微分トポロジー ’22： Note 2

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双曲デーン手術定理とその精密化（サーベイ）

Thurston の双曲デーン手術定理、および、その精密化や亜種について、それらの関係や流れ、応用についての概要を説明する。

§2. Refinements & descendants

Hyperbolic Dehn surgery thm.

M: complete hyp. 3-mfd. of finite vol. (w/ cusps)
($=: M_{\infty,\infty,\cdots\infty}$)

$M_{d_1,\cdots,d_n}$: the 3-mfd. obtained by
generalized Dehn surgery.
(metric complition)

$(d_1,\cdots,d_n) \in S^2 \times \cdots \times S^2$ $(S^2 = \mathbb{R}^2 \cup \{\infty\})$

$\exists U$: nhd. of $(\infty,\cdots,\infty)$ s.t.
$M_{d_1,\cdots,d_n}$ admits a hyp. str.
for $^\forall (d_1,\cdots,d_n) \in U$

Furthermore:

① $d_i = (p_i, q_i)$, coprime, → "usual" Dehn surgery
& hyp. str. is genuine.

①' $d_i = r(p_i, q_i)$, (p_i, q_i) : coprime

→ $M_{d_1,\cdots,d_n} \underset{\text{topogically}}{\cong}$ 3-mfd. obtained by "usual" Dehn surgery along (p_i, q_i)'s.

$\exists$ cone hyp. str. on $M_{d_1,\cdots,d_n}$
w/ axis of cone sing.
is <u>the cores of attached solid tori</u>.

② For (p_i, q_i) as above,
$|p_i| + |q_i| \to \infty$, then

- the len(the core geod.) $\to 0$
- $\mathrm{Vol}(M_{d_1,\cdots,d_n}) < \mathrm{Vol}\, M$ $((d_1,\cdots,d_n) \neq (\infty,\cdots,\infty))$
  & $\mathrm{Vol}(M_{d_1,\cdots,d_n}) \to \mathrm{Vol}\, M$.

[Universal bound]

C. D. Hodgson\ and\ S. P. Kerckhoff, Universal bounds for hyperbolic Dehn surgery, Ann. of Math. (2) {\bf 162} (2005), no.~1, 367--421. MR2178964
arXiv:math/0204345

THEOREM 1.3. *Let X be a complete, finite volume, orientable, hyperbolic 3-manifold with one cusp, and let T be a horospherical torus which is embedded as a cross-section to the cusp of X. Let γ be a simple closed curve on T whose Euclidean geodesic length on T is denoted by L. If the normalized length of γ, $\hat{L} = \dfrac{L}{\sqrt{\text{area}(T)}}$, is at least 7.515, then the closed manifold $X(\gamma)$ obtained by Dehn filling along γ is hyperbolic.*

COROLLARY 1.4. *Let X be a complete, orientable, hyperbolic 3-manifold with one cusp. Then at most 60 Dehn fillings on X yield manifolds which admit no complete hyperbolic metric.*

C. D. Hodgson\ and\ S. P. Kerckhoff, The shape of hyperbolic Dehn surgery space, Geom. Topol. {\bf 12} (2008), no.~2, 1033--1090. MR2403805
arXiv:0709.3566

Theorem 1.1 *Consider a complete, finite volume hyperbolic structure on the interior of a compact, orientable 3–manifold X with one torus boundary component. Let T be a horospherical torus which is embedded as a cross-section to the cusp of the complete structure. Consider $\mathcal{HDS}(X)$ as a subset of $H_1(T;\mathbb{R}) \cong \mathbb{R}^2$ where the latter is endowed with the Euclidean metric induced from the universal cover of T with its flat metric scaled to have unit area. Then $\mathcal{HDS}(X)$ contains the complement of a disk of radius 7.5832, centered at the origin. Equivalently, any $c \in H_1(T;\mathbb{R})$ whose normalized length $\widehat{L}(c)$ is bigger than 7.5832 is in $\mathcal{HDS}(X)$.*

J. F. Brock\ and\ K. W. Bromberg, On the density of geometrically finite Kleinian groups, Acta Math. {\bf 192} (2004), no.~1, 33--93. MR2079598

THEOREM 1.3. (The drilling theorem) *Let M be a geometrically finite hyperbolic 3-manifold. For each $L>1$, there is an $l>0$ so that if c is a geodesic in M with length $l_M(c)<l$, there is an L-bi-Lipschitz diffeomorphism of pairs*

$$h\colon (M \backslash \mathbf{T}(c), \partial \mathbf{T}(c)) \longrightarrow (M_0 \backslash \mathbf{P}(c), \partial \mathbf{P}(c)),$$

where $M\backslash\mathbf{T}(c)$ denotes the complement of a standard tubular neighborhood of c in M, M_0 denotes the complete hyperbolic structure on $M\backslash c$, and $\mathbf{P}(c)$ denotes a standard rank-2 cusp corresponding to c.

