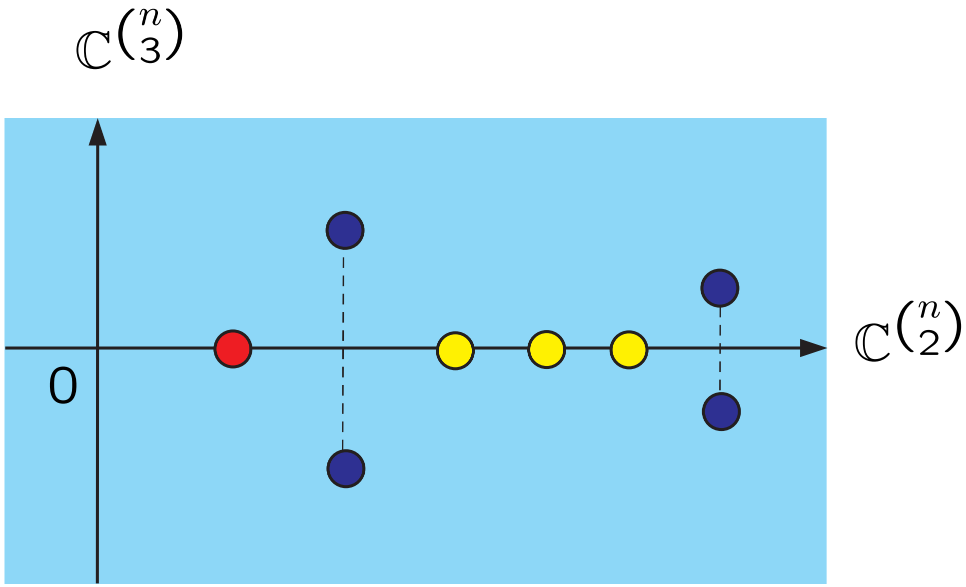
The structure of the trace-free (traceless) $\mathrm{SL}_2(\mathbb{C})$-character varieties of the knot groups

$S_0(K) =$

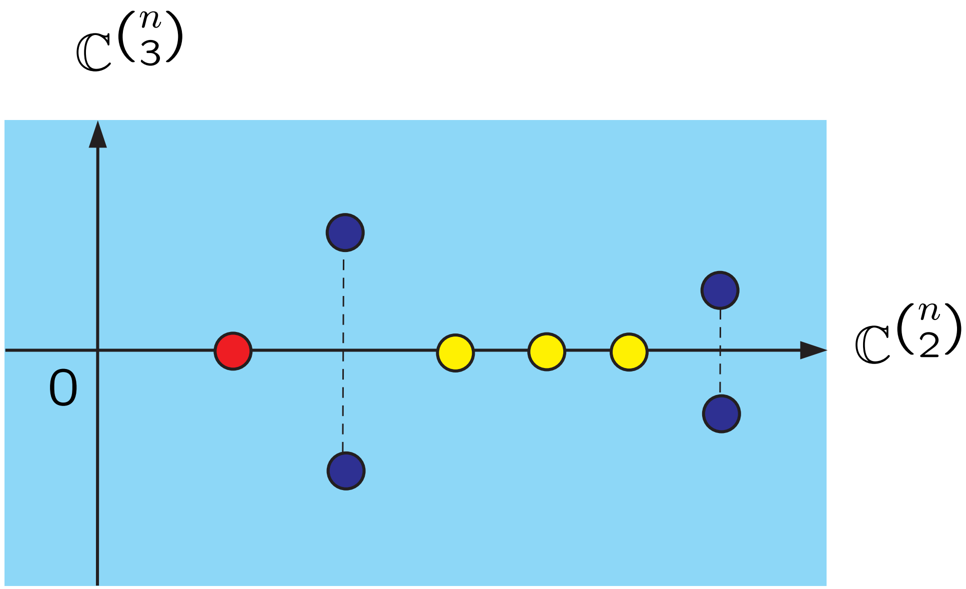


Fumikazu Nagasato (Meijo U)

2023年3月10日 @ 微分トポロジー '23

Today's key object

- **Trace-free (traceless) representation** $\rho : G(K) \to \mathrm{SL}_2(\mathbb{C})$
- **The $\mathrm{SL}_2(\mathbb{C})$-character variety** $X(K)$ of a knot K
- The trace-free (**traceless**) $\mathrm{SL}_2(\mathbb{C})$-character varieties (**trace-free slice**) $S_0(K)$ of a knot K.

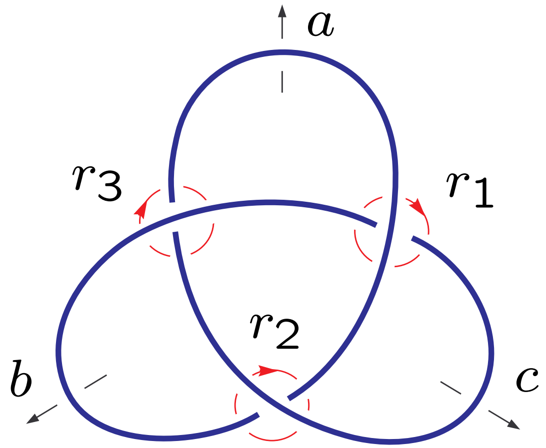
$S_0(K) =$ 

Trace-free (traceless) representations

$\rho : G(K) \to \mathrm{SL}_2(\mathbb{C})$, a group homomorphism

s.t. $\mathrm{tr}\rho(\text{meridian of } K) = 0$.

