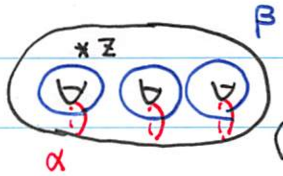
微分トポロジー

Heegaard Floer homology とcosmetic surgery：近年の発展の概説

§1. Cheating tour of Heegaard Floer theory

• (pointed) Heegaard diagram $\rightsquigarrow$ closed, oriented 3-manifold Y



$(\Sigma;\alpha,\beta,z)$

$\wr$

(spin^c structure 毎に) filtered graded chain complex CF^∞ $(=CF^\infty(\Sigma;\alpha,\beta,z))$

over $\mathbb{Z}[U,U^{-1}]$ (well-defined up to chain homotopy)

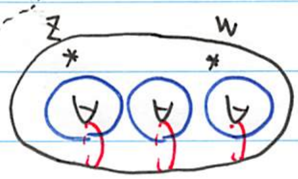
$\wr$ quotient/subcomplex

various complexes CF^+, CF^-, $\widehat{CF}$, ...

$\wr$ homology

HF^+, HF^-, $\widehat{HF}$, HF_{red}

• Doubly pointed Heegaard diagram $\rightsquigarrow$ knot $K \subset Y$



$(\Sigma;\alpha,\beta,z,w)$

$\wr$

bi filtered graded chain complex $CFK^\infty(K)$ $(=CFK^\infty(\Sigma;\alpha,\beta,z,w))$

over $\mathbb{Z}[U,U^{-1}]$

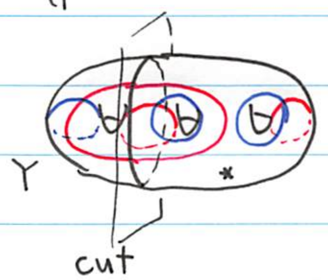
$\wr$

various complexes CFK^+, CFK^-, $\widehat{CFK}$, ...

$\wr$ homology

HF^+, HF^-, $\widehat{HF}$

• (pointed) Bordered Heegaard diagram

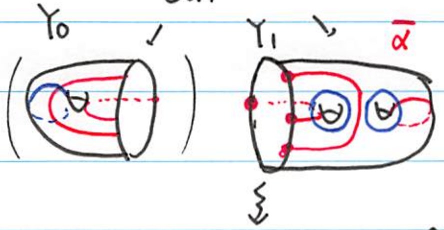


$\rightsquigarrow$ 3-manifold with boundary Y

with marking $\phi : F \xrightarrow{\cong} \partial Y$

$\updownarrow$

some algebra $\mathcal{A}(F)$ encoding 



type D module $\widehat{CFD}(Y)$ DG $\mathcal{A}(F)$-module

type A module $\widehat{CFA}(Y)$ A_∞ $\mathcal{A}(F)$-module

(← chain complex + product (action))

Theorem (Lipshitz-Ozsvath-Thurston)

$$\widehat{HF}(Y) = H\left(\widehat{CFA}(Y_0) \otimes_{\mathcal{A}(F)} \widehat{CFD}(Y_1)\right)$$

(homology)

※ (Holomorphic disk などのギロンを無視しても)

Heegaard Floer で現れる対象は複雑で分かりにくい...

Hanselmann-Rassumussen-Watson

$\widehat{CFD}(Y)$ を F 上の immersed curve として (ほぼ)等価に記述できる

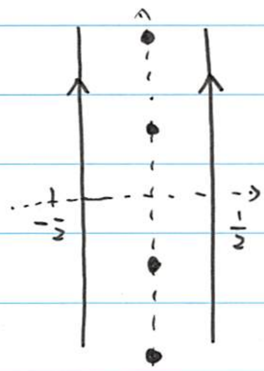
⋮

Hanselmann

cosmetic surgery への応用.

§2. Immersed curve invariant.

