Lens space surgery realization

Motoo Tange

University of Tsukuba

2022/3/21

Lens space

Definition 1

$K \subset S^3$: a knot
$S^3_{p/q}(K)$: p/q-Dehn surgery on K
p/q:slope

Problem

When can a knot K produce a lens space?

Theorem 2 (Culler-Gordon-Luecke-Shallen)

$S^3_{p/q}(K)$ is a lens space.
Then
K unknot or torus knot or other.
The other has integer slope.

Lens space knot

Definition 3

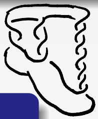
$K \subset S^3$: a knot
If $S^3_p(K)$ is a lens space, K is a lens space knot.

Definition 4

$K \subset S^3$: a knot
If $S^3_p(K)$ is an L-space, K is an L-space knot.

Y: $\mathbb{Q}HS^3$ Y is an L-space
if $HF^+(Y,\mathfrak{s}) \cong HF^+(S^3,\mathfrak{s})$ for any spin^c structure.

Lens space knots

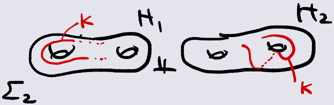


Lemma 5 (Fintushel-Stern)

$Pr(-2,3,7)$ is lens space knot.

Definition 6 (double-primitive)

$K \subset S^3$: double-primitive knot
if $K \subset \Sigma_2 \subset S^3$ (Heegaard surface)
$[K] \in \pi_1(H_i)$: a primitive for $i = 1, 2$



Theorem 7

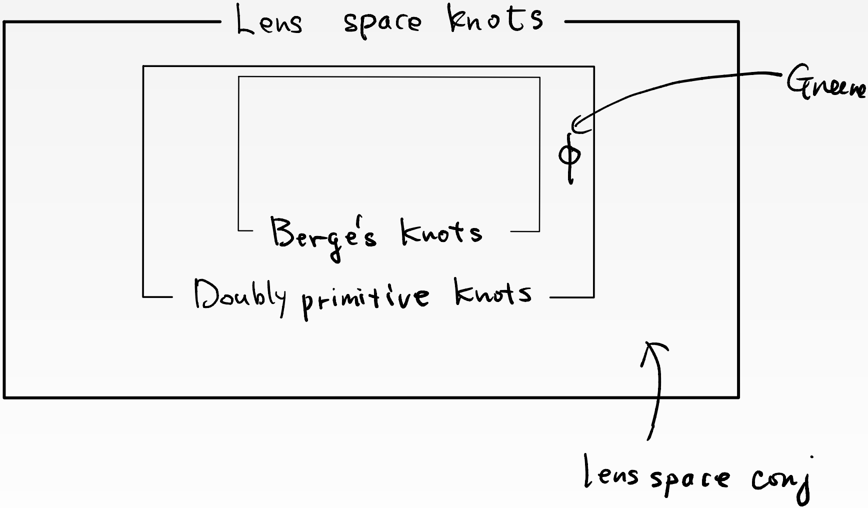
Known all lens space knots in S^3 are double-primitive.

Theorem 8

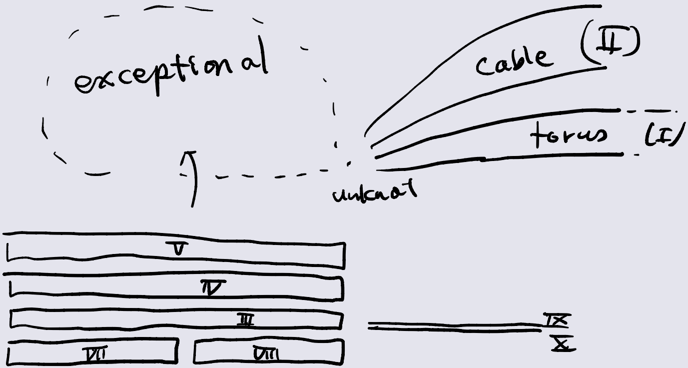
'All double primitive knots' are listed as follows:

