectively computable positive constant $C_4(b,\xi,\varepsilon)$ depending only on b, ξ, and ε such that

$$\gamma_b(\xi;N) \geq (1-\varepsilon)\mu(\xi)N^{1/D}$$

for any integer N with $N \geq C_4(b,\xi,\varepsilon)$, where

$$\mu(\xi) = \frac{1}{2b-3}\left(\frac{\rho}{r}\right)^{1/D}\nu_b\left(\frac{A_D}{u}\right)^{-1/D}.$$

Note that if $b = 2$ and if u is a prime number, then the assumptions in Theorem 2.1 coincide with the prime divisor assumptions which we mentioned in Section 1. We give a numerical example of Theorem 2.1. Consider the case where $b = 3$ and $\xi = 1/2\sqrt{2}$. Then the minimal polynomial of ξ is $A_2X^2 + A_1X + A_0 = 8X^2 - 1$. Thus, $u = 8$ satisfies the assumptions in Theorem 2.1. In fact, u divides $A_1(b-1) = 0$ and $A_2(b-1)^0 = 8$, but not $A_0(b-1)^2 = -4$. We get

$\eta = 1/2 = (0.1^\omega)_3$. In particular, $r = \rho = 1$. Moreover, since the SSDE of A_2/u is $(1)_b$, we have $\nu_b(A_2/u) = 1$. Hence,

$$\mu\left(\frac{1}{2\sqrt{2}}\right) = \frac{1}{3}.$$

Let ε be any real number less than 1. Then Theorem 2.1 implies that

$$\gamma_3\left(\frac{1}{2\sqrt{2}}; N\right) \geq \frac{1-\varepsilon}{3}\sqrt{N}$$

for any sufficiently large N.

However, we cannot apply Theorem 2.1 in the case of $b = 3$ and $\xi' = 1/\sqrt{2}$. In fact, the minimal polynomial of ξ' is $A_2'X^2 + A_1'X + A_0' = 2X^2 - 1$. Suppose that u' satisfies the third assumption in Theorem 2.1. Then u' divides $A_2'(b-1)^0 = 2$. Hence, u' does not fulfill the second assumption in Theorem 2.1.

In the rest of this section, we consider the number of occurrences of words in the (ordinary) binary expansions of algebraic irrational numbers ξ. Recall that the binary expansion of ξ is (1.1) with $b = 2$. For any nonempty finite word w on the alphabet $\{0, 1\}$, put

$$f(\xi, w; N) := \mathrm{Card}\{-N \leq h \leq R - |w| + 1 \mid s_{h+|w|-1}^{(2)}(\xi) \ldots s_h^{(2)}(\xi) = w\}.$$

If Borel's conjecture is true, then we have

$$f(\xi, w; N) \sim \frac{N}{2^{|w|}}$$

as N tends to infinity.

First we consider the case where the length of w is 1. Let D be the degree of ξ. Then (1.2) implies that

$$f(\xi, 1; N) \gg N^{1/D}$$

for any sufficiently large N. Take a positive integer M such that $2^M > \xi$. Using (1.2) again, we get

$$f(\xi, 0; N) = f(2^M - \xi, 1; N) + O(1) \gg N^{1/D}.$$

Next we consider the case of $|w| = 2$. We have

$$f(\xi, 01; N) = \frac{1}{2}\gamma_2(\xi; N) + O(1), \tag{2.6}$$

$$f(\xi, 10; N) = \frac{1}{2}\gamma_2(\xi; N) + O(1). \tag{2.7}$$

Thus, by (1.3), (2.6), and (2.7), we get

$$f(\xi, 01; N) \gg \frac{(\log N)^{3/2}}{(\log\log N)^{1/2}},\ f(\xi, 10; N) \gg \frac{(\log N)^{3/2}}{(\log\log N)^{1/2}}$$

for any sufficiently large N. If ξ satisfies the prime divisor assumption, then (1.4), (2.6) and (2.7) imply that

$$f(\xi, 01; N) \gg N^{1/D},\ f(\xi, 10; N) \gg N^{1/D}$$

for every sufficiently large N. However, for any algebraic irrational number ξ, it has neither been proven that

$$\lim_{N\to\infty} f(\xi, 00; N) = \infty$$

nor that

$$\lim_{N\to\infty} f(\xi, 11; N) = \infty.$$

On the other hand, for any positive irrational number ξ, we have

$$\lim_{N\to\infty} (f(\xi, 00; N) + f(\xi, 11; N)) = \infty$$

In fact, if

$$\lim_{N\to\infty} (f(\xi, 00; N) + f(\xi, 11; N)) < \infty$$

then the binary expansion of ξ is written as $\xi = (s_M \dots s_0.s_{-1} \dots s_{-L}(01)^\omega)_2$, which is a contradiction.

We now give lower bounds of the number $f(\xi, 00; N) + f(\xi, 11; N)$ for certain classes of algebraic irrational numbers ξ as follows:

THEOREM 2.2. *Let ξ be a positive algebraic irrational number with minimal polynomial $A_D X^D + \cdots + A_0 \in \mathbb{Z}[X]$, where $A_D > 0$. Assume that there is a positive odd integer u' satisfying the following two assumptions:*

1. *u' does not divide $3^D A_0$;*
2. *u' divides $3^{D-h} A_h$ for any $h = 1, \dots, D$.*

Let r' be the least period of the SSDE of

$$\eta' := -\frac{3^D A_0}{u'}$$

and let ρ' be the cost of the period of η'. Then, for any positive real number ε less than 1, there exists an effectively computable positive constant $C_5(\xi, \varepsilon)$ depending only on ξ and ε such that

$$f(\xi, 00; N) + f(\xi, 11; N) \geq (1 - \varepsilon)\mu'(\xi) N^{1/D}$$

for any integer N with $N \geq C_5(\xi, \varepsilon)$, where

$$\mu'(\xi) = \frac{1}{6} \left(\frac{\rho'}{r'}\right)^{1/D} \nu_2 \left(\frac{A_D}{u'}\right)^{-1/D}.$$