$Y = S^3 \setminus N(K)$: knot exterior



$T = \partial Y = \mathbb{R}^2/\mathbb{Z}^2$

$\overline{T} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ with marked points $(0, \frac{1}{2}+\mathbb{Z})$ (puncture)

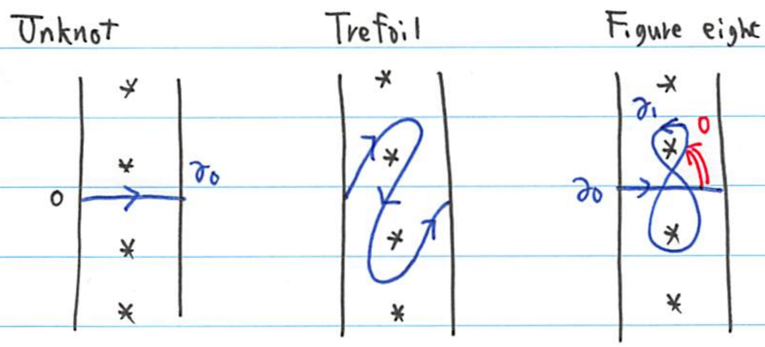
$\mathbb{R}/\mathbb{Z}$ ↑ longitude, $\mathbb{R}$ ↑ meridian

knot K について次のような Invariant

$\widehat{\Gamma}(K) = \{\gamma_0, \dots, \gamma_n,$ 「grading arrow」$\}$ が定まる

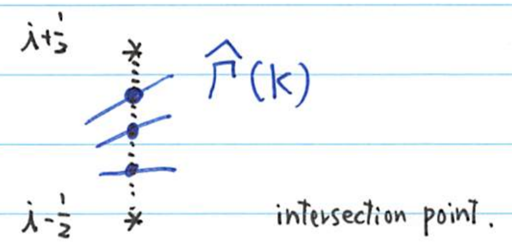
- $\gamma_i \subset \overline{T}$: immersed curve,  を持たない
- γ_0 は $-\frac{1}{2} \times \mathbb{R}$ とちょうど1回 正に交わる
  γ_i $(i>0)$ は $-\frac{1}{2} \times \mathbb{R}$ と交わらない
- γ_i $(i>0)$ の各 marked point についての winding number の和 $= 0$
- 「grading arrow」は γ_i と γ_j をつなぐ weight のついた矢印

<u>example.</u>



$\widehat{\Gamma}(K)$ からどのように HF の情報が読みとれるか？

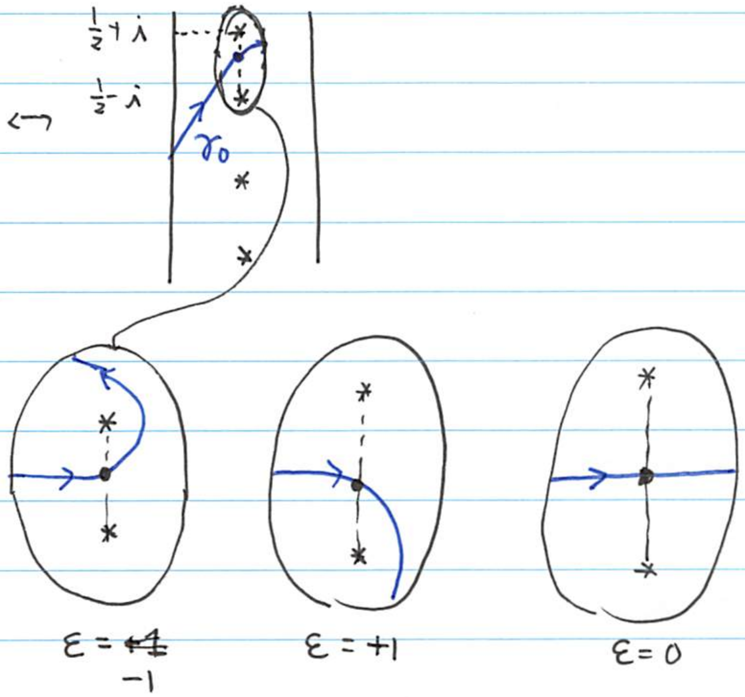
• Alexander grading i part of $\widehat{HFK}$ $\longleftrightarrow$