$(I)_{\pm}:\ p=ik\pm1,\ \gcd(i,k)=1$
$(II)_{\pm}:\ p=ik\pm1,\ \gcd(i,k)=2,\ i,k\geq 4$
$(III)(a)_{\pm}:\ p=\pm(2k-1)d\ (k^2),\ d|k+1,\ \frac{k+1}{d}$: odd
$(III)(b)_{\pm}:\ p=\pm(2k+1)d\ (k^2),\ d|k-1,\ \frac{k-1}{d}$: odd
$(IV)(a)_{\pm}:\ p=\pm(k-1)d\ (k^2),\ d|2k+1$
$(IV)(b)_{\pm}:\ p=\pm(k+1)d\ (k^2),\ d|2k-1$
$(V)(a)_{\pm}:\ p=\pm(k+1)d\ (k^2),\ d|k+1,\ d$: odd
$(V)(b)_{\pm}:\ p=\pm(k-1)d\ (k^2),\ d|k-1,\ d$: odd
$(VII):\ k^2+k+1\equiv 0\ (p)$
$(VIII):\ k^2-k-1\equiv 0\ (p)$
$(IX):\ p=\frac{1}{11}(2k^2+k+1)\ k\equiv 2\ \ (11)$
$(X):\ p=\frac{1}{11}(2k^2+k+1)\ k\equiv 3\ \ (11)$

Each of the family is called a *Berge's knot* or *Berge's lens space*.


Lens space knots
Greene
ϕ
Berge's knots
Doubly primitive knots
lens space conj

lens space knots



Lens space surgeries from the Poincare Sphere

Theorem 9 (T.)

Y: an L-space homology sphere (not S^3)
Each of the following list gave a double-primitive knot $K \subset Y$ such that $Y_p(K) = L(p, q)$.

	$L(p,q)$
A_1	$L(14\ell^2 + 7\ell + 1, (7\ell + 2)^2)$
A_2	$L(20\ell^2 + 15\ell + 3, (5\ell + 2)^2)$
B	$L(30\ell^2 + 9\ell + 1, (6\ell + 1)^2)$
C_1	$L(42\ell^2 + 23\ell + 3, (7\ell + 2)^2)$
C_2	$L(42\ell^2 + 47\ell + 13, (7\ell + 4)^2)$
D_1	$L(52\ell^2 + 15\ell + 1, (13\ell + 2)^2)$
D_2	$L(52\ell^2 + 63\ell + 19, (13\ell + 8)^2)$
E_1	$L(54\ell^2 + 15\ell + 1, (27\ell + 4)^2)$
E_2	$L(54\ell^2 + 39\ell + 7, (27\ell + 10)^2)$

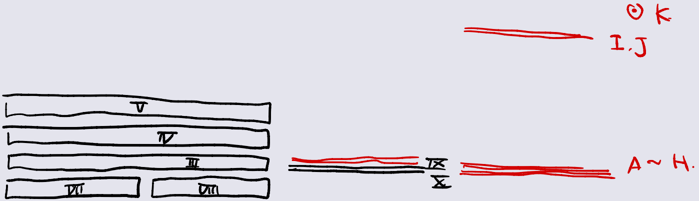
F_1	$L(69\ell^2+17\ell+1,(23\ell+3)^2)$
F_2	$L(69\ell^2+29\ell+3,(23\ell+5)^2)$
G_1	$L(85\ell^2+19\ell+1,(17\ell+2)^2)$
G_2	$L(85\ell^2+49\ell+7,(17\ell+5)^2)$
H_1	$L(99\ell^2+35\ell+3,(11\ell+2)^2)$
H_2	$L(99\ell^2+53\ell+7,(11\ell+3)^2)$
I_1	$L(120\ell^2+16\ell+1,(12\ell+1)^2)$
I_2	$L(120\ell^2+36\ell+3,(12\ell+2)^2)$
I_3	$L(120\ell^2+20\ell+1,(20\ell+2)^2)$
J	$L(120\ell^2+104\ell+22,(12\ell+5)^2)$
K	$L(191,15^2)$