We consider the case of $\xi = 1/\sqrt{5}$. The minimal polynomial of ξ is $A_2 X^2 + A_1 X + A_0 = 5X^2 - 1$. It is easily seen that $u = 5$ satisfies the assumptions in Theorem 2.1. Observe that the SSDE of η is

$$\eta = \frac{1}{5} = \left(0.(010\bar{1})^\omega\right)_2.$$

In particular, we get $r = 4$ and $\rho = 2$. We have $\nu_2(A_2/u) = \nu_2(1) = 1$. Thus,

$$\mu\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{2}}$$

Let ε be an arbitrary positive real number less than 1. Theorem 2.1 implies that

$$\gamma_2\left(\frac{1}{\sqrt{5}}; N\right) \geq \frac{1-\varepsilon}{\sqrt{2}}\sqrt{N}$$

for all sufficiently large N. Hence, using (2.6) and (2.7), we obtain

$$f\left(\frac{1}{\sqrt{5}}, 01; N\right) \geq \frac{1-\varepsilon}{2\sqrt{2}}\sqrt{N},\ f\left(\frac{1}{\sqrt{5}}, 10; N\right) \geq \frac{1-\varepsilon}{2\sqrt{2}}\sqrt{N}$$

for every sufficiently large N. On the other hand, $u' = 5$ satisfies the assumptions in Theorem 2.2. The SSDE of η' is

$$\eta' = \frac{9}{5} = \left(10.(0\bar{1}01)^{\omega}\right)_2.$$

Thus, we get $r' = 4$ and $\rho' = 2$. Since $\nu_2(A_2/u') = \nu_2(1) = 1$, we obtain

$$\mu'\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{6\sqrt{2}}.$$

Hence, using Theorem 2.2, we deduce that

$$f\left(\frac{1}{\sqrt{5}}, 00; N\right) + f\left(\frac{1}{\sqrt{5}}, 11; N\right) \geq \frac{1-\varepsilon}{6\sqrt{2}}\sqrt{N}$$

for any sufficiently large N.

3 Symmetric signed expansions of real numbers

In this section we prove for any integral base $b \geq 2$ that each real number ξ has at least one SSDE. In the case of $b = 2$, Dajani, Kraaikamp, and Liardet [8] showed that any real number has the SSDE, or signed separated binary (SSB) expansion, and studied their ergodic properties. Now we assume that b is odd. Then the SSDE of ξ is given as follows: There exists a nonnegative integer R such that $|\xi| < b^R/2$. We have

$$0 < \xi + \frac{b^R}{2} < b^R.$$

Write the ordinary base-b expansion of $\xi + b^R/2$ by

$$\xi + \frac{b^R}{2} = \sum_{h=-\infty}^{R-1} s_h^{(b)}\left(\xi + \frac{b^R}{2}\right) b^h.$$

Since

$$\frac{b^R}{2} = \sum_{h=-\infty}^{R-1} \frac{b-1}{2} b^h,$$

we obtain

$$\xi = \sum_{h=-\infty}^{R-1} \left(s_h^{(b)} \left(\xi + \frac{b^R}{2} \right) - \frac{b-1}{2} \right) b^h.$$

Let us define the sequence $(\sigma_h^{(b)}(\xi))_{h=-\infty}^{\infty}$ by

$$\sigma_h^{(b)}(\xi) = \begin{cases} 0 & (h \geq R), \\ s_h^{(b)} \left(\xi + b^R/2 \right) - (b-1)/2 & (h \leq R-1). \end{cases} \tag{3.1}$$

This sequence satisfies the digit conditions defined in Section 2 because $|\sigma_h^{(b)}(\xi)| \leq (b-1)/2$ for every h. Hence, the SSDE of ξ is

$$\xi = \sum_{h=-\infty}^{\infty} \sigma_h^{(b)}(\xi) b^h.$$

We now consider the case where b is even. Recall that any integer n has a unique SSDE $\sum \sigma_h^{(b)}(n) b^h$. Heuberger and Prodinger [11, p.388] showed for any $h \in \mathbb{N}$ that $\sigma_h^{(b)}(n)$ is represented as

$$\begin{aligned} \sigma_h^{(b)}(n) &= \sum_{j=0}^{b-1} \left(\left\lfloor \frac{n}{b^{h+1}} + \frac{j}{b} + \frac{I(j < b/2) + b/2}{b(b+1)} \right\rfloor \right. \\ &\qquad \left. - b \left\lfloor \frac{n}{b^{h+2}} + \frac{j}{b} + \frac{I(j < b/2) + b/2}{b(b+1)} \right\rfloor \right), \end{aligned} \tag{3.2}$$

where

$$I(j < b/2) = \begin{cases} 1 & (j < b/2), \\ 0 & (j \geq b/2). \end{cases}$$

We construct the SSDE of a real number ξ, using (3.2). For any integer h, put

$$\begin{aligned} \sigma_h^{(b)}(\xi) &= \sum_{j=0}^{b-1} \left(\left\lfloor \frac{\xi}{b^{h+1}} + \frac{j}{b} + \frac{I(j < b/2) + b/2}{b(b+1)} \right\rfloor \right. \\ &\qquad \left. - b \left\lfloor \frac{\xi}{b^{h+2}} + \frac{j}{b} + \frac{I(j < b/2) + b/2}{b(b+1)} \right\rfloor \right). \end{aligned} \tag{3.3}$$

THEOREM 3.1. *Let b be an integer greater than 1 and ξ a real number. Then the sequence $(\sigma_h^{(b)}(\xi))_{h=-\infty}^{\infty}$ satisfies the digit conditions. Moreover,*

$$\xi = \sum_{h=-\infty}^{\infty} \sigma_h^{(b)}(\xi) b^h, \tag{3.4}$$

where $\sigma_h^{(b)}(\xi) = 0$ for any sufficiently large h.

Proof. We only have to show Theorem 3.1 in the case where b is even. Put $\underline{\sigma} = (\sigma_h^{(b)}(\xi))_{h=-\infty}^{\infty}$. We verify that $\underline{\sigma}$ satisfies the digit conditions. First we consider the case where ξ is an integer. Heuberger and Prodinger [11] showed that (2.2), (2.3), and (2.4) hold for any nonnegative integer h. Observe for any integer k that $\sigma_{h-k}^{(b)}(\xi) = \sigma_h^{(b)}(\xi b^k)$. Thus, (2.2), (2.3), and (2.4) holds for every integer h. Let

$$\Xi := \left\{ \frac{n}{b^l} \middle| n, l \in \mathbb{Z} \right\}.$$

If $\xi \in \Xi$, then $\underline{\sigma}$ satisfies the digit conditions because

$$\sigma_h^{(b)}\left(\frac{\xi}{b^l}\right) = \sigma_{h+l}^{(b)}(\xi)$$

for all integers h and l.

Let ξ be an arbitrary real number. Note for each integer h that $\sigma_h(\xi)$ is an integer and that the function $\sigma_h(\cdot)$ is right-continuous. In particular, there exists a positive real number $\delta(h,\xi)$ depending only on h and ξ such that

$$\sigma_h(\xi + x) = \sigma_h(\xi)$$

for any real number x with $0 \leq x < \delta(h,\xi)$. Hence, we deduce for any real number ξ that $\underline{\sigma}$ fulfills the digit conditions because Ξ is dense in $\mathbb{R}$.