• τ- invariant

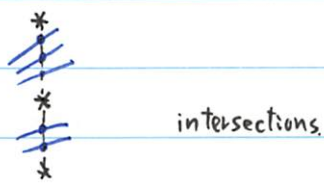
$\tau(K) = i$ $\longleftrightarrow$

• ε - invariant

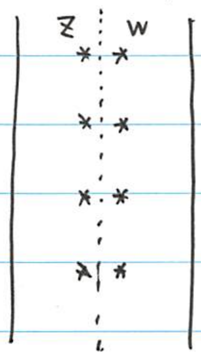


• Maslov grading / Knot Floer chain complex CFK^∞

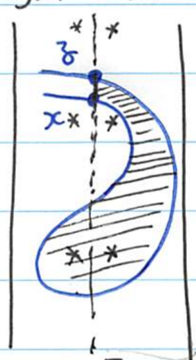
$CFK^\infty = \mathbb{Z}[U, U^{-1}]$-module generated by



slightly split/perturb marked points



bigon from x to z

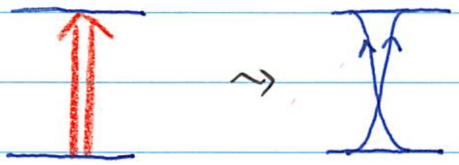


$n_w = \#\ w$ in bigon

$n_z = \#\ z$ 〃

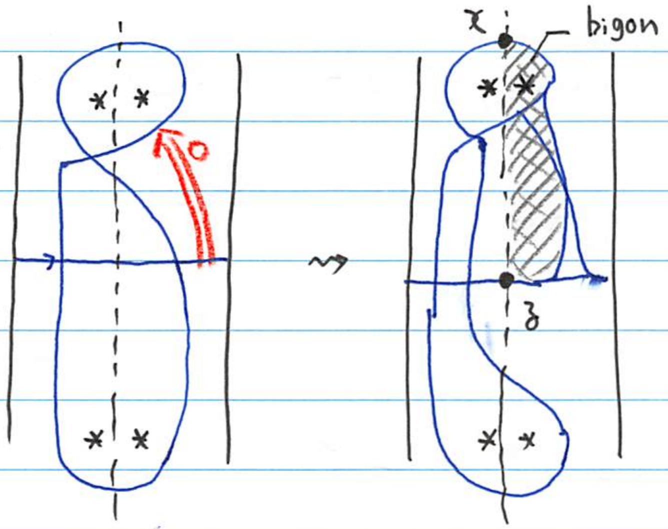
$$\partial x = \sum_z \left[\begin{array}{l} x\text{から}z\text{への} \\ \text{bigon たちで決まる} \end{array} \right] z$$

grading arrow を通る bigon を考えることで
(relative) Maslov grading も分かる.



と見る.

例



x と z の Maslov grading の差 $= -1 + 2n_w +$ (weight の和)

∘ $\widehat{\Gamma}(K)$ は (ほぼ) CFK^{∞} , $\widehat{CFD}$ と等価であった.

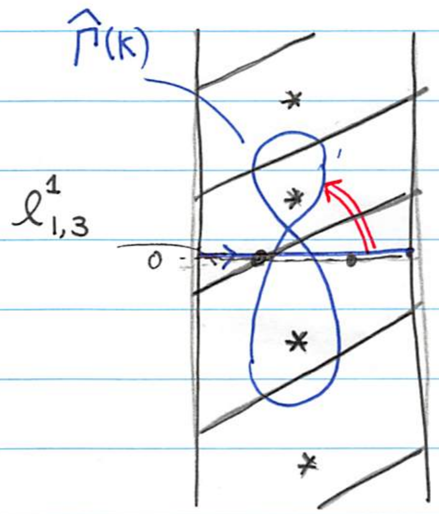
CFK^{∞} ↓ surgery formula

$\widehat{CFD}$ ↓ glueing formula

↓

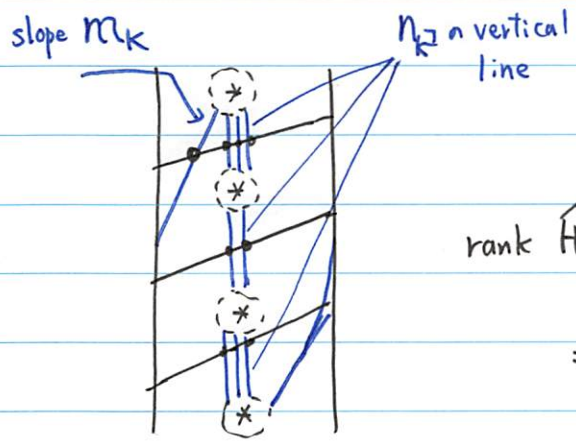
$\widehat{\Gamma}(K)$ version surgery formula.