Each of them is realized by double-primitive knot in $\Sigma(2,3,5)$.

lens space knots

Y : L-space $\mathbb{Z}HS^3$

$K \subset Y$ Lens space knot.



lens space knots

Strategies to classify lens space surgeries

$S^3_p(K) = L(p,q)$

Δ_K. α conditions

Berge's list

lens space knots

Kadokami - Yamada - Ichihara - Saito - Teragaito - T

(A) $\Delta_K = t^{-\frac{1}{2}(k-1)(h-1)} \dfrac{(t^{kh}-1)(t-1)}{(t^k-1)(t^h-1)} \mod t^p - 1$

$= a_{-g} t^{-g} + a_{-g+1} t^{-g+1} + \cdots + a_{g-1} t^{g-1} + a_g t^g$

(B) Ozsváth - Szabó's

$\Delta_K = (-1)^m + \sum_{j=1}^{k} (-1)^{m-j} (t^{n_j} + t^{-n_j}) \quad n_1 < \cdots < n_k = g.$

(C) Ozsváth Szabó's ineq.

$2g - 1 \leq p$

$\Delta_K = 0\, t^{p/2} + \cdots + 0 \cdot t^{-g-1} + t^{-g} - t^{-g+1} + \cdots + t^g + 0\, t^{g+1} + \cdots + t^{p/2}$

lens space knots

Me.

(A C) – (B) ····> partial. ?

Greene

(B, C) & changemaker (4-dim argue) ····> complete. !!

(automatically A?

Conjecture 10 (Goda-Teragaito)

K : Hyperbolic knot, $S^3_p(K)$: a lens space
Then

$$\frac{p-1}{2} \leq 2g-1 \leq p-9$$

Theorem 11 (Rasmussen)

K : a lens space knot
Then

$$\frac{p-5}{2} \leq 2g-1$$

Theorem 12

K : L-space knot
$S^3_p(K)$ bounds a neg. def. 4-manifold X (H_1 torsion free)
then

$$2g - 1 \leq p - \sqrt{p} - 1 \quad \textit{(1st ineq.)}$$

If X is sharp, then

$$2g - 1 \leq p - \sqrt{3p+1} \quad \textit{(2nd ineq.)}$$

If $X = X(p,q)$ ($S^3_p(K)$: lens space), then

$$2g(K) - 1 \leq p - 2\sqrt{\frac{4p+1}{5}} \quad \textit{(3rd ineq.)}$$

Theorem 13 (Greene(2010))

If $S^3_p(K) = L(p,q)$,
then $\exists B$: a Berge's knot s.t. $S^3_p(K) = S^3_p(B)$

In particular. Berge's lens spaces are complete lens space constructed by integral Dehn surgery in S^3.

Theorem 14 (T.(2010))

Y : an L-space homology sphere
$Y_p(K)$ a lens space and $|q-k|$: small
Then the lens space is one of the following

$$\begin{cases} \textit{Berge's knot} \\ \textit{my knots in } A \textit{ to } H \end{cases}$$

Y is realized as S^3 or $\Sigma(2,3,5)$.

Conjecture 15

Y: an L-space homology sphere
$Y_p(K)$ is a lens space then $Y_p(K)$ is
Berge's family or Hedden's family or my family.

Y: L-space homology sphere

Cabling conj

Conjecture 16 (Cabling conj.)

If $S^3_p(K)$ is reducible, then K is a cable knot $C_{q,r}(K')$ $p = qr$ (cabling slope)
$S^3_{qr}(K) = S^3_{q/r}(K')\#L(r, q)$.

Theorem 17 (Gordon-Luecke)

If $S^3_p(K)$ is reducible, then p is integer. A lens space summand is contained.

Theorem 18

If $S^3_p(K)$ is reducible, then at most three conn. comp's are contained. If it has three, then two are lens space of coprime orders and the third is a homology sphere.