EX. The case $K = 3_1$



$$G(3_1) = \langle a, b, c \mid aca^{-1} = b, bab^{-1} = c \rangle$$
$$\cong \langle a, b \mid aba = bab \rangle$$

▶ $(\rho_1(a), \rho_1(b)) = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right)$ **abelian**

➡ $\rho_1(aba) = \rho_1(a)\rho_1(b)\rho_1(a) = \rho_1(b)\rho_1(a)\rho_1(b) = \rho_1(bab)$

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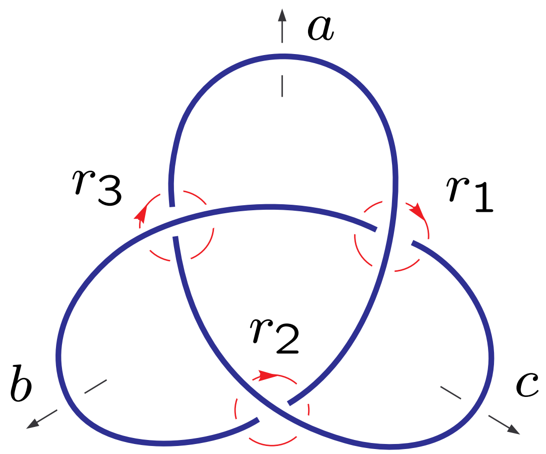


$$G(3_1) = \langle a, b, c \mid aca^{-1} = b, bab^{-1} = c \rangle$$
$$\cong \langle a, b \mid aba = bab \rangle$$

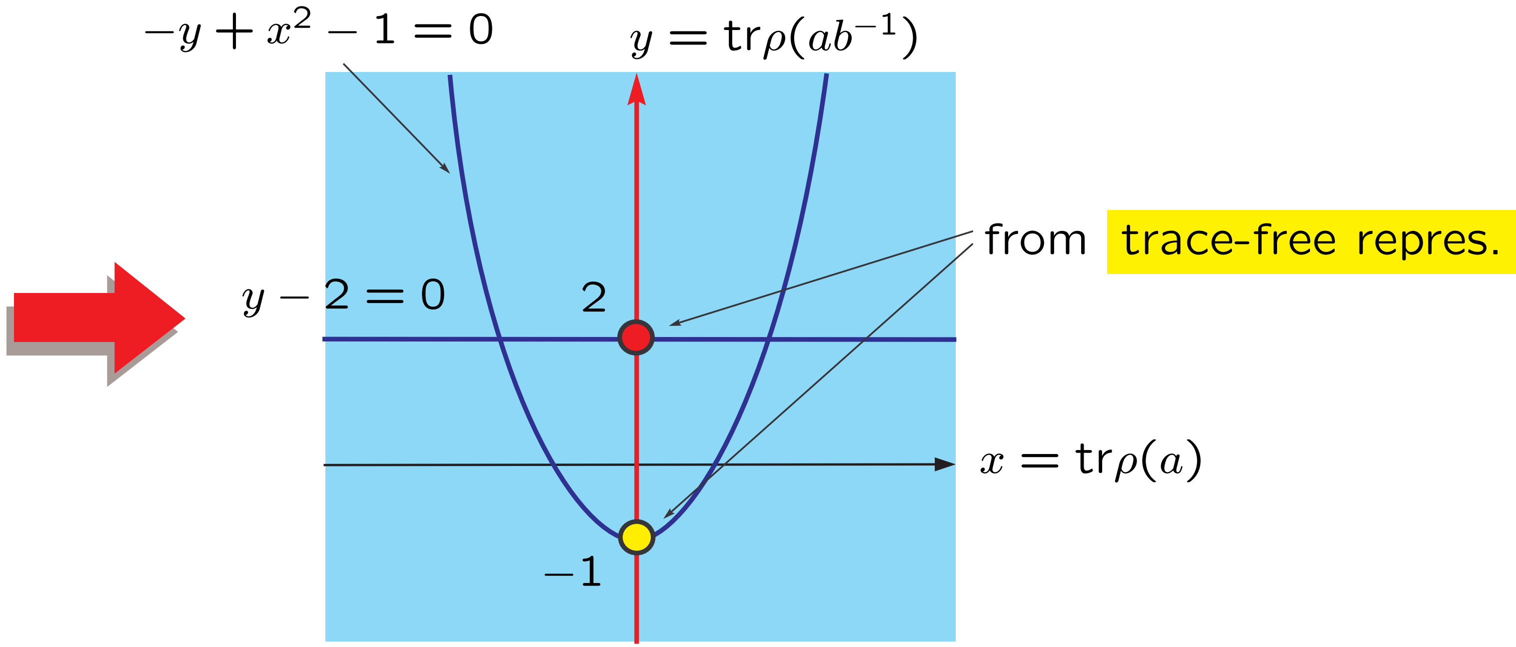
▶ $(\rho_2(a), \rho_2(b)) = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} -i/2 & 3 \\ -1/4 & i/2 \end{bmatrix} \right)$ **non-abelian**

$\Rightarrow$ $\rho_2(aba) = \rho_2(a)\rho_2(b)\rho_2(a) = \begin{bmatrix} i/2 & 3 \\ -1/4 & -i/2 \end{bmatrix} = \rho_2(b)\rho_2(a)\rho_2(b) = \rho_2(bab)$

▶ Plot $\rho : G(3_1) \to \mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ through $\big(\mathrm{tr}\rho(a), \mathrm{tr}\rho(ab^{-1})\big)$

- abelian repres: $\rho_1(ab^{-1}) = \mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- non-abelian repres: $\rho_2(ab^{-1}) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} i/2 & -3 \\ 1/4 & -i/2 \end{bmatrix} = -\begin{bmatrix} 1/2 & 3i \\ i/4 & 1/2 \end{bmatrix}$
- $\mathcal{R}(3_1) := \{\rho : G(3_1) \to \mathrm{SL}_2(\mathbb{C}), \text{representations}\}$

$\mathcal{R}(3_1)$



The blue curves are so-called **the character variety** of 3_1 (on $\mathbb{R}^2$).