Let δ be a positive real number less than 1. Put

$$s_h^{(b)}(\delta;\xi) := \left\lfloor \frac{\xi}{b^h} + \delta \right\rfloor - b \left\lfloor \frac{\xi}{b^{h+1}} + \delta \right\rfloor.$$

Note that $s_h^{(b)}(\delta;\xi) = 0$ for any sufficiently large h because $\delta > 0$. $\sigma_h^{(b)}(\xi)$ is written as

$$\sigma_h^{(b)}(\xi) = \sum_{j=0}^{b-1} s_h^{(b)}\left(\delta(j); \frac{\xi}{b}\right), \tag{3.5}$$

where

$$\delta(j) = \frac{j}{b} + \frac{I(j < b/2) + b/2}{b(b+1)} \in (0,1).$$

Thus, $\sigma_h^{(b)}(\xi) = 0$ for every sufficiently large h. Hence, for the proof of Theorem 3.1, it suffices to check (3.4).

Let again δ be a positive real number less than 1. Then

$$\begin{aligned}\sum_{h=0}^{\infty} s_h^{(b)}(\delta;\xi) b^h &= \sum_{h=0}^{\infty}\left(\left\lfloor \frac{\xi}{b^h} + \delta \right\rfloor - b \left\lfloor \frac{\xi}{b^{h+1}} + \delta \right\rfloor\right) b^h \\ &= \lfloor \xi + \delta \rfloor. \end{aligned} \tag{3.6}$$

Since

$$s_h^{(b)}(\delta;\xi) = \delta - b\delta - \left\{ \frac{\xi}{b^h} + \delta \right\} + b \left\{ \frac{\xi}{b^{h+1}} + \delta \right\},$$

the sequence $s_{-h}^{(b)}(\delta;\xi)$ $(h = 0, 1, \ldots)$ is bounded. Thus, we get

$$\begin{aligned}&\sum_{h=-\infty}^{-1} s_h^{(b)}(\delta;\xi) b^h \\ &= \sum_{h=-\infty}^{-1} b^{1+h}\left(\left\{ \frac{\xi}{b^{h+1}} + \delta \right\} - \delta\right) - \sum_{h=-\infty}^{-1} b^h \left(\left\{ \frac{\xi}{b^h} + \delta \right\} - \delta\right) \\ &= \{\xi + \delta\} - \delta. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we obtain that

$$\sum_{h=-\infty}^{\infty} s_h^{(b)}(\delta;\xi)b^h = \xi.$$

Therefore, we conclude that

$$\sum_{h=-\infty}^{\infty} \sigma_h^{(b)}(\xi)b^h = \sum_{j=0}^{b-1}\sum_{h=-\infty}^{\infty} s_h^{(b)}\left(\delta(j);\frac{\xi}{b}\right)b^h = \xi.$$

Finally, we proved Theorem 3.1. $\square$

Theorem 3.1 implies that any real number ξ has at least one SSDE. We determine the condition when ξ has distinct SSDEs. Let

$$b_1 := \begin{cases} (b-1)/2 & (\text{if } b \text{ is odd}), \\ b/2 & (\text{if } b \text{ is even}). \end{cases}$$

Moreover, if b is even, then put $b_2 := b_1 - 1$.

LEMMA 3.2. *Let ξ be a real number.*
(1) *Assume that b is odd. If ξ has distinct SSDEs, then they are given by*

$$\xi = (\ldots (s-1)b_1^{\omega})_b = \left(\ldots s\overline{b_1}^{\omega}\right)_b,$$

where s is an integer.
(2) *Suppose that b is even. If ξ has distinct SSDEs, then they are written as*

$$\xi = (\ldots (s-1)(b_1 b_2)^{\omega})_b = \left(\ldots s(\overline{b_2}\ \overline{b_1})^{\omega}\right)_b$$

or

$$\xi = (\ldots (s-1)(b_2 b_1)^{\omega})_b = \left(\ldots s(\overline{b_1}\ \overline{b_2})^{\omega}\right)_b,$$

where s is an integer.

Proof. Assume that b is odd. If necessary, multiplying ξ by b^{-R} with suitable $R \in \mathbb{N}$, we may assume that $|\xi| < 1/2$ and that the SSDE of ξ is denoted as

$$\xi = \sum_{h=-\infty}^{-1} \sigma_h b^h.$$

Then the ordinary base-b expansion of $\xi + 1/2$ is given by

$$\xi + \frac{1}{2} = \sum_{h=-\infty}^{-1}\left(\sigma_h + \frac{b-1}{2}\right)b^h. \tag{3.8}$$

Thus, if ξ has distinct SSDEs, then $\xi + 1/2$ has distinct (ordinary) b-ary expansions of the form

$$\xi + \frac{1}{2} = (\ldots (A-1)(b-1)^{\omega})_b = (\ldots A0^{\omega})_b, \tag{3.9}$$

where A is an integer with $1 \leq A \leq b-1$. Comparing (3.8) and (3.9), we deduce that the SSDEs of ξ are written as

$$\xi = (\ldots (s-1)b_1^\omega)_b = \left(\ldots s\overline{b_1}^\omega\right)_b,$$

where $s = A - b_1$.

In what follows, suppose that b is even. Let

$$\mathcal{W} := \{\mathbf{a} = (a_h)_{h=-\infty}^{-1} \mid \mathbf{a} \text{ satisfies the digit conditions}\}.$$

Then $\mathcal{W}$ is a nonempty compact subset of $\{-b/2, \ldots, b/2\}^{\mathbb{N}}$ endowed with the weak topology. For any $\mathbf{a} = (a_h)_{h=-\infty}^{-1} \in \mathcal{W}$, put

$$\varphi(\mathbf{a}) := \sum_{h=-\infty}^{-1} a_h b^h.$$

Now we show that

$$|\varphi(\mathbf{a})| \leq (0.(b_1 b_2)^\omega)_b = \frac{b+2}{2(b+1)}. \tag{3.10}$$

Since $\varphi : \mathcal{W} \to \mathbb{R}$ is continuous and since $\mathcal{W}$ is compact, the image $\varphi(\mathcal{W})$ has the maximal element

$$x = \sum_{h=-\infty}^{-1} w_h b^h,$$

where $\mathbf{w} = (w_h)_{h=-\infty}^{-1} \in \mathcal{W}$. Note that

$$\frac{1}{b} \cdot \frac{b}{2} + \frac{1}{b^2}\left(\frac{b}{2} - 1\right) + \frac{x}{b^2} = (0.b_1 b_2 w_{-1} w_{-2} \ldots)_b \in \varphi(\mathcal{W})$$

because the sequence $b_1 b_2 w_{-1} w_{-2} \ldots$ satisfies the digit conditions. We get

$$\begin{aligned} x &= \frac{w_{-1}}{b} + \frac{w_{-2}}{b^2} + \sum_{h=3}^{\infty} \frac{w_{-h}}{b^h} \leq \frac{w_{-1}}{b} + \frac{w_{-2}}{b^2} + \frac{x}{b^2} \\ &\leq \frac{1}{b} \cdot \frac{b}{2} + \frac{1}{b^2}\left(\frac{b}{2} - 1\right) + \frac{x}{b^2} \leq x, \end{aligned} \tag{3.11}$$

where for the first and last inequalities of (3.11) we use the maximality of x. In particular, the equalities hold in (3.11). Using the equalities of (3.11) and the maximality of x, we obtain

$$\varphi(\mathbf{a}) \leq x = \frac{b+2}{2(b+1)}.$$

Since $\varphi(\mathcal{W})$ is symmetric with respect to the origin, we deduce (3.10).