D.Futer, J.S. Purcell, S.Schleimer, Effective bilipschitz bounds on drilling and filling, preprint arXiv:1907.13502, To appear in Geometry & Topology

Theorem 1.2. *Fix $0 < \epsilon \leq \log 3$ and $J > 1$. Let M be a finite-volume hyperbolic 3–manifold and Σ a geodesic link in M whose total length ℓ satisfies*

$$\ell \leq \min\left\{ \frac{\epsilon^5}{6771\cosh^5(0.6\epsilon + 0.1475)}, \frac{\epsilon^{5/2}\log(J)}{11.35} \right\}.$$

Then, setting $N = M - \Sigma$, and equipping it with its complete hyperbolic metric, there are natural J–bilipschitz inclusions

$$\varphi\colon M^{\geq\epsilon} \hookrightarrow N^{\geq\epsilon/1.2}, \qquad \psi\colon N^{\geq\epsilon} \hookrightarrow M^{\geq\epsilon/1.2}.$$

Here $M^{\geq\epsilon}$ and $N^{\geq\epsilon}$ are the ϵ–thick parts of M and N, respectively. The compositions $\varphi\circ\psi$ and $\psi\circ\varphi$ are the identity wherever both maps are defined. Furthermore, φ and ψ are equivariant with respect to the symmetry group of the pair (M,Σ).

Corollary 7.20. *Let M be a complete, finite volume hyperbolic 3–manifold. Let $\Sigma \cup \gamma$ be a geodesic link in M, where γ is connected. Let $\mathcal{L}_M(\gamma) = \mathrm{len}_M(\gamma) + i\tau_M(\gamma)$ be the complex length of γ in the complete metric on M, and suppose that* $\max(\mathrm{len}_M(\Sigma), \mathrm{len}_M(\gamma)) \leq 0.0735$. *Then γ is also a geodesic in the complete metric on $N = M - \Sigma$, of complex length $\mathcal{L}_N(\gamma)$. Furthermore,*

$$1.9793^{-1} \leq \frac{\mathrm{len}_N(\gamma)}{\mathrm{len}_M(\gamma)} \leq 1.9793 \qquad \textit{and} \qquad |\tau_N(\gamma) - \tau_M(\gamma)| \leq 0.05417.$$

[Volume change]

W. D. Neumann\ and\ D. Zagier, Volumes of hyperbolic three-manifolds, Topology {\bf 24} (1985), no.~3, 307--332. MR0815482

$$Q_i(p,q) = (\text{length of } p\,m_i + q\ell_i)^2/(\text{volume of } T_i).$$

THEOREM 1A. *With the above notations,*

$$\mathrm{Vol}(M_\kappa) = \mathrm{Vol}\, M - \pi^2 \sum_{i=1}^{h} \frac{1}{Q_i(p_i,q_i)} + 0\left(\sum \frac{1}{p_i^4 + q_i^4}\right).$$

C. D. Hodgson\ and\ S. P. Kerckhoff, The shape of hyperbolic Dehn surgery space, Geom. Topol. {\bf 12} (2008), no.~2, 1033--1090. MR2403805
arXiv:0709.3566

Theorem 1.3 *Let X be a compact, orientable 3–manifold as in Theorem 1.2, and let V_∞ denote the volume of the complete hyperbolic structure on the interior of X. Let $c = (c_1, \dots, c_k) \in H_1(\partial X; \mathbb{R})$ be a surgery coefficient with normalized lengths $\widehat{L}_i = \widehat{L}(c_i)$ satisfying*

$$\sum_i \frac{1}{\widehat{L}_i^2} < \frac{1}{C^2} \text{ where } C = 7.5832,$$

and let $M(c)$ be the filled hyperbolic manifold with Dehn surgery coefficient c. Then the decrease in volume $\Delta V = V_\infty - \mathrm{vol}(M(c))$ during hyperbolic Dehn filling is at most 0.198.