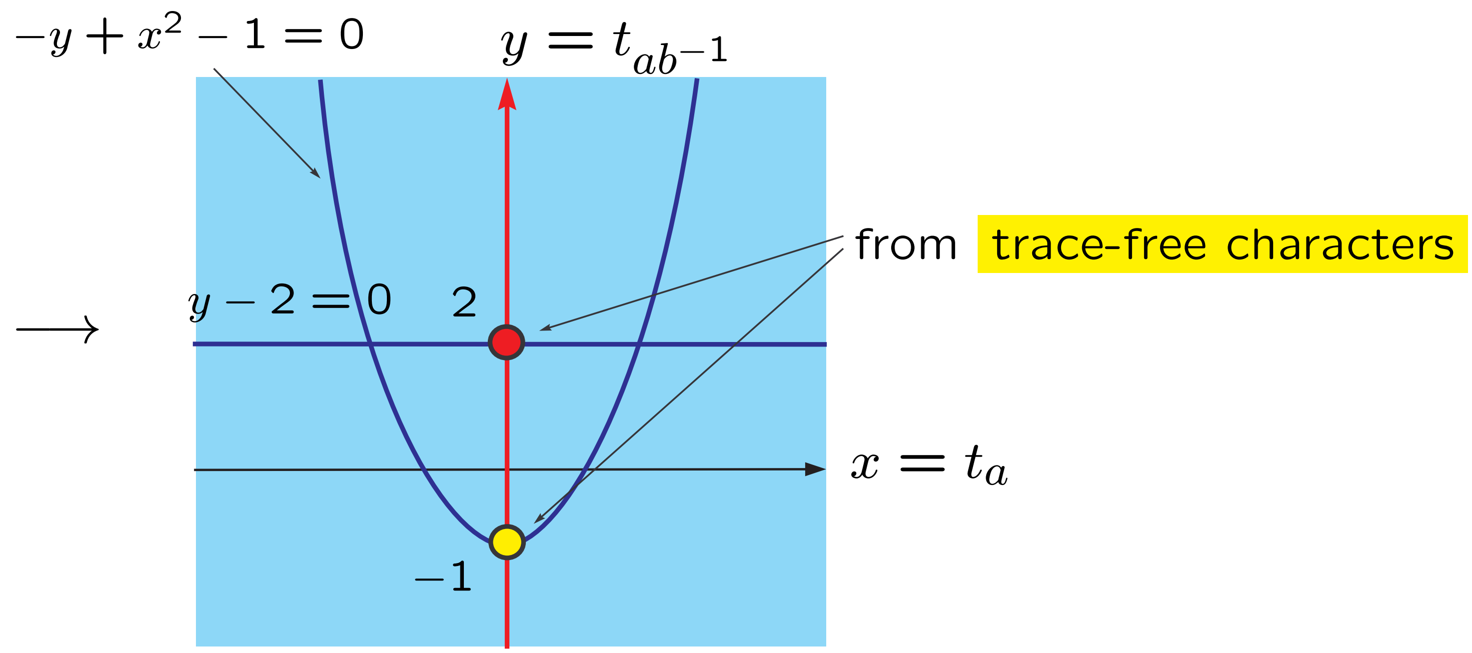
Reconstructing the character variety of 3_1 by **the characters**

- $\chi_\rho : G(3_1) \to \mathbb{C}$, $\chi_\rho(g) := \mathrm{tr}\rho(g)$, **the character** of ρ.
- $\mathcal{X}(3_1) := \{\chi_\rho : G(K) \to \mathbb{C} \mid \rho \in \mathcal{R}(3_1)\}$
- For $g \in G(3_1)$, set **the trace function** $t_g : \mathcal{X}(3_1) \to \mathbb{C}$,

$$t_g(\chi_\rho) := \mathrm{tr}\rho(g).$$

- Set the map $t : \mathcal{X}(3_1) \to \mathbb{C}^2$ by $t(\chi_\rho) := \big(t_a(\chi_\rho), t_{ab^{-1}}(\chi_\rho)\big)$.

$t : \mathcal{X}(3_1) \longrightarrow$



MEMO: why the characters? (background)

- $\rho_1,\ \rho_2 : G \to \mathrm{SL}_2(\mathbb{C})$ are said to be **isomorphic**, if $\exists C \in \mathrm{SL}_2(\mathbb{C})$ s.t. $\rho_2(g) = C^{-1}\rho_1(g)C$ $(\forall g \in G)$ **(a coordinate change!)**

  - $(\rho_1(a), \rho_1(b)) = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right)$ **abelian**
  - $(\rho_2(a), \rho_2(b)) = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} -i/2 & 3 \\ -1/4 & i/2 \end{bmatrix} \right)$ **non-abelian (irredubible)**
  - $(\rho_3(a), \rho_3(b)) = \left(\begin{bmatrix} -4-i & 0 \\ -4i & 4+i \end{bmatrix}, \begin{bmatrix} -3/2+9i/2 & -19/4+i \\ -4+2i & -3/2-9i/2 \end{bmatrix} \right)$

Actually, $\rho_3 \sim \rho_2$: $\exists C := \begin{bmatrix} 2 & -1+2i \\ 1 & i \end{bmatrix}$

s.t. $(C^{-1}\rho_3(a)C, C^{-1}\rho_3(b)C) = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} -i/2 & 3 \\ -1/4 & i/2 \end{bmatrix} \right) = (\rho_2(a), \rho_2(b))$

- For **irreducible representations** $\rho_1,\ \rho_2 : G(K) \to \mathrm{SL}_2(\mathbb{C})$,

  $\rho_1 \sim \rho_2 \Leftrightarrow \chi_{\rho_1}(g) = \chi_{\rho_2}(g)$ $(\forall g \in G(K))$. (**cf. [CS]**)

([CS] M. Culler and P. Shalen, *Varieties of group presentations and splittings of 3-manifolds*, Ann. of Math. **117** (1983), 109-146.)

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- For **irreducible representations** $\rho_1,\ \rho_2 : G(K) \to \mathrm{SL}_2(\mathbb{C})$,

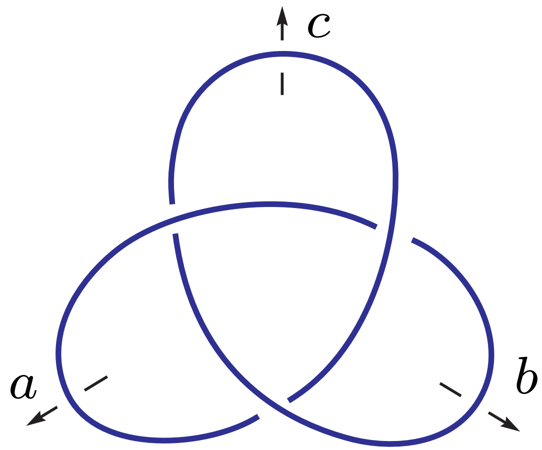
$\rho_1 \sim \rho_2 \Leftrightarrow \chi_{\rho_1}(g) = \chi_{\rho_2}(g)$ $(\forall g \in G(K))$. (**[CS]**)