We define the sequence $\mathbf{u} = (u_h)_{h=-\infty}^{-1} \in \mathcal{W}$ as follows:

$$u_h = \begin{cases} b/2 & (h \text{ is odd}), \\ b/2 - 1 & (h \text{ is even}). \end{cases}$$

Then we have $\varphi(\mathbf{u}) = x$. Let again $\mathbf{a} = (a_h)_{h=-\infty}^{-1} \in \mathcal{W}$. We claim that if $\varphi(\mathbf{a}) = x$, then $\mathbf{a} = \mathbf{u}$. We also claim that if $\varphi(\mathbf{a}) = -x$, then $\mathbf{a} = (-u_h)_{h=-\infty}^{-1}$. Suppose that $\varphi(\mathbf{a}) = x$ and that $\mathbf{a} \neq \mathbf{u}$. Put

$$-R = \max\{h \leq -1 \mid a_h \neq u_h\}.$$

It is easily seen that $\mathbf{u}$ is the maximal element of $\mathcal{W}$ with respect to the lexicographic order. So we have

$$u_{-R} \geq 1 + a_{-R}.$$

Observe for any integer h that

$$u_h b^h + u_{h-1} b^{h-1} - a_h b^h - a_{h-1} b^{h-1} \geq -b^h, \tag{3.12}$$

by the definition of $\mathbf{u}$ and the digit conditions on $\mathbf{a}$. Hence, using (3.12) and $\varphi(\mathbf{a}) = \varphi(\mathbf{u})$, we obtain

$$\begin{aligned} 0 &= \sum_{h=-\infty}^{-1} u_h b^h - \sum_{h=-\infty}^{-1} a_h b^h \geq b^{-R} + \sum_{h=-\infty}^{-R-1} (u_h - a_h) b^h \\ &\geq b^{-R} - \sum_{h=0}^{\infty} b^{-R-1-2h} > 0, \end{aligned}$$

which is a contradiction. Therefore, we proved the first claim. The second claim follows from the first one and the following observation: for any $\mathbf{a} \in \mathcal{W}$, we have $\varphi(-\mathbf{a}) = -\varphi(\mathbf{a})$, where $-\mathbf{a} = (-a_h)_{h=-\infty}^{-1}$.

We now check the second statement of Lemma 3.2. Write the distinct SSDEs of ξ by

$$\xi = \sum_{h=-\infty}^{R} \sigma_h b^h = \sum_{h=-\infty}^{R} \sigma'_h b^h. \tag{3.13}$$

Put

$$M = \max\{h \in \mathbb{Z} \mid \sigma_h \neq \sigma'_h\}.$$

Multiplying ξ by b^{-M}, we may assume that $M = 0$. Namely, $\sigma_h = \sigma'_h$ for any positive integer h. If necessary, considering the numbers $\xi' = \xi - \sum_{h=1}^{R} \sigma_h b^h$ instead of ξ, we may assume that $\sigma_h = \sigma'_h = 0$ for any positive integer h. Thus, (3.10) and (3.13) imply that

$$\begin{aligned} |\sigma_0 - \sigma'_0| &= \left| \sum_{h=-\infty}^{-1} \sigma'_h b^h - \sum_{h=-\infty}^{-1} \sigma_h b^h \right| \\ &\leq 2x = \frac{b+2}{b+1} < 2. \end{aligned}$$

Hence, we get

$$|\sigma_0 - \sigma'_0| = 1.$$

Without loss of generality, we may assume that

$$\sigma_0 = \sigma_0' + 1. \tag{3.14}$$

Note that $(\sigma_{-1}, \sigma_{-1}') \neq (-b/2, b/2)$. In fact, if $(\sigma_{-1}, \sigma_{-1}') = (-b/2, b/2)$, then the digit conditions on $(\sigma_h)_{h=-\infty}^{0}$ and $(\sigma_h')_{h=-\infty}^{0}$ imply that

$$1 - \frac{b}{2} \leq \sigma_0 \leq 0,\ 0 \leq \sigma_0' \leq \frac{b}{2} - 1,$$

which contradicts (3.14). Assume that $\sigma_{-1} = -b/2$. Using (3.10) and $\sigma_{-1}' \leq -1 + b/2$, we obtain that

$$\begin{aligned}
\xi &= \sum_{h=-\infty}^{0} \sigma_h b^h = 1 + \sigma_0' + \sum_{h=-\infty}^{-1} \sigma_h b^h \geq 1 + \sigma_0' - x \\
&= \sigma_0' + \frac{1}{b}\left(\frac{b}{2} - 1\right) + \frac{1}{b} x \\
&\geq \sigma_0' + \sigma_{-1}' b^{-1} + \frac{1}{b} \sum_{h=-\infty}^{-1} \sigma_{h-1}' b^h = \sum_{h=-\infty}^{0} \sigma_h' b^h = \xi.
\end{aligned}$$

Thus, the equalities hold in the inequalities above. Using (3.14), we obtain

$$\sum_{h=-\infty}^{-1} \sigma_h b^h = -x,\ \sigma_{-1}' = -1 + \frac{b}{2},\ \sum_{h=-\infty}^{-1} \sigma_{h-1}' b^h = x.$$

By our claims in the proof of Lemma 3.2,

$$\sigma_h = -u_h,\ \sigma_{h-1}' = u_h$$

for any negative integer h. Namely, we deduce that the distinct SSDEs of ξ are represented as

$$\xi = (\sigma_0' . (b_2 b_1)^{\omega})_b = \left((1 + \sigma_0') . (\overline{b_1}\ \overline{b_2})^{\omega}\right)_b .$$

Similarly we prove Lemma 3.2 in the case of $\sigma_{-1} \geq 1 - b/2$. By (3.10)

$$\begin{aligned}
\xi &= \sum_{h=-\infty}^{0} \sigma_h b^h \\
&\geq 1 + \sigma_0' + \frac{1}{b}\left(1 - \frac{b}{2}\right) + \frac{1}{b} \sum_{h=-\infty}^{-1} \sigma_{h-1} b^h \\
&\geq 1 + \sigma_0' + \frac{1}{b}\left(1 - \frac{b}{2}\right) - \frac{1}{b} x \\
&= \sigma_0' + x \geq \sum_{h=-\infty}^{0} \sigma_h' b^h = \xi.
\end{aligned}$$

Since the equalities hold in the inequalities above, we obtain

$$\sigma_{-1} = 1 - \frac{b}{2},\ \sum_{h=-\infty}^{-1} \sigma_{h-1} b^h = -x,\ \sum_{h=-\infty}^{-1} \sigma_h' b^h = x.$$

Our claims in the proof of Lemma 3.2 imply that

$$\sigma_{h-1} = -u_h,\ \sigma'_h = u_h$$

for any negative integer h. Hence, the distinct SSDEs of ξ are written as

$$\xi = (\sigma'_0.(b_1 b_2)^\omega)_b = \left((1+\sigma'_0).(\overline{b_2}\ \overline{b_1})^\omega\right)_b .$$

Therefore, we verified Lemma 3.2. □

In the rest of this section we prove the ultimate periodicity of the SSDEs of rational numbers.