(graph theoretic argument)

Theorem 19 (Matignon-Sayari)

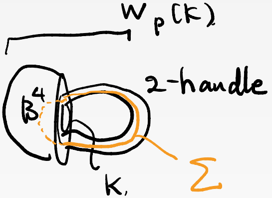
If $S^3_p(K)$ is reducible, then K is a cable knot or $p \leq 2g - 1$.

Theorem 20 (Greene)

If $S^3_p(K)$ is a conn-sum of lens spaces, then K is one of the following:

$$K = \begin{cases} T(q,r) & p = qr \\ C_{q,r}(T(s,t)) & q = rst \pm 1 \end{cases}$$

Preliminaries



$\mathrm{Spin}^c(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$

$W_p(K)$: 2-handle attachment
$\partial W_p(K) = S^3_p(K)$
$\mathfrak{t} \in \mathrm{Spin}^c(S^3_p(K))$ $\exists \mathfrak{s} \in W_p(K)$ $\mathfrak{s}|_{S^3_p(K)} = \mathfrak{t}$

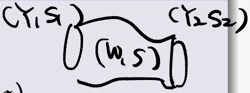
$H^2(W_p(K)) \to H^2(S^3_p(K)) \to H_1(W_p(K)) = 0.$

$[\Sigma] \in H_2(W_p(K)) = \mathbb{Z}$ (generator)
$\langle c_1(\mathfrak{s}), [\Sigma] \rangle + p = 2i$
where $i \mod p$ does not depend on the choice of ext. of $\mathfrak{s}$.
$\mathrm{Spin}^c(S^3_p(K)) \to \mathbb{Z}/p\mathbb{Z}$ (bij)

Y :$\mathbb{Q}HS^3$.

d-invariant (correction term)

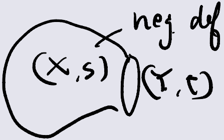
$d : \{(Y, \mathfrak{s}) | \mathfrak{s} \in \mathrm{Spin}^c(Y)\} \to \mathbb{Q}$
rational spin^c cobordism invariant.
(minimal degree of tower component of $HF^+(Y, \mathfrak{s})$)



Theorem 21

Y : $\mathbb{Q}HS^3$
X^4: neg def. 4-manifold s.t. $Y = \partial X^4$.
then
$\forall \mathfrak{s} \in Spin^c(X)$ for $\mathfrak{s}|_Y = \mathfrak{t}$.

$$c_1^2(\mathfrak{s}) + b_2(X) \leq 4d(Y, \mathfrak{t}).$$



The absolute grading is moved by $\frac{c_1^2(\mathfrak{s}) - 3\sigma(X) - 2\chi(X)}{4}$

Def (Sharp) X: neg def 4-mfd is sharp

$\Leftrightarrow$ $\forall \mathfrak{t} \in \mathrm{Spin}^c(Y)$ $\exists \mathfrak{s} \in \mathrm{Spin}^c(X)$ $\mathfrak{s}|_Y = \mathfrak{t}$

$c_1^2(\mathfrak{s}) + b_2(X) = 4d(Y, \mathfrak{t})$

Example L: non-split alt link

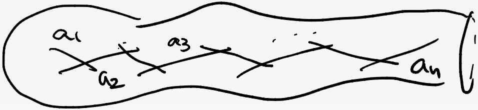
$\Rightarrow$ $\Sigma_2(L)$: is an L-space

$\exists$ $X(L)$: sharp 4-mfd.