- The map $t : \mathcal{X}(3_1) \to \mathbb{C}^2$, $t(\chi_\rho) = \left(t_a(\chi_\rho), t_{ab^{-1}}(\chi_\rho)\right)$ is **1 to 1** on the set of the irreducible characters. (true for $\forall K$)

The $SL_2(\mathbb{C})$-character variety $X(K)$ of K (quickly)

K ➡ $G(K)$ ➡ $\mathcal{X}(K)$ ➡ a closed algebraic set $X(K) \subset \mathbb{C}^N$

the character variety of K **[CS]**



⬇

$G(3_1) \cong \langle a, b \mid ab = b^{-1}aba \rangle$

⬇

$\rho : G(3_1) \to SL_2(\mathbb{C})$

a pair $(\rho(a), \rho(b)) = (A, B)$

➡ $AB = B^{-1}ABA$ ➡

$\mathrm{tr}(AB) = \mathrm{tr}(B^{-1}ABA)$

$\mathrm{tr}(XY) = \mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(XY^{-1})$

$SL_2(\mathbb{C})$-trace identity

⬆

$\{\mathrm{tr}(AB^{-1}) - 2\}\{\mathrm{tr}(AB^{-1}) + \mathrm{tr}(A)^2 - 1\} = 0$

$x := \mathrm{tr}(A)$ ⬆ $y := \mathrm{tr}(AB^{-1})$

$(y-2)(-y+x^2-1) = 0$

⬆

$X(3_1) = \{(x, y) \in \mathbb{C}^2 \mid (y-2)(-y+x^2-1) = 0\}$

The $SL_2(\mathbb{C})$-character variety $X(K)$ of K (quickly)

- Here is how to choose the coordinates for $X(K)$ in general.

"Embedding theorem" for $X(F_n)$ (cf. [GM])

For a free group $F_n := \langle g_1, \cdots, g_n \rangle$, $t_g : \mathcal{X}(F_n) \to \mathbb{C}$ can be described by a polynomial in t_{g_i} $(1 \leq i \leq n)$, $t_{g_i g_j}$ $(1 \leq i < j \leq n)$, $t_{g_i g_j g_k}$ $(1 \leq i < j < k \leq n)$ by $SL_2(\mathbb{C})$-trace identity.

$*$ **[GM]** determines the defining polynomials of $X(F_n)$.

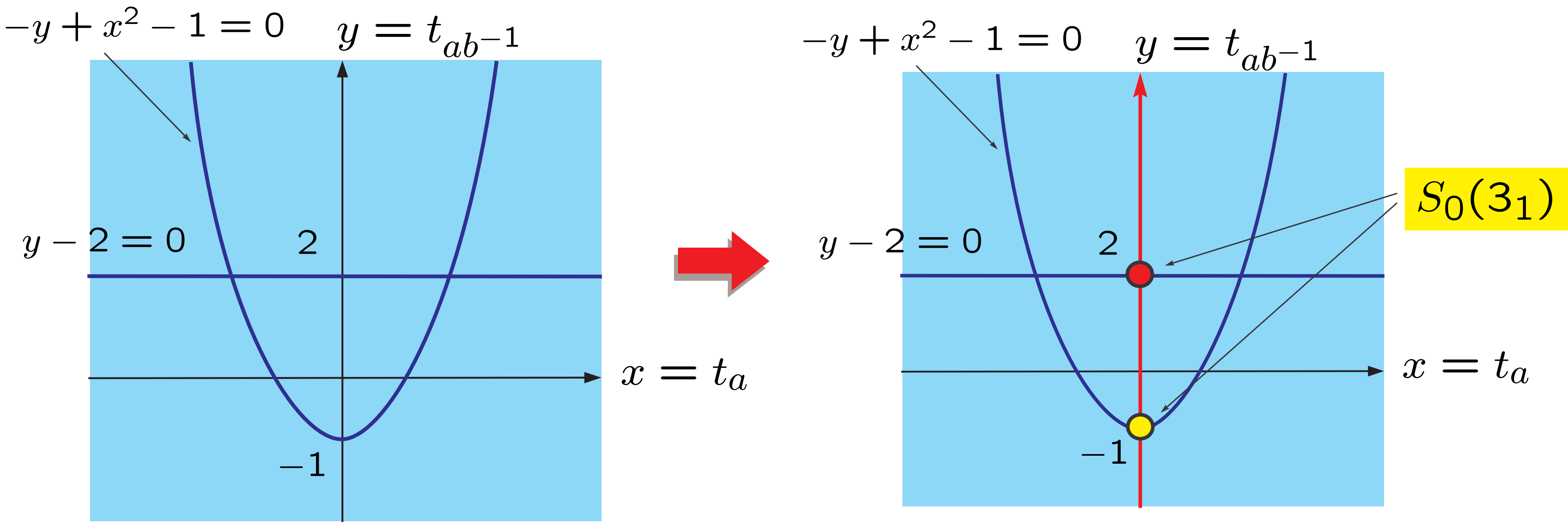
- For $G(K) = \langle g_1, \cdots, g_n \mid r_1, \cdots, r_n \rangle$, the character variety $X(K)$ can be realized in $\mathbb{C}^{n + \binom{n}{2} + \binom{n}{3}}$ by

$$t : \mathcal{X}(K) \to \mathbb{C}^{n+\binom{n}{2}+\binom{n}{3}}, \quad t(\chi_\rho) := \Big(t_{g_i}(\chi_\rho); t_{g_i g_j}(\chi_\rho); t_{g_i g_j g_k}(\chi_\rho)\Big)$$

(**[GM]** F. González-Acuña and J.M. Montesinos: *On the character variety of group representations in* $SL(2,\mathbb{C})$ *and* $PSL(2,\mathbb{C})$, Math. Z., **214** (1993), 627–652.)

The trace-free $SL_2(\mathbb{C})$-character variety $S_0(K)$

$$X(3_1) = \left\{(x,y) \in \mathbb{C}^2 \,\middle|\, (y-2)\left(-y+x^2-1\right) = 0\right\}$$



genuin $X(3_1) \subset \mathbb{C}^2$

$S_0(3_1) = X(3_1) \cap \{t_a = 0\}$

Definition (Trace-free slice $S_0(K)$)

$S_0(K) := X(K) \cap \{t_\mu = 0\}$ **(w/o multiplicity)**,

where μ is a meridian of K.