LEMMA 3.3. *The SSDE of any rational number ξ is ultimately periodic.*

Proof. We may assume that the SSDE of ξ is unique by Lemma 3.2. In particular, the digits of the SSDE of ξ are given by (3.1) and (3.3). In the case where b is odd, the SSDE of ξ is ultimately periodic because the ordinary b-ary expansion of $\xi + b^R/2$ is ultimately periodic, where R is a nonnegative integer with $|\xi| < b^R/2$.

Suppose that b is even. We use the same notation as in the proof of Theorem 3.1. For any j with $0 \le j \le b-1$, the sequence $s_{-h}^{(b)}(\delta(j); \xi/b)$ $(h = 0, 1, \ldots)$ is ultimately periodic because

$$s_{-h}^{(b)}\left(\delta(j); \frac{\xi}{b}\right) = \delta(j) - b\delta(j) - \{\xi b^{h-1} + \delta(j)\} + b\{\xi b^{h-2} + \delta(j)\}.$$

Hence, using (3.5), we deduce that the SSDE of ξ is ultimately periodic. □

4 Preliminaries

We study the cost function $\nu_b(\cdot)$ defined in Section 2, where b is an integer greater than 1. Heuberger and Prodinger [11] showed that the SSDE of an integer n gives the minimal cost among the signed digit representations of n. That is, assume that $n = \sum_{h=0}^R a_h b^h$, where $R \in \mathbb{N}$ and $a_h \in \mathbb{Z}$ for any h with $0 \le h \le R$. Then

$$\nu_b(n) \le \sum_{h=0}^{R} |a_h|. \tag{4.1}$$

In particular, we have

$$\nu_b(n) \le |n| \tag{4.2}$$

because

$$n = \underbrace{1 + \cdots + 1}_{|n|} \text{ or } n = \underbrace{-1 - \cdots - 1}_{|n|}.$$

Note that $\nu_b(n) = \nu_b(-n)$ for any integer n.

Bailey, Borwein, Crandall, and Pomerance [3] proved the following: Let n be a nonnegative integer with ordinary binary expansion $n = \sum_{h=0}^M s_h 2^h$, where

$s_h \in \{0,1\}$ for $h = 0, 1, \ldots, M$. Write the cost, or the Hamming weight, of this expansion by $\lambda(n) := \sum_{h=0}^{M} s_h$. Let m and n be any nonnegative integers. Then we have

$$\lambda(m+n) \leq \lambda(m) + \lambda(n),\ \lambda(mn) \leq \lambda(m)\lambda(n),$$

which we call the convexity relations in this paper. We show for any integral base $b \geq 2$ that the function $\nu_b(\cdot)$ also fulfills the convexity relations, which are generalizations of Lemma 2.1 in [12].

LEMMA 4.1. *Let m and n be integers. Then we have*

$$\nu_b(m+n) \leq \nu_b(m) + \nu_b(n) \tag{4.3}$$

and

$$\nu_b(mn) \leq \nu_b(m)\nu_b(n). \tag{4.4}$$

Proof. We can easily check (4.3) and (4.4) in the case of $mn = 0$. So we may assume that $mn \neq 0$. For simplicity, put $\nu(k) := \nu_b(k)$ for an integer k. Let

$$\Lambda := \{\pm b^h \mid h \in \mathbb{N}\}.$$

Then there exist $t_1, \ldots, t_{\nu(m)}, t'_1, \ldots, t'_{\nu(n)} \in \Lambda$ such that

$$m = \sum_{i=1}^{\nu(m)} t_i,\ n = \sum_{j=1}^{\nu(n)} t'_j.$$

We get

$$m + n = \sum_{i=1}^{\nu(m)} t_i + \sum_{j=1}^{\nu(n)} t'_j$$

and

$$mn = \sum_{i=1}^{\nu(m)} \sum_{j=1}^{\nu(n)} t_i t'_j.$$

Hence, using (4.1), we obtain (4.3) and (4.4) because $t_i t'_j \in \Lambda$ for any i and j. $\square$

Combining (4.2) and (4.3), we get

$$\nu_b(m+n) - \nu_b(m) \leq \nu_b(n) \leq |n|$$

and

$$\nu_b(m) - \nu_b(m+n) \leq \nu_b(-n) \leq |n|.$$

Hence, for all integers m and n,

$$|\nu_b(m+n) - \nu_b(m)| \leq |n|. \tag{4.5}$$

LEMMA 4.2. *Let ξ be a positive real number and B a positive integer. Then, for any $k \in \mathbb{Z}^+$ and $N \in \mathbb{N}$, we have*

$$\nu_b(\lfloor B^k b^N \xi^k \rfloor) \leq \nu_b(B\lfloor b^N \xi \rfloor)^k + 2^{k+1} B^k \max\{1, \xi^k\}.$$

Proof. Write the ordinary base-b expansion of ξ by

$$\xi = \sum_{h=-\infty}^{R} s_h b^h,$$

where $R \in \mathbb{N}$ and $0 \leq s_h \leq b-1$ for any h with $h \leq R$. Put

$$\xi_1 := \sum_{h=-N}^{R} s_h b^h, \ \xi_2 := \sum_{h=-\infty}^{-N-1} s_h b^h.$$

Then we have

$$\xi_1 \leq \xi, \ \xi_2 \leq b^{-N} (\leq 1). \tag{4.6}$$

Observe that

$$\begin{aligned} B^k b^N \xi^k &= B^k b^N (\xi_1 + \xi_2)^k \\ &= B^k b^N \xi_1^k + B^k b^N \sum_{i=1}^{k} \binom{k}{i} \xi_1^{k-i} \xi_2^i. \end{aligned} \tag{4.7}$$

For any real numbers x and y, it is easily seen that

$$|\lfloor x + y \rfloor - (\lfloor x \rfloor + \lfloor y \rfloor)| \leq 1 \tag{4.8}$$

and that

$$|\lfloor x - y \rfloor - (\lfloor x \rfloor - \lfloor y \rfloor)| \leq 1. \tag{4.9}$$

