13:13 3月20日(日) 100%

メモ

2022年3月20日 12:15

微分トポロジー '22： Note 1

2022.3.20, 11:20–12:20

市原 一裕

双曲デーン手術定理とその精密化（サーベイ）

Thurston の双曲デーン手術定理、および、その精密化や亜種について、それらの関係や流れ、応用についての概要を説明する。

§1. Hyperbolic Dehn Surgery Theorem.

① Theorem [Thurston, Bull. AMS, 1982, 2.6 Thm]

$L \subset M^3$: link, $M \setminus L$: hyp. str.

Then

"most mfds obtained from M by Dehn surgery along L have hyp. strs."

Lec. note : Thm 5.8.2 (§5.8, p.102, in pdf)

\# of cusps

$M = M_{\infty, \cdots, \infty}$ admits a hyp. str.

Then

$\exists U$: nhd of $(\infty, \cdots, \infty)$ in $S^2 \times \cdots \times S^2$ $(\mathbb{R}^2 \cup \{\infty\})$

s.t. $M_{d_1, \cdots, d_k}$ admits a hyp. str.

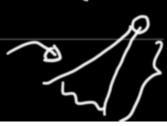
for $\forall (d_1, \cdots, d_k) \in U$.

w/ $\boxed{\text{cone sing}}$

Assume : $d_i = (\alpha_i, \beta_i)$ satisfies "certain condition."

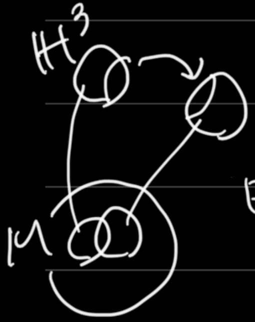
② hyp. str. hyp. str as (G, X)-str.

• M : a non-compact, hyp. 3-mfd. (cusp?)

H^3  having a finite number of "ends" 

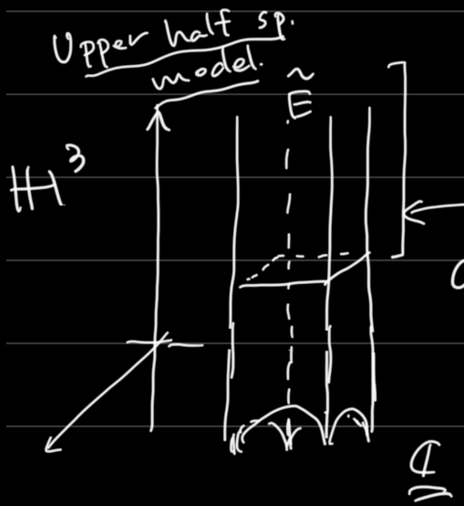
② hyp. str. — hyp. str as (G,X)-str.

- M : a non-compact, hyp. 3-mfd. (cusp?) having a finite number of "ends" $E_1, \dots, E_k$;



$\underline{E_i \cong T^2 \times [0,\infty)}$, is isometric to $\mathbb{H}^3 / \langle \rho_1, \rho_2 \rangle$ (para. fix ∞)

if the str. is complete ← (later)



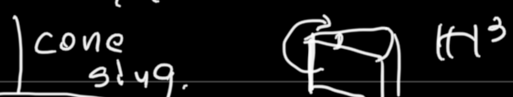
$\tilde{M}$: univ. cover of M

- $\underline{M_{d_1, \dots, d_k}}$: the space obtained by "generalized Dehn surgery" with parameters $d_1, \dots, d_k$

$$\underline{d_i \in \mathbb{R}^2 \simeq H_1(T^2, \mathbb{R})} \text{ (in general)}$$

$^\forall d_i$: primitive in $H_1(T^2, \mathbb{Z}) \Rightarrow M_{d_1, \dots, d_k}$: closed mfd.
("usual" Dehn surgery)

not primitive ⟶ hyp. cone-mfd.