$\partial X(L) = \Sigma_2(L)$

$H_1(X(L)) = 0$

Example $K_{p,q}$: 2-bridge knot. $\Sigma(K_{p,q}) = L(p,q)$

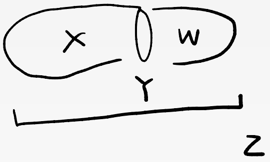


$X_{p,q}$

$$p/q = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots \frac{1}{a_n}}}$$

$H_1(X_{p,q})$

Lem $Y = S^3_p(K)$ bounds neg def 4-mfd X
$p > 0$



$W = -W_p(K)$

$\forall i \in Spin^c(Y)$, $\exists \mathfrak{s} \in Spin^c(Z)$

$$c_1^2(\mathfrak{s}) + n + 1 \leq 4d(S^3_p(K), i) - 4d(S^3_p(U), i)$$

if X : sharp, then "$\leq$" $\to$ "$=$"

K: an L-space knot.

$$\Delta_K = \sum_{i=-g}^{g} a_i T^i \qquad g = \deg(\Delta_K)$$

Thm (Ozsvath-Szabo)

K: an L-space knot

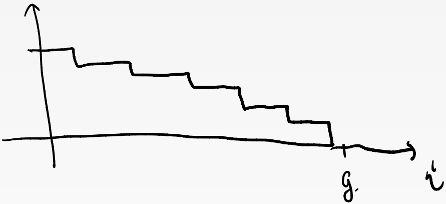
$\Rightarrow a_i = 0$ or ± 1

alt sign in order. //

ex. $(a_{-g}, a_{-g+1}, \cdots, a_{g-1}, a_g)$

$= (1, -1, 0, 1, -1, 0, 0, 1, \cdots)$

$$t_i - t_{i+1} = \sum_{j\geq 1} a_{i+j} = 0 \text{ or } 1$$



$|i| \geq g.$

$\Rightarrow t_i = 0$

Thm (Owen-Strle) K: L-space knot $p \in \mathbb{Z}_{>0}$

$$-2t_i = d(S^3_p(K), i) - d(S^3_p(U), i)$$

$\in$ L-sp.

$$T^+_{(0)} = HF^+(S^3) \longrightarrow HF^+(S^3_0(K), i) = \frac{\mathbb{Z}[U]}{U^{t_i}}$$

$$HF^+(S^3_p(K), i)$$

$\therefore \quad c_1^2(\mathfrak{s}) + n + 1 \leqq -8t_i$

$\therefore$

<u>Lem.</u> K : L-space knot

$S^3_p(K) = \partial X$

$\uparrow$ neg def H-cntd. $H_1 = 0$

$\Rightarrow c^2(\mathfrak{s}) + n + 1 \leqq -8t_i$

for $\forall i$ $|i| < p/2$. $\langle c, s \rangle + p = 2i \ (2p)$

Prop K: L-space knot, $S^3_p(K)$: L-sp.

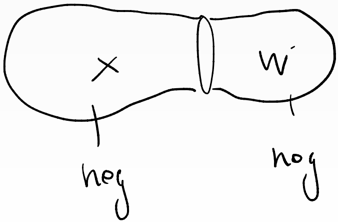
$S^3_p(K)$ bounds a smooth. neg. def 4-fld, $H_1=0$

$$\Rightarrow \quad 2g \le p - |\sigma|_1$$

$$|\sigma|_1 = \sum_{i=1}^{n+1} |\sigma_i|$$

$$\sigma = (\sigma_0 \cdots \sigma_n)$$

Z:



$$Q_Z = \langle -1 \rangle^{n+1}$$

(Donaldson's Thm)

<u>proof</u>. $c^2+n+1 \leqq -8t_i$ $\quad \sigma = (\sigma_0 \cdots \sigma_n)$

$$c_1 : \mathrm{Spin}^c(Z) \to \mathrm{Char}(Z) \subset H^2(Z)$$

$$c_1(\mathfrak{s}) \in \{\pm 1\}^{n+1} \iff c^2+n+1 = 0.$$

$$\iff t_i \leqq 0$$

$$\iff t_i = 0$$

$$\langle c_1 \sigma \rangle + p = 2i \geqq 2g.$$

$$2g \leqq p + \langle c_1 \sigma \rangle$$

Here $c = s(\sigma)$

$$s(\sigma)_j = \begin{cases} +1 & \sigma_j \geq 0 \\ -1 & \text{o/w} \end{cases}$$

$$\Rightarrow \langle c, \sigma \rangle = -\sum |\sigma_i| = -|\sigma|_1$$

$$\therefore 2g \leq p - |\sigma|_1$$

If X: sharp

$$2g = p - |\sigma|_1$$

Greene's 1st ineq.

$$p = |\langle \sigma, \sigma \rangle| \leq |\sigma|_1^2 \quad \therefore |\sigma| \geq \sqrt{p}$$

$$\therefore 2g \leq p - \sqrt{p}$$

Assume X: sharp.