Thus, using (4.6), (4.7), and (4.8), we get

$$\lfloor B^k b^N \xi_1^k \rfloor \leq \lfloor B^k b^N \xi^k \rfloor$$

and

$$\begin{aligned} \lfloor B^k b^N \xi^k \rfloor &\leq \lfloor B^k b^N \xi_1^k \rfloor + \left\lfloor B^k b^N \sum_{i=1}^{k} \binom{k}{i} \xi_1^{k-i} \xi_2^i \right\rfloor + 1 \\ &\leq \lfloor B^k b^N \xi_1^k \rfloor + \left\lfloor B^k \sum_{i=0}^{k} \binom{k}{i} \max\{1, \xi^k\} \right\rfloor + 1 \\ &= \lfloor B^k b^N \xi_1^k \rfloor + \lfloor 2^k B^k \max\{1, \xi^k\} \rfloor + 1. \end{aligned}$$

In particular, we have

$$|\lfloor B^k b^N \xi^k \rfloor - \lfloor B^k b^N \xi_1^k \rfloor| \leq \lfloor 2^k B^k \max\{1, \xi^k\} \rfloor + 1.$$

Hence, using (4.5), we obtain

$$\nu_b(\lfloor B^k b^N \xi^k \rfloor) \le \nu_b(\lfloor B^k b^N \xi_1^k \rfloor) + \left\lfloor 2^k B^k \max\{1, \xi^k\} \right\rfloor + 1. \tag{4.10}$$

In what follows, we estimate the value $\nu_b(\lfloor B^k b^N \xi_1^k \rfloor)$. Note that $Bb^N \xi_1$ is an integer. Lemma 4.1 implies that

$$\nu_b(B^k b^{kN} \xi_1^k) \le \nu_b(Bb^N \xi_1)^k = \nu_b(B\lfloor b^N \xi \rfloor)^k. \tag{4.11}$$

Denote the SSDE of $B^k b^{kN} \xi_1^k$ by

$$B^k b^{kN} \xi_1^k = \sum_{h=0}^{M} \sigma_h b^h.$$

Put

$$\theta_1 := \sum_{h=(k-1)N}^{M} \sigma_h b^{h-(k-1)N}$$

and

$$\theta_2 := \sum_{h=0}^{-1+(k-1)N} \sigma_h b^{h-(k-1)N}.$$

Then we have $|\theta_2| < 1$ because $|\sigma_h| \le b/2$ for any h. Using $\theta_1 \in \mathbb{Z}$ and

$$\theta_1 + \theta_2 = B^k b^N \xi_1^k,$$

we get

$$|\lfloor B^k b^N \xi_1^k \rfloor - \theta_1| \le 1. \tag{4.12}$$

By (4.5), (4.11), and (4.12), we obtain

$$\begin{aligned}
\nu_b(\lfloor B^k b^N \xi_1^k \rfloor) &\le 1 + \nu_b(\theta_1) = 1 + \sum_{h=(k-1)N}^{M} |\sigma_h| \\
&\le 1 + \sum_{h=0}^{M} |\sigma_h| = 1 + \nu_b(B^k b^{kN} \xi_1^k) \\
&\le 1 + \nu_b(B\lfloor b^N \xi \rfloor)^k. \qquad (4.13)
\end{aligned}$$

Combining (4.10) and (4.13), we conclude that

$$\nu_b(\lfloor B^k b^N \xi^k \rfloor) \le \nu_b(B\lfloor b^N \xi \rfloor)^k + 2^{k+1} B^k \max\{1, \xi^k\}.$$

□

In the rest of this section we give lower bounds of the costs of certain classes of integers.

LEMMA 4.3. *Let η be a rational number. Let r be the least period of the SSDE of η and let ρ be the cost of the period of η. Assume that ρ is positive. Then there exists an effectively computable positive constant $C_6(b,\eta)$ depending only on b and η such that*

$$\nu_b(\lfloor b^N \eta \rfloor) \geq \frac{\rho}{r} N - C_6(b,\eta)$$

for any $N \in \mathbb{N}$.

Proof. Denote the SSDE of η by

$$\eta = \sum_{h=-\infty}^{M-1} \sigma_h b^h = (\sigma_{M-1} \ldots \sigma_0 . \sigma_{-1} \ldots \sigma_{-L} v^{\omega})_b \,,$$

where L is a positive integer and v is a finite word of length r. For the proof of Lemma 4.3, we may assume that $N \geq L$. Put

$$\eta_N := \sum_{h=-N}^{M-1} \sigma_h b^h.$$

The SSDE of $b^N \eta_N \in \mathbb{Z}$ is written as

$$b^N \eta_N = \sigma_{M-1} \ldots \sigma_{-L} \underbrace{v \ldots v}_{\lfloor (N-L)/r \rfloor} v',$$

where v' is the prefix of v of length $r\{(N-L)/r\}$. In particular,

$$\nu_b(b^N \eta_N) \geq \rho \left\lfloor \frac{N-L}{r} \right\rfloor . \tag{4.14}$$

Since

$$|b^N \eta - b^N \eta_N| = \left| b^N \sum_{h=-\infty}^{-N-1} \sigma_h b^h \right| = \left| \sum_{h=-\infty}^{-1} \sigma_{h-N} b^h \right| \leq 1,$$

we get

$$|\lfloor b^N \eta \rfloor - b^N \eta_N| \leq 2. \tag{4.15}$$

Hence, using (4.5), (4.14), and (4.15), we obtain

$$\nu_b(\lfloor b^N \eta \rfloor) \geq \nu_b(b^N \eta_N) - 2 \geq \frac{\rho}{r} N - C_6(b,\eta).$$

□

5 Proofs of the main results

We give lower bounds of the number $\gamma_b(\xi; N)$ of digit changes in base $b \geq 2$, using the cost function $\nu_b(\cdot)$.

LEMMA 5.1. *Let b be an integer greater than 1 and ξ a positive real number. Then*

$$\nu_b\big((b-1)\lfloor \xi b^N \rfloor\big) \le (2b-3)\gamma_b(\xi;N)+1$$

for any $N \in \mathbb{N}$.