$$\widehat{HF}(S^3_K(p/q), i) = \underbrace{HF}_{\text{intersection Floer homology}}(\widehat{\Gamma}(K), \underbrace{l^i_{p,q}}_{(0, -\frac{1}{2}+\frac{i}{q}(+\varepsilon)) \text{ を通る傾き } p/q \text{ の直線}})$$



If $\widehat{\Gamma}(K)$ and $l^i_{p,q}$ minimally intersects,
$HF(\widehat{\Gamma}(K), l^i_{p,q})$ is generated by intersection points.
(relative grading は grading arrow から読みとれる)

ex.) rank formula



put $\widehat{\Gamma}(K)$ in the above form.

$$\operatorname{rank} \widehat{HF}(S^3_K(p/q)) = \widehat{\Gamma}(K) \text{ と } l^i_{p,q} \text{ たちの交点数}$$

$$= \underbrace{|p - m_K q|}_{\text{slope } m_K \text{ curve との交点数}} + \underbrace{n_K |q|}_{\text{vertical line との交点数}}$$

§3 Cosmetic surgery

(Purely) cosmetic surgery conjecture

$$S^3_K(r) \cong S^3_K(r') \Rightarrow r = r'$$

Heegaard Floer homology による結果

- Ozsvath - Szabo

Rational surgery formula of $HF^+(S^3_K(r))$, $\widehat{HF}(S^3_K(r))$

slope r + CFK^∞ determines HF^+, $\widehat{HF}$

$\rightsquigarrow$ rank formula $\quad \operatorname{rank}(\widehat{HF}(S^3_K(p/q))) = |p - m_K q| + n_K |q|$

If

$\rightsquigarrow$ $\operatorname{rank}\widehat{HF}(S^3_K(p/q)) = \operatorname{rank}\widehat{HF}(S^3_K(p/q'))$

$\Rightarrow$ • $qq' < 0$ or $S^3_K(p/q)$ は L-space

- Ni - Wu

surgery formula of d-invariant ($\leftarrow$ absolute grading of certain generator)

If $p/q > 0$

comes from surgery formula of HF^+

$$\boxed{d(S^3_K(p/q), i) = d(L(p,q), i) - 2\max\{\underset{\overset{\rotatebox{90}{$\leq$}}{0}}{V_{\lfloor \frac{i}{q} \rfloor}}, \underset{\overset{\rotatebox{90}{$\leq$}}{0}}{H_{\lfloor \frac{i-p}{q} \rfloor}}\} \leq d(L(p,q), i)}$$

equality $\Leftrightarrow H_0 = V_0 = 0$

Casson invariant

If $\begin{cases} \lambda(S^3_K(p/q)) = \lambda(S^3_K(p/q')) \\ d(S^3_K(p/q)) = d(S^3_K(p/q')) \end{cases}$

$\rightsquigarrow$

$\Rightarrow$

$$d(S^3_K(p/q)) \leqq d(L(p,q)) = -2p\,\lambda(L(p,q))$$

$qq' < 0$ より $\qquad \| \qquad\qquad\qquad\qquad\qquad\qquad \|$

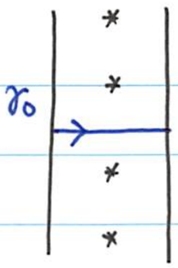
$$d(S^3_K(p/q')) \geqq d(L(p,q')) = -2p\,\lambda(L(p,q'))$$