③ Def (Developing map)

$\mathbb{H}^3$: simply conn. Riem. mfd. sec. const. curv. -1.

isometry $\in \mathrm{Isom}^+ \mathbb{H}^3$

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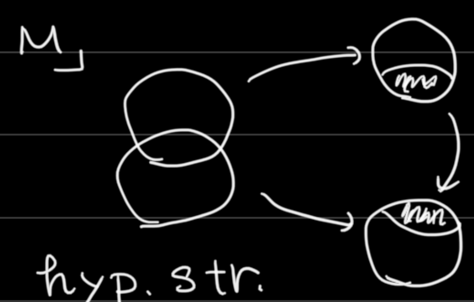
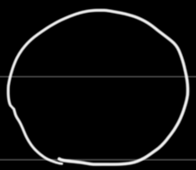


$\mathbb{H}^3$: simply conn. Riem. mfd. sec. const. curv. -1.

isometry $\in \mathrm{Isom}^+ \mathbb{H}^3$ (restriction)

hyp. str.

$\rightsquigarrow$



$\tilde{M} \xrightarrow{\exists D} \mathbb{H}^3$

univ. cover $\downarrow$

M

D: developing map (defined by "analytic contination")

Def.

hyp. str. on M is complete if $D : \tilde{M} \to \mathbb{H}^3$, homeo.

covering for general (G,X)-str.

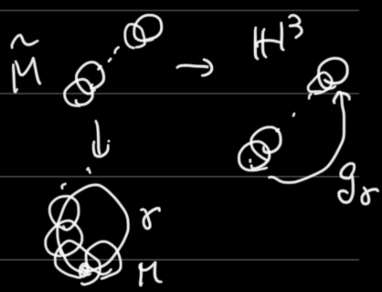
$\to$ In this case, M is a comple metric sp. (with path metric).

④ **Holonomy** : $\pi_1 M \curvearrowright \tilde{M}$, deck trans.

$\forall \gamma \in \pi_1 M \longmapsto \exists g \in \mathrm{Isom}^+ \mathbb{H}^3$

s.t. $D \circ \gamma = g_\gamma \circ D$

$\rightsquigarrow$ $H : \pi_1 M \longrightarrow \mathrm{Isom}^+ \mathbb{H}^3$



If the hyp. str. is complete, then H is a discrete, faithful rep.

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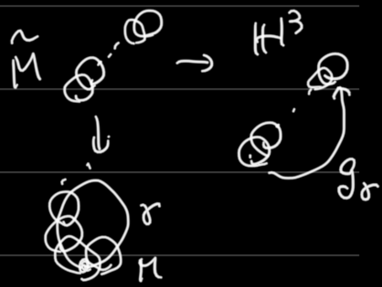


If the hyp. str. is complete,

then H is a <u>discrete, faithful rep.</u>

& $M \underset{\text{isom.}}{\simeq} \mathbb{H}^3/\Gamma$, $\Gamma = \overline{H(\pi_1 M)}$

<u>klein. gr.</u>

$< \mathrm{Isom}^+ \mathbb{H}^3$

$\simeq \underline{PSL(2,\mathbb{C})}$

Lec. not 4.5 & 4.4

⑤ <u>Def</u> (generalized Dehn surgery invariant)

$\hat{M}$: compact 3-mfd w/ $\underline{P_1, .., P_k \cong T^2}$, as boundary.

<u>Fix (a_i, b_i) on P_i</u>, generators of $\pi_1(P_i) \simeq \mathbb{Z} + \mathbb{Z}$

M : the interior of $\hat{M}$, <u>admitting a hyp. str.</u>

(non-compact, hyp. 3-mfd)

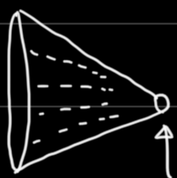
[the hyp. str. is NOT assumed to be complete.

& "<u>deformed</u> from the complete one"]

$\bar{M}$: the space obtained by "<u>metric completion</u>"

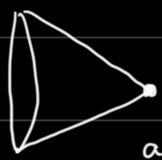
(add "equiv. class of Cauchy seq." to the original sp.)

⑥ 2-dim. picture



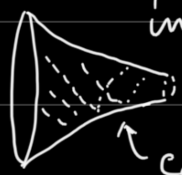
$S^1 \times E^1$ "ideal pt."

↓ completion



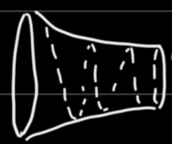
add 1pt.

incomplete hyperbolic



Cauchy seq. "horocycle" like "spiral"

completion

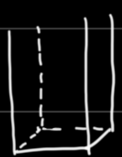


add "S^1"

3-dim. (local picture) incomplete

∞

complete



D_i → p.174 ~ 177 Motegi's book!

parallelgram (Euclidean)

(Assumed to be an ideal triangulation) ??

⑦ Generalized Dehn surgery invariant (Coefficent)

If all (α_i, β_i) happen to be primitive elements of $\mathbb{Z} \oplus \mathbb{Z}$, then $\bar{M}$ is the topological manifold $M_{(\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k)}$ with a non-singular hyperbolic structure, so that our extended definition is compatible with the original. If each ratio α_i/β_i is the rational number p_i/q_i in lowest terms, then $\bar{M}$ is topologically the manifold $M_{(p_1,q_1),\ldots,(p_k,q_k)}$. The hyperbolic structure, however, has singularities at the component circles of $\bar{M} - M$ with cone angles of $2\pi(p_i/\alpha_i)$ [since the holonomy $\tilde{H}$ of the primitive element $p_i a_i + q_i b_i$ in $\pi_1(P_i)$ is a pure rotation of angle $2\pi(p_i/\alpha_i)$].

hyp. cone mfd.

[For geneal case, $\bar{M}$ is not a mfd.]

Condition: $d_i = (\alpha_i, \beta_i) \in \mathbb{R}^2$

$\alpha_i \cdot \tilde{H}(a_i) + \beta_i \cdot \tilde{H}(b_i) = $ (rotation by $\pm 2\pi$)

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$$\alpha_i \cdot \tilde{H}(a_i) + \beta_i \cdot \tilde{H}(b_i) = (\text{rotation by } \pm 2\pi)$$

(a_i, b_i) : meridian - longitude on P_i

when the str. is incomplete

D_i : "End" for P_i in M.

$H|_{P_i} : D_i \longrightarrow I_i$

$$\begin{array}{ccc} \hat{D}_i & \longrightarrow & \tilde{I}_i \\ \downarrow & \circlearrowleft & \downarrow \\ D_i & \longrightarrow & I_i \end{array}$$

$\leftarrow$ dev. map.

univ. cov

$$\tilde{H} : \pi_1 D_i \longrightarrow \mathrm{Isom}^+ \tilde{I}_i \cong \mathbb{R}^2$$

$\pi_1 D_i \cong \pi_1 P_i \cong \mathbb{Z} + \mathbb{Z}$

⑧

References:

W. P. Thurston, The geometry and topology of 3-manifolds, lecture notes, Princeton University, 1980,
currently available at: http://library.msri.org/nonmsri/gt3m/

8

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W. P. Thurston, The geometry and topology of 3-manifolds, lecture notes, Princeton University, 1980,
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茂手木公彦. **デーン手術 —3次元トポロジーへのとびら—** , 東京, 共立出版, 2022年, 292ページ, (ひろがるトポロジー).

C. Petronio and J. Porti, Negatively oriented ideal triangulations and a proof of Thurston's hyperbolic Dehn filling theorem, Expo. Math. 18 (2000), no.~1, 1--35.

M. Boileau and J. Porti, Geometrization of 3-orbifolds of cyclic type, Asterisque No. 272 (2001), 208 pp. **Appendix B. Trurston's hyperbolic Dehn filling theorem**

"Gaps" in the "proof" of Thurston.

① It is still open: $\forall$ hyp. 3-mfd. (comp. fin. vol.) admits a "genuine" ideal triangulation.
(Rem. $\exists$ ideal polyhedral decomp. by Epstein-Penner)

② In Thurston's Lec. Note, no proof of "smoothness of the complete pt. in the deformation sp. of hyp. str."

See P.-P. for ① & B.-P. for ② in detail.