Proof. Without loss of generality, we may assume that $\lfloor \xi b^N \rfloor \ge 1$. For simplicity, put $\gamma := \gamma_b(\xi;N)$. Then the ordinary base-b expansion of $\lfloor \xi b^N \rfloor$ is written as

$$\lfloor \xi b^N \rfloor = \Big(\underbrace{a_1 \ldots a_1}_{l(1)} \underbrace{a_2 \ldots a_2}_{l(2)} \ldots \underbrace{a_\gamma \ldots a_\gamma}_{l(\gamma)} \Big)_b,$$

where $a_i \in \{0,1,\ldots,b-1\}$ and $l(i) \in \mathbb{Z}^+$ for each i. Thus, we get

$$(b-1)\lfloor \xi b^N \rfloor = \sum_{i=1}^{\gamma} a_i \left(b^{e_i} - b^{f_i}\right),$$

where e_i and f_i are nonnegative integers for any i. Using (4.1), we obtain

$$\nu_b\big((b-1)\lfloor \xi b^N \rfloor\big) \le 2\sum_{i=1}^{\gamma} |a_i|. \tag{5.1}$$

For any i with $1 \le i \le \gamma-1$, at least one of a_i and a_{i+1} are less than or equal to $b-2$. Hence,

$$2\sum_{i=1}^{\gamma} |a_i| \le 2\left(\left\lceil \frac{\gamma}{2} \right\rceil + (b-2)\gamma\right) \le (2b-3)\gamma + 1. \tag{5.2}$$

Therefore, (5.1) and (5.2) imply Lemma 5.1. □

Proof of Theorem 2.1.
Let $N \in \mathbb{N}$. Since $A_D\xi^D + \cdots + A_0 = 0$, we get

$$\eta b^N = \sum_{h=1}^{D} \frac{A_h(b-1)^{D-h}}{u}(b-1)^h b^N \xi^h. \tag{5.3}$$

In what follows, we estimate the cost of the integral part of (5.3) in two ways. By the first and second assumptions of Theorem 2.1, the cost ρ is positive. Thus, by Lemma 4.3,

$$\nu_b(\lfloor \eta b^N \rfloor) \ge \frac{\rho}{r} N - C_6(b,\eta). \tag{5.4}$$

The third assumption of Theorem 2.1 implies that u divides $A_h(b-1)^{D-h}$ for any h with $1 \le h \le D$. Using (4.5), (4.8), (4.9) and Lemma 4.1, we get

$$\begin{aligned}
\nu_b(\lfloor \eta b^N \rfloor) &= \nu_b\left(\left\lfloor \sum_{h=1}^{D} \frac{A_h(b-1)^{D-h}}{u}(b-1)^h b^N \xi^h \right\rfloor\right) \\
&\le \nu_b\left(\sum_{h=1}^{D} \frac{A_h(b-1)^{D-h}}{u} \lfloor (b-1)^h b^N \xi^h \rfloor\right) + \sum_{h=1}^{D} \frac{|A_h|(b-1)^{D-h}}{u} \\
&\le \sum_{h=1}^{D} \nu_b\left(\frac{A_h(b-1)^{D-h}}{u} \lfloor (b-1)^h b^N \xi^h \rfloor\right) + \sum_{h=1}^{D} \frac{|A_h|(b-1)^{D-h}}{u} \\
&\le \sum_{h=1}^{D} \nu_b\left(\frac{A_h(b-1)^{D-h}}{u}\right) \nu_b\left(\lfloor (b-1)^h b^N \xi^h \rfloor\right) \\
&\qquad + \sum_{h=1}^{D} \frac{|A_h|(b-1)^{D-h}}{u}. \qquad (5.5)
\end{aligned}$$

Using Lemmas 4.2 and 5.1, we get

$$\nu_b(\lfloor (b-1)^h b^N \xi^h \rfloor) \le \big((2b-3)\gamma_b(\xi;N)+1\big)^h + 2^{h+1}(b-1)^h \max\{1,\xi^h\} \quad (5.6)$$

for any h with $1 \le h \le D$. Combining (5.4), (5.5), and (5.6), we obtain

$$\frac{\rho}{r} N \le P\big(\gamma_b(\xi;N)\big), \quad (5.7)$$

where $P(X) \in \mathbb{R}[X]$ is a polynomial with leading term

$$(2b-3)^D \nu_b\left(\frac{A_D}{u}\right) X^D.$$

Hence, for any positive real number u, there exists an effectively computable positive constant $C_7(b,\xi,u)$ depending only on b, ξ, and u such that

$$\gamma_b(\xi;N) \ge u$$

for any N with $N \ge C_7(b,\xi,u)$. Let ε be an arbitrary positive number less than 1. Then there exists an effectively computable positive constant $C_4(b,\xi,\varepsilon)$ depending only on b, ξ, and ε such that

$$P\big(\gamma_b(\xi;N)\big) \le (1-\varepsilon)^{-D}(2b-3)^D \nu_b\left(\frac{A_D}{u}\right) \gamma_b(\xi;N)^D. \quad (5.8)$$

for any integer N with $N \ge C_4(b,\xi,\varepsilon)$ because $(1-\varepsilon)^{-D} > 1$. Using (5.7) and (5.8), we conclude that

$$\gamma_b(\xi;N) \ge (1-\varepsilon)\mu(\xi) N^{1/D}$$

for any $N \ge C_4(b,\xi,\varepsilon)$. Therefore, we proved Theorem 2.1.

In what follows, we assume that $b=2$. We use the same notation as in Section 2. Put

$$\begin{aligned}
F(\xi;N) &:= f(\xi,00;N) + f(\xi,11;N) \\
&= \mathrm{Card}\{-N \le h \le R-1 \mid s_h^{(2)}(\xi) = s_{h+1}^{(2)}(\xi)\}.
\end{aligned}$$

LEMMA 5.2. *Let ξ be a positive real number. Then*

$$\nu_2(3\lfloor \xi 2^N \rfloor) \leq 6F(\xi; N) + 2$$

for any $N \in \mathbb{N}$.

Proof. Let $v = s_{L-1} \dots s_1 s_0$ be a finite word on the alphabet $\{0, 1\}$. For any nonnegative real number x, put

$$v^x := \underbrace{v \dots v}_{\lfloor x \rfloor} v',$$

where v' is the prefix of v of length $\lfloor \{x\} |v| \rfloor$. Recall that

$$(v)_2 = \sum_{h=0}^{L-1} s_h 2^h.$$

For the proof of Lemma 5.2, we may assume that $\lfloor 2^N \xi \rfloor \geq 1$. Then the binary expansion of $\lfloor \xi 2^N \rfloor$ is written as

$$\lfloor \xi 2^N \rfloor = \left(v_1^{x_1} w_1^{y_1} v_2^{x_2} w_2^{y_2} \dots v_{l-1}^{x_{l-1}} w_{l-1}^{y_{l-1}} v_l^{x_l} \right)_2 \tag{5.9}$$

or

$$\lfloor \xi 2^N \rfloor = \left(v_1^{x_1} w_1^{y_1} v_2^{x_2} w_2^{y_2} \dots v_{l-1}^{x_{l-1}} w_{l-1}^{y_{l-1}} v_l^{x_l} w_l^{y_l} \right)_2, \tag{5.10}$$

where, for each i, $v_i \in \{01, 10\}$, $2x_i, y_i \in \mathbb{Z}^+$, and $w_i \in \{0, 1\}$ is the last letter of $v_i^{x_i}$. The last block of the right-hand side of (5.10) is different from that of (5.9). For instance, assume that $v_i = 10$ and that $x_i = 3$. Then we have $v_i^{x_i} = 101010$ and $w_i = 0$. By the definition of w_i,