The trace-free $SL_2(\mathbb{C})$-character variety $S_0(K)$

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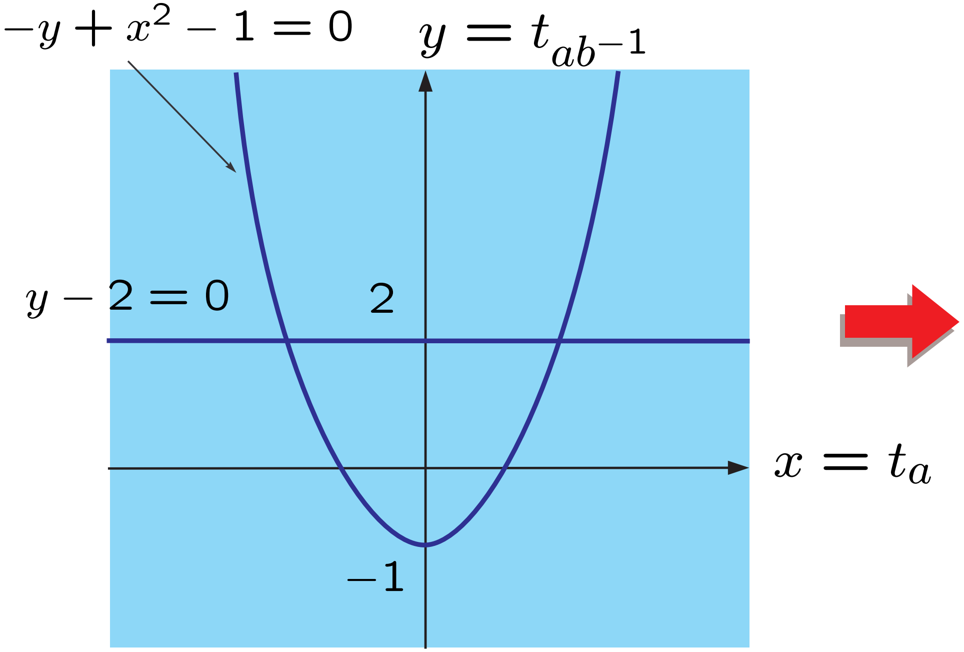

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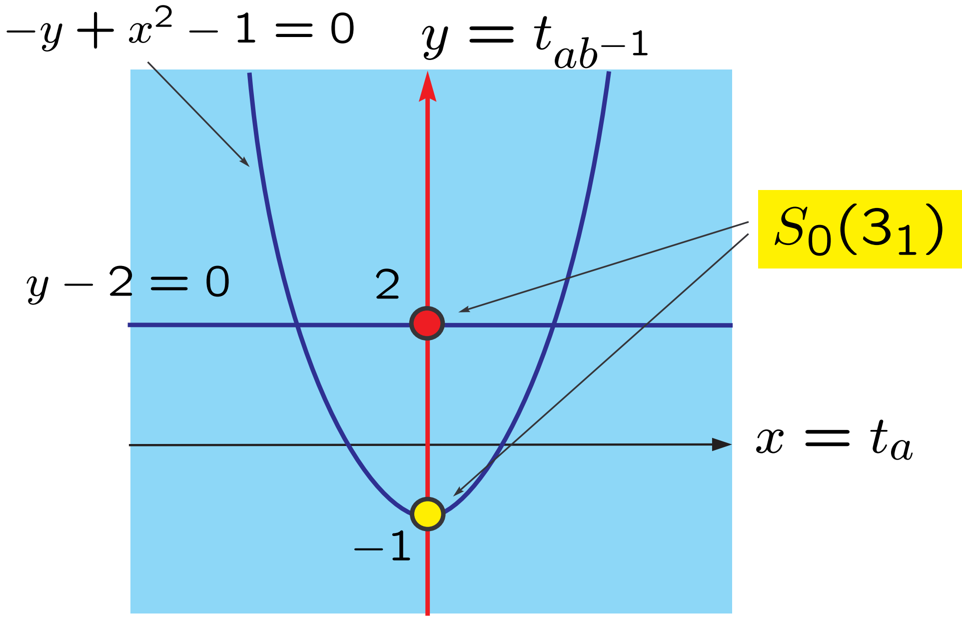


$S_0(3_1) = X(3_1) \cap \{t_a = 0\}$

The aim of this talk

Understand the geometric structure of **the trace-free slice** $S_0(K)$ through the topology of K.

Background of $S_0(K)$

The character variety $X(K)$ of a knot K is a powerful tool to research topological properties of knots (knot exteriors)

- **the Culler-Shalen theory** ➡ **essential surfaces** in E_K

- **the A-polynomial**: given by a projection of $X(K)$ to $\mathbb{C}^2$

  ➡ an unknot detector, **boundary slopes** of knots

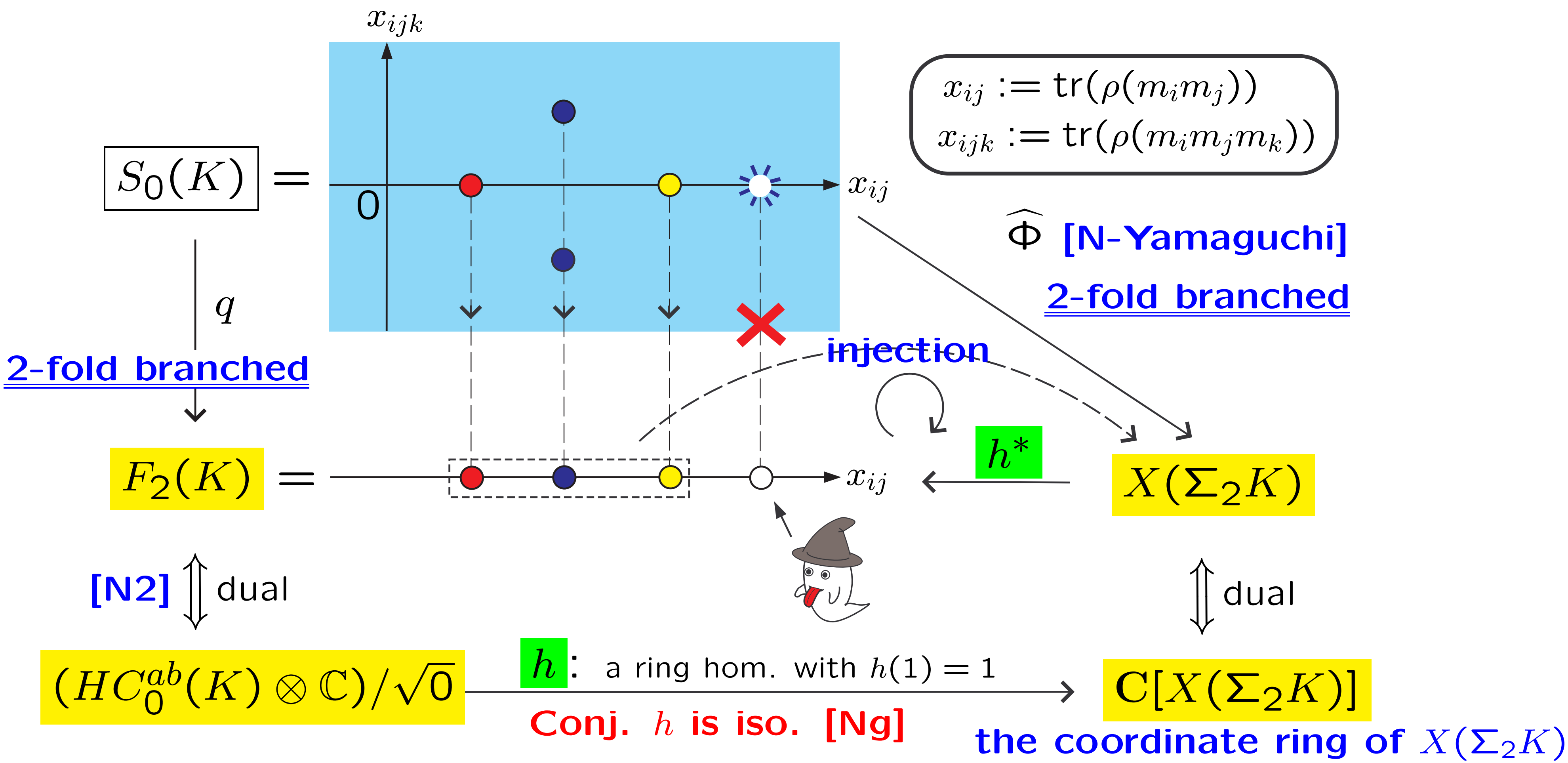
  A projection of $X(K)$ also has a topological information

- **the Casson-Lin invariant**: **trace-free slice** of "$X_{\mathrm{SU}(2)}(K)$"

  ➡ **the signature of** K

  A subset of $X(K)$ has also a topological information

Today's landscape around the trace-free slice $S_0(K)$



[N1] F. Nagasato, *Trace-free characters and abelian knot contact homology I*, available on ArXiv. (will be updated)

[N2] F. Nagasato, *Varieties via a filtration of the KBSM and knot contact homology*, Topology Appl. **264** (2019), 251-275.

[Ng] L. Ng, *Knot and braid invariants from contact homology II*, Geom. Topol. **9** (2005), 1603-1637.

The geometric structures of the trace-free slice $S_0(K)$

S1. **Metabelian characters** and **the knot determinant** $|\Delta_K(-1)|$

S2. **The defining polynomials** of $S_0(K)$

S3. **A symmetric structure** of $S_0(K)$ with involution $\tilde{\iota}$

$$\begin{pmatrix} * \text{ 2-fold branched cover with base space } F_2(K) \\ \text{branched at the metabelian characters} \end{pmatrix}$$

S4. A deep inside $S_0(K)$ from **degree 0 abelian knot contact homology** $HC_0(K)$ and **ghost characters**

S5. **Ghost characters** as an obstruction for $F_2(K) \cong X(\Sigma_2 K)$

S6. Finding **ghost characters**

S7. Finding **non τ-equivariant (proper) representation**

S1. Metabelian characters and the knot determinant

- A representation $\rho : G(K) \to \mathrm{SL}_2(\mathbb{C})$ is called **metabelian** if $\rho([G(K), G(K)])$ is **an abelian subgroup** of $\mathrm{SL}_2(\mathbb{C})$.

EX. Any abelian representation is metabelian.

Proposition [N], [N-Yamaguchi], [X.-S.Lin], [Klassen]

For any knot K, irr. binary dihedral char.

$$\#\{\ \circ\ \} = \#\{\text{irreducible metabelian characters}\} = \frac{|\Delta_K(-1)| - 1}{2}$$

[N] *Finiteness of a section of the* $\mathrm{SL}(2,\mathbb{C})$*-character variety of knot groups*, Kobe J. Math. **24** (2007), 125-136.

[N-Yamaguchi] *On the geometry of the slice of trace-free* $\mathrm{SL}_2(\mathbb{C})$*-characters of a knot group*, Math. Ann. **354** (2012), 967-1002.

[X.-S. Lin] X-.S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin., Engl. **17** (2001), 361-380.

[Klassen] E. Klassen, *Representations of knot groups in* $SU(2)$, Trans. Am. Math. Soc. **326** (1991), 795-828.

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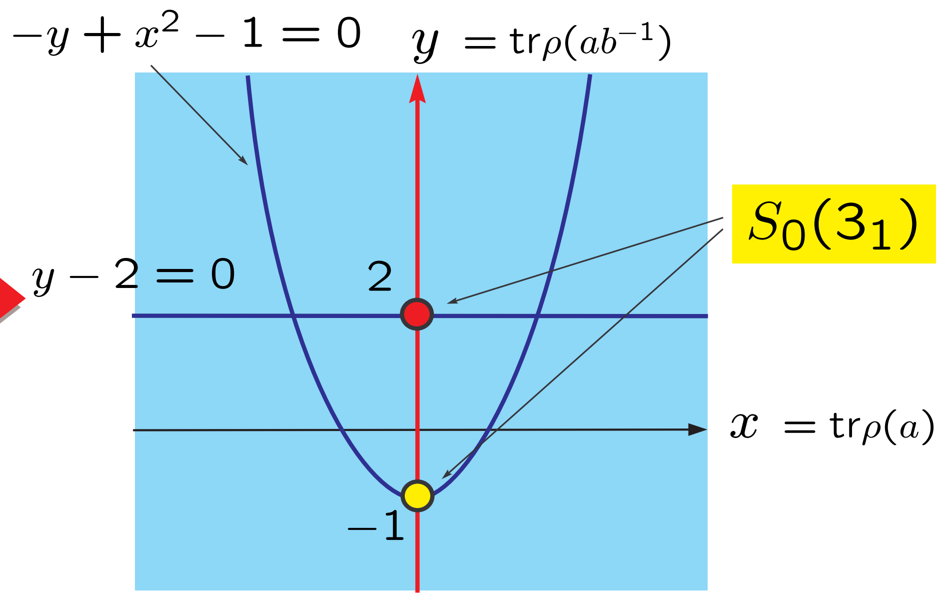
$$\frac{|\Delta_{3_1}(-1)|-1}{2} = \frac{3-1}{2} = 1$$



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Proposition [Burde], [de Rham], [CCGLS], [HPP]

$\exists$ **reducible non-abelian representation** $\rho_\lambda : G(K) \to \mathrm{SL}_2(\mathbb{C})$ satisfying $\rho_\lambda(\mu) \sim \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ if and only if $\Delta_K(\lambda^2) = 0$.

[Burde] G. Burde: *Darstellungen von Knottengruppen*, Math. Ann. **173** (1967), 24-33.

[de Rham] G. de Rham: *Introduction aux polynômes d'un nœud*, Enseign. Math. **13** (1967), 187–194.

[CCGLS] D. Cooper, M. Culler, H. Gillett, D. Long, P. Shalen, *Plane curves associated to character varieties of knot complements*, Invent. Math. **118**, 47-84 (1994)

[HPP] M. Heusener, J. Porti, E. S. Peiŕo, *Deformations of reducible representations of 3-manifold groups into* $\mathrm{SL}_2(\mathbb{C})$, J. Reine Angew. Math. **530**, 191-227 (2001).

S1. Metabelian characters and the knot determinant

- A representation $\rho : G(K) \to SL_2(\mathbb{C})$ is called **metabelian** if $\rho([G(K), G(K)])$ is **an abelian subgroup** of $SL_2(\mathbb{C})$.

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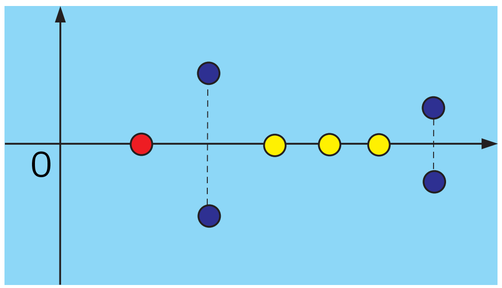
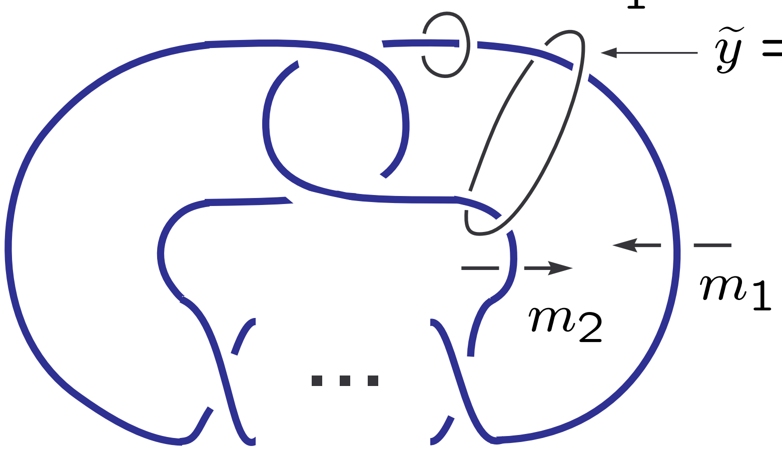
- $\mathcal{S}_0(K)$ consists of
  - ● : **the abelian character** (a kind of trivial one)
  - ○ : **the irreducible mentabelian characters**
  - ● : **the irreducible non-metabelian characters**

$S_0(K) =$ 

- Let's take a look at this structure for twist knots and more.

Observatons of $S_0(K_m)$ for twist knots K_m

$K_m =$ 

$x := t_{m_1}$

$y := t_{m_1 m_2^{-1}}$

Theorem **[Gelca-N (JKTR)], [N (Bull. Korean Math.)]**

$$X(K_m) = \left\{ (x,y) \in \mathbb{C}^2 \;\middle|\; (y-2)R_m(-x,-y) = 0 \right\},$$

where $R_m(x,y) := S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$

$$S_{n+2}(z) = zS_{n+1}(z) - S_n(z),\ S_1(z) = z,\ S_0(z) = 1.$$

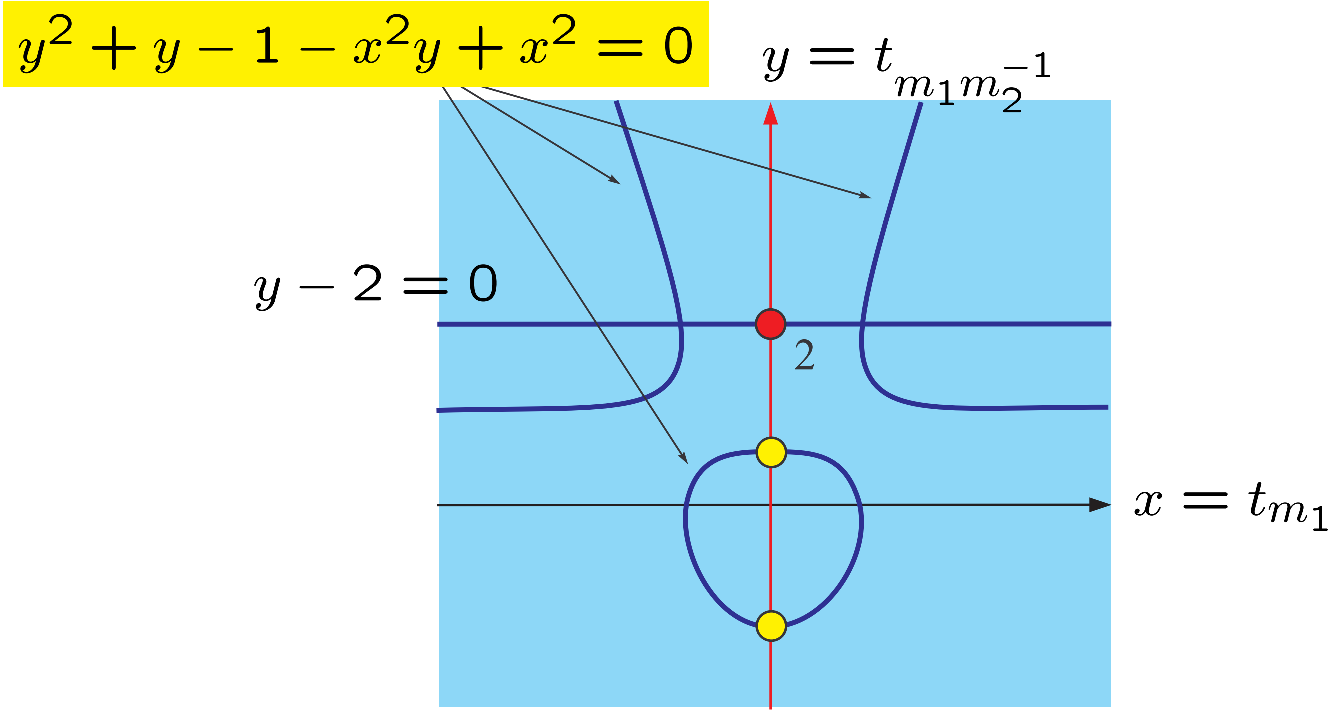
EX.

$$R_1(-x,-y) = -y - 1 + x^2$$

$$R_2(-x,-y) = y^2 + y - 1 - x^2 y + x^2$$

The trace-free slice $S_0(4_1)$

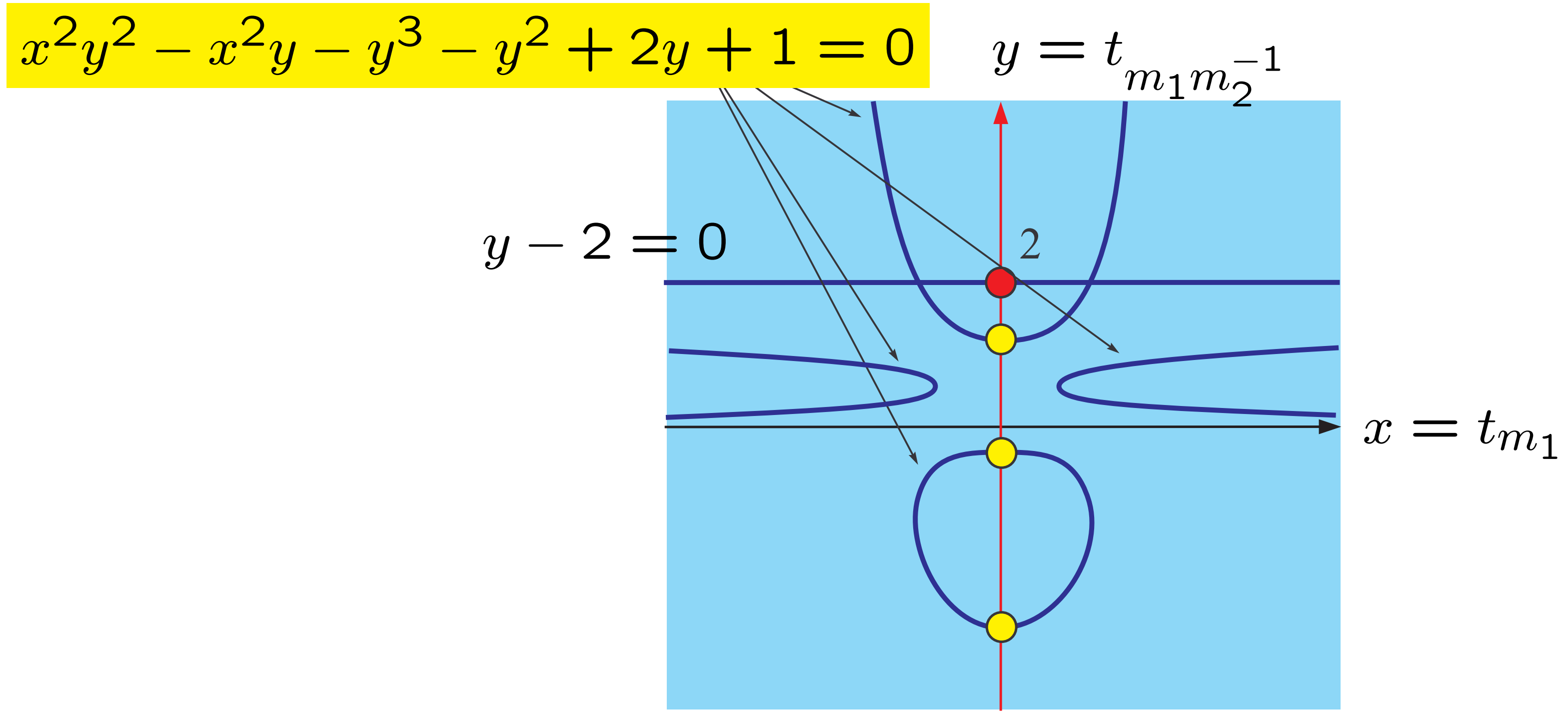
- $X(4_1) = \left\{(x,y) \in \mathbb{C}^2 \,\middle|\, (y-2)\left(y^2+y-1-x^2y+x^2\right) = 0\right\}$



- $S_0(4_1) = X(4_1) \cap \{t_{m_1} = 0\} = \left\{2, \frac{-1\pm\sqrt{5