全て等号になる

$\Rightarrow H_0 = V_0 = 0 \Rightarrow m_K = 0 \Rightarrow |q| = |q'|$,

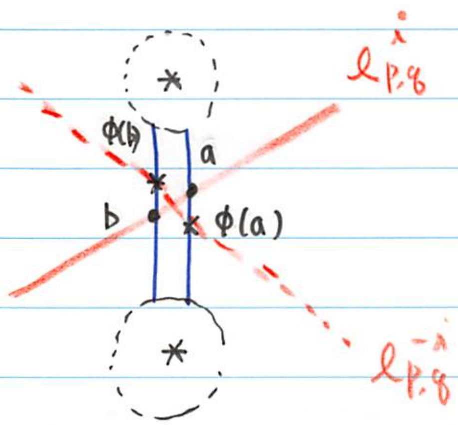
$\Rightarrow \tau = 0$

↳ 特に $\widehat{\Gamma}(K)$ において γ_0 は



の形

Hanselmann



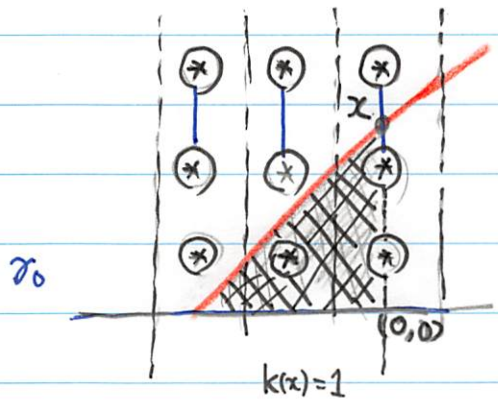
canonical 1-to-1 correspondence

$$l^{i}_{p,q}\cap\widehat{\Gamma}(K) \overset{\phi}{\longleftrightarrow} l^{-i}_{p,q}\cap\widehat{\Gamma}(K)$$

$$x \longmapsto \phi(x)$$

(this gives ungraded isomorphism

$$\phi : \widehat{HF}(S^3_K(p/q), i) \cong \widehat{HF}(S^3_K(-p/q), i)$$



Lemma. x と $\phi(x)$ の ~~Maslov~~ (relative) grading の差

$$\Delta_{rel}(x) = 1 - 2|A(x)| - 4k(x)$$

xの高さ (= Alexander grading)

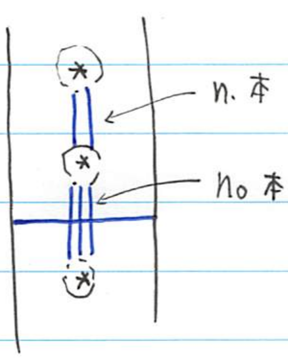
~~△~~内の Marked point の数

Hanselmann

$S^3_K(p/q) \cong S^3_K(-p/q)$

$\Rightarrow$

- $p/q > 1 \Rightarrow p/q = 2$, $g(K) = 2$, $n_0 = 2n_1$
- $p/q < 1 \Rightarrow p = 1$, $q = \dfrac{n_0 + 2\sum n_s}{4\sum_{s=1} s^2 n_s}$



$$th(K) \geq 2q\, g(g-1) - 2g$$

(Outline)

$$S^3_K(p/q) \cong S^3_K(-p/q)$$

$\Rightarrow$ $\exists$ permutation $\sigma : \{1, \cdots, p\} \to \{1, \cdots, p\}$

$$\widehat{HF}(S^3_K(p/q), i) \cong \widehat{HF}(S^3_K(-p/q), \sigma(i))$$

※ Spin^c structure の番号付けは surgery 表示に depend.
なので $\widehat{HF}(S^3_K(p/q), i) = \widehat{HF}(S^3_K(-p/q), i)$ とは限らない!!

analysis of rank

$\Rightarrow$ $\sigma(i) = i$ for $i = 0, \cdots, r$ ($r = 8$ if $p > 8$)

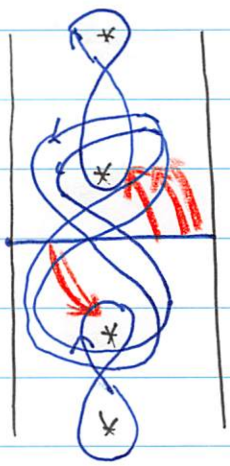
$\Rightarrow$ $\phi : \widehat{HF}(S^3_K(p/q), i) \to \widehat{HF}(S^3_K(-p/q), i)$

が grading を保つ同型とできるか調べる

• Knot 9_{44}

$K = 9_{44}$ $\qquad HF^+(S^3_K(1)) \cong HF^+(S^3_{-1}(1))$ Heegaard Floer では区別できない・・・

P(K)



Cosmetic surgery conjecture は Heegaard Floer だけでは解けない!!
(少なくとも現在の)