$$F(\xi; N) = \sum_{i \geq 1} y_i.$$

First we assume that $\lfloor \xi 2^N \rfloor$ is denoted as (5.9). Let i be a positive integer. Then we get

$$3 \left(v_i^{x_i} \right)_2 = (11 \dots 1)_2 = 2^k - 1$$

or

$$3 \left(v_i^{x_i} \right)_2 = (11 \dots 10)_2 = 2^k - 2,$$

where k is a positive integer. In particular,

$$\nu_2 \left(3 \left(v_i^{x_i} \right)_2 \right) \leq 2. \tag{5.11}$$

Similarly,

$$\nu_2((w_i^{y_i})_2) \leq 2$$

because $(w_i^{y_i})_2 = 0$ or $(w_i^{y_i})_2 = 2^m - 1$, where m is a positive integer. Thus, by Lemma 4.1

$$\nu_2 \left(3(w_i^{y_i})_2 \right) \leq \nu_2(3) \nu_2 \left((w_i^{y_i})_2 \right) \leq 4. \tag{5.12}$$

In particular, if $l=1$, then we have

$$\nu_2(3\lfloor 2^N\xi\rfloor)=\nu_2\left(3(v_1^{x_1})_2\right)\le 2\le 6F(\xi;N)+2.$$

So we may assume that $l\ge 2$. (5.9) is rewritten as

$$\lfloor \xi 2^N\rfloor=\sum_{i=1}^{l}2^{t_i}(v_i^{x_i})_2+\sum_{i=1}^{l-1}2^{u_i}(w_i^{y_i})_2,$$

where t_i and u_i are nonnegative integers for each i. Using (5.11), (5.12), and Lemma 4.1, we obtain

$$\begin{aligned}\nu_2\big(3\lfloor\xi 2^N\rfloor\big) &\le \sum_{i=1}^{l}\nu_2\big(3\cdot 2^{t_i}(v_i^{x_i})_2\big)+\sum_{i=1}^{l-1}\nu_2\big(3\cdot 2^{u_i}(w_i^{y_i})_2\big)\\ &= \sum_{i=1}^{l}\nu_2\big(3(v_i^{x_i})_2\big)+\sum_{i=1}^{l-1}\nu_2\big(3(w_i^{y_i})_2\big)\\ &\le 2l+4(l-1)=6(l-1)+2\\ &\le 6\sum_{i=1}^{l-1}y_i+2=6F(\xi;N)+2.\end{aligned}$$

Next we consider the case where $\lfloor\xi 2^N\rfloor$ is written as (5.10). Namely,

$$\lfloor \xi 2^N\rfloor=\sum_{i=1}^{l}2^{t_i}(v_i^{x_i})_2+\sum_{i=1}^{l}2^{u_i}(w_i^{y_i})_2,$$

where t_i and u_i are nonnegative integers for every i. By (5.11), (5.12) and Lemma 4.1,

$$\begin{aligned}\nu_2\big(3\lfloor\xi 2^N\rfloor\big) &\le \sum_{i=1}^{l}\nu_2\big(3\cdot 2^{t_i}(v_i^{x_i})_2\big)+\sum_{i=1}^{l}\nu_2\big(3\cdot 2^{u_i}(w_i^{y_i})_2\big)\\ &= \sum_{i=1}^{l}\nu_2\big(3(v_i^{x_i})_2\big)+\sum_{i=1}^{l}\nu_2\big(3(w_i^{y_i})_2\big)\\ &\le 2l+4l=6l\le 6\sum_{i=1}^{l}y_i=6F(\xi;N).\end{aligned}$$

Hence, we proved Lemma 5.2. □

Proof of Theorem 2.2.
Let $N\in\mathbb{N}$. Using $A_D\xi^D+\cdots+A_0=0$, we get

$$\eta' 2^N=\sum_{h=1}^{D}\frac{3^{D-h}A_h}{u'}3^h 2^N\xi^h. \tag{5.13}$$

We estimate the cost of the integral part of (5.13) in two ways. Since u' is odd and since u' satisfies the first assumption of Theorem 2.2, the cost ρ' is positive. Hence, Lemma 4.3 implies that

$$\nu_2(\lfloor\eta' 2^N\rfloor)\ge\frac{\rho'}{r'}N-C_6(2,\eta'). \tag{5.14}$$

By the second assumption of Theorem 2.2, u' divides $3^{D-h}A_h$ for each h with $1 \le h \le D$. Using (4.5), (4.8), (4.9) and Lemma 4.1, we obtain

$$\begin{aligned}
\nu_2(\lfloor \eta' 2^N \rfloor) &= \nu_2\left(\left\lfloor \sum_{h=1}^{D} \frac{3^{D-h}A_h}{u'} 3^h 2^N \xi^h \right\rfloor\right) \\
&\le \nu_2\left(\sum_{h=1}^{D} \frac{3^{D-h}A_h}{u'} \left\lfloor 3^h 2^N \xi^h \right\rfloor\right) + \sum_{h=1}^{D} \frac{3^{D-h}|A_h|}{u'} \\
&\le \sum_{h=1}^{D} \nu_2\left(\frac{3^{D-h}A_h}{u'} \left\lfloor 3^h 2^N \xi^h \right\rfloor\right) + \sum_{h=1}^{D} \frac{3^{D-h}|A_h|}{u'} \\
&\le \sum_{h=1}^{D} \nu_2\left(\frac{3^{D-h}A_h}{u'}\right) \nu_2\left(\left\lfloor 3^h 2^N \xi^h \right\rfloor\right) + \sum_{h=1}^{D} \frac{3^{D-h}|A_h|}{u'}. \qquad (5.15)
\end{aligned}$$

By Lemmas 4.2 and 5.2, for each h with $1 \le h \le D$,

$$\nu_2(\lfloor 3^h 2^N \xi^h \rfloor) \le (6F(\xi; N) + 2)^h + 2^{h+1} 3^h \max\{1, \xi^h\}. \qquad (5.16)$$

Combining (5.14), (5.15), and (5.16), we obtain that

$$\frac{\rho'}{r'} N \le P'\big(F(\xi; N)\big),$$

where $P'(X) \in \mathbb{R}[X]$ is a polynomial whose leading term is

$$6^D \nu_2\left(\frac{A_D}{u'}\right) X^D.$$

Let ε be any positive number less than 1. Then, in the same way as in the proof of Theorem 2.1, we deduce the following: There exists an effectively computable positive constant $C_5(\xi, \varepsilon)$ depending only on ξ and ε such that

$$F(\xi; N) \ge (1 - \varepsilon)\mu'(\xi) N^{1/D}$$

for any integer N with $N \ge C_5(\xi, \varepsilon)$. Finally we verified Theorem 2.2.

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