I. Agol, Volume change under drilling, Geom. Topol. {\bf 6} (2002), 905--916. MR1943385

D. Futer, E. Kalfagianni\ and\ J. S. Purcell, Dehn filling, volume, and the Jones polynomial, J. Differential Geom. {\bf 78} (2008), no.~3, 429--464. MR2396249

Theorem 1.1. *Let M be a complete, finite–volume hyperbolic manifold with cusps. Suppose $C_1, \dots, C_k$ are disjoint horoball neighborhoods of some subset of the cusps. Let $s_1, \dots, s_k$ be slopes on $\partial C_1, \dots, \partial C_k$, each with length greater than 2π. Denote the minimal slope length by $\ell_{\min}$. If $M(s_1, \dots, s_k)$ satisfies the geometrization conjecture, then it is a hyperbolic manifold, and*

$$\mathrm{vol}(M(s_1, \dots, s_k)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(M).$$

[Variations]

S. A. Bleiler\ and\ C. D. Hodgson, Spherical space forms and Dehn filling, Topology {\bf 35} (1996), no.~3, 809--833. MR1396779
Gromov-Thurston's 2 π-theorem

THEOREM 9 (The "2π" Theorem). *Let M be a complete hyperbolic 3-manifold of finite volume and $P_1, \dots, P_v$ disjoint horoball neighbourhoods of the cusps of M. Suppose r_i is a slope on ∂P_i represented by a geodesic α_i with length in the Euclidean metric satisfying $\mathrm{length}(\alpha_i) > 2\pi$, for each $i = 1, \dots, v$. Then $M(r_1, \dots, r_v)$ has a metric of negative curvature.*

THEOREM 4. *If M is of type H, then there exist at most 24 finite fillings on M. Moreover, if r and s are slopes such that $M(r)$ and $M(s)$ have finite fundamental group then $\Delta(r, s) < 23$. The same results hold, more generally, replacing finite fillings by Dehn fillings which do not admit negatively curved metrics.*

I. Agol, Bounds on exceptional Dehn filling, Geom. Topol. {\bf 4} (2000), 431--449. MR1799796
M. Lackenby, Word hyperbolic Dehn surgery, Invent. Math. {\bf 140} (2000), no.~2, 243--282. MR1756996

Theorem 3.1. *Let M be a compact orientable 3-manifold with interior having a complete finite volume hyperbolic structure. Let $s_1, \dots, s_n$ be slopes on ∂M, with one s_i on each component of ∂M. Suppose that there is a horoball neighbourhood N of the cusps of $M - \partial M$ on which each s_i has length more than 6. Then, the manifold obtained by Dehn filling along $s_1, \dots, s_n$ is irreducible, atoroidal and not Seifert fibred, and has infinite, word hyperbolic fundamental group.*

Theorem 8.1 *Let N be a hyperbolic 3–manifold, and C a distinguished embedded torus cusp. The intersection number between exceptional boundary slopes on C is ≤ 10, and there are at most 12 exceptional boundary slopes.*

M. Lackenby\ and\ R. Meyerhoff, The maximal number of exceptional Dehn surgeries, Invent. Math. {\bf 191} (2013), no.~2, 341--382. MR3010379

Theorem 4.1 *Let M be a compact orientable 3-manifold, with boundary a torus and with interior admitting a complete finite-volume hyperbolic structure. Let s be a slope on ∂M with length at least*

$$\frac{\pi e_2}{\arcsin(e_2/2)}$$

if $e_2 \leq \sqrt{2}$, and length at least

$$\frac{2\pi e_2}{2\arcsin(\sqrt{1-e_2^{-2}}) + e_2^2 - 2\sqrt{e_2^2 - 1}}$$

if $e_2 > \sqrt{2}$. Then, $M(s)$ is hyperbolic.

Theorem 1.1 *Let M be a compact orientable 3-manifold with boundary a torus, and with interior admitting a complete finite-volume hyperbolic structure. Then the number of exceptional slopes on ∂M is at most 10.*

Theorem 1.2 *Let M be a compact orientable 3-manifold with boundary a torus, and with interior admitting a complete finite-volume hyperbolic structure. If s_1 and s_2 are exceptional slopes on ∂M, then their intersection number $\Delta(s_1, s_2)$ is at most 8.*