$p - |\sigma|_1 \leq 2i \leq p. \Rightarrow \exists c \in \{\pm 1\}^{n+1}$

$\langle c, \sigma \rangle + p = 2i \geq 2g$

$i \to -i \quad \langle -c, \sigma \rangle + p = -2i \ (2p)$

$\therefore \ p - |\sigma|_1 \leq \overset{\forall}{2i} \leq p + |\sigma|_1$

$\exists c \in \{\pm 1\}^{n+1}$

$\langle c, \sigma \rangle + p = 2i$

$-|\sigma|_1 \leq \underset{j}{\underset{\shortparallel}{2i - p}} \leq |\sigma|_1 \qquad -|\sigma|_1 \leq \overset{\forall}{j} \leq |\sigma|_1.$

$j \equiv p \equiv |\sigma|_1 \ (2)$

$\exists\, c \in \{\pm 1\}^{n+1}, \quad j = 2i - p = \langle c, \sigma \rangle$

$-|\sigma|_1 \leq {}^{\forall} j \leq |\sigma|_1 \quad (j \equiv p \equiv |\sigma|_1 \ (2))$

$\overset{1:1}{\longleftrightarrow} \quad 0 \leq {}^{\forall} k \leq |\sigma|_1$

$k = \dfrac{j + |\sigma|_1}{2}$

Here $\sigma = (\sigma_0, \sigma_1 \cdots \sigma_n)$

may assume. $0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_n$

- $0 \leq {}^{\forall}k \leq |\sigma|_1$

$$k = \frac{j + |\sigma|_1}{2} = \frac{\langle c, \sigma\rangle + |\sigma|_1}{2}.$$

$$\left(c = (-1, \dots, -1) + 2x. \right)$$

$x : \{0 \cdots n\} \to \{0, 1\}$

$$= \frac{\langle (-1, \dots, -1) + 2x, \ \sigma \rangle + |\sigma|_1}{2}$$

$$= \langle x, \sigma \rangle$$

$$= \sum_{i \in A} \sigma_i$$

$A = \{ i \mid x(i) = 1 \}.$

Lem. $0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_n$ int seq.

$0 \leq {}^\forall k \leq |\sigma|_1 = \sigma_0 + \sigma_1 + \cdots + \sigma_n$

$\exists A \subset \{0, \cdots, n\}$.

$$\sum_{i \in A} \sigma_i = k$$

$$\Leftrightarrow \quad {}^\forall i \quad \sigma_i \leq \sigma_0 + \sigma_1 + \cdots + \sigma_{i-1} + 1.$$

Greene's 2nd ineq.

$$(\sigma_0 + \sigma_1 + \cdots + \sigma_n + 1)^2 = 1 + \sum_{i=1}^{n} \left(\sigma_i^2 + 2\sigma_i(\sigma_0 + \cdots + \sigma_{i-1})\right)$$

$$\geq 1 + \sum_{i=0}^{n} (\sigma_i^2 + 2\sigma_i^2) \geq 1 + 3\sum_{i=0}^{n} \sigma_i^2 = 1 + 3p.$$

$$\sigma_0+\sigma_1+\cdots+\sigma_n \geq \sqrt{3p+1}-1$$

$$\therefore \quad 2g = p - |\sigma|_1$$

$$\leq p - \sqrt{3p+1} + 1$$

$$2g-1 \leq p - \sqrt{3p+1}$$

<u>Def</u> (Changemaker vector)

$\sigma = (\sigma_0 \cdots \sigma_n) \in \mathbb{Z}^{n+1}$ is changemaker.

$\Leftrightarrow$ $0 \leq {}^{\forall} k \leq \sigma_0 + \cdots + \sigma_n$, $\exists A \subset \{0, \cdots, n\}$

s.t. $\sum_{i \in A} \sigma_i = k$.

Thm. $K \subset S^3$: L-space knot.

$S^3_p(K)$: bounds a sharp 4-mfd. X

$$\Rightarrow H_2(X) \oplus H_2(W) \hookrightarrow -\mathbb{Z}^{n+1}$$

full rank.

$[\Sigma] \in H_2(W)$

$[\Sigma] \longrightarrow \sigma = (\sigma_0, \cdots, \sigma_n)$

σ : changemaker.

$|\sigma| = -\sum \sigma_i^2 = -p$

Greene's result

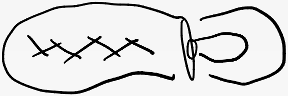
<u>Thm</u>. $S^3_p(K) = L(p,q)$

then. $S^3_p(K) = S^3_p(B)$

In particular, Berge's lens spaces are complete list constructed by integral Dehn surgery.

Strategy.

$K \subset S^3$. $\quad S^3_p(K) = L(p,q)$

 $= -\mathbb{Z}^{n+1}$

$\Lambda(p,q)$ $\quad [\Sigma] \longrightarrow (\sigma_0, \cdots, \sigma_n)$

$(\sigma)^\perp = L$ (changemaker lattice)

Classify all cases where changemaker lattice is a linear lattice!

Actually, if $(\sigma)^\perp$ is linear
changemaker

, then. $L(p,q)$ is one of

Berge's list !

First. $\sigma_0 = 1$ holds.

$\sigma_0 = 0$ then $(\sigma)^\perp$ is not linear.

$\sigma_0 \geqq 2$ then σ is not changemaker.

Suppose $\sigma_j = \sigma_0 + \cdots + \sigma_{j-1} + 1$

then.

$$v_j = -e_j + 2e_0 + \sum_{i=1}^{j-1} e_i \quad \text{(tight)}$$

Suppose $\sigma_j \leq \sigma_0 + \cdots + \sigma_{j-1}$

$$\sigma_j = \sum_{i \in A} \sigma_i \qquad A \subset \{0, \ldots, j-1\}$$

A : maximal. $A' < A$ lexico order

$$v_j = -e_j + \sum_{i \in A} e_i \quad \text{(gappy)}$$

(A: non-consecutive)

$$v_j = -e_j + \sum_{i \in A} e_i \quad \text{(just right)}$$

$(A :$ consecutive$)$

Lemma $v_1 \cdots v_n \cdot (\sigma)^{\perp}$

$= L$

v_i either of tight.

gappy, just right

v_i : irreducible

$\Big($ x: reducible $x = y + z$ $\quad y, z \in L$, $y, z \neq 0$. $\quad \langle y, z \rangle \geq 0$ $\Big)$

Lem $S = \{v_1 \cdots v_n\}$ std. basis

$\Rightarrow$ at most one tight vector.

at most two gappy vectors

No tight vectors

no gappy vectors

I_+, II_-, $\mathrm{III}(a)_-$ $\mathrm{IV}(b)$ $\mathrm{V}(a)_-$ X IX

one gappy vectors

$\mathrm{III}(a)_-$ $\mathrm{III}(b)_-$, $\mathrm{IV}(a)_-$ $\mathrm{IV}(b)_\pm$ $\mathrm{V}(a)_+$ $\mathrm{V}(b)_-$ VI

One tight vectors

I_- II_+, $\mathrm{III}(a)_+$ $\mathrm{III}(b)_+$ $\mathrm{IV}(a)_+$ $\mathrm{IV}(b)_+$ $\mathrm{V}(a)_+$ $\mathrm{V}(b)_+$

VIII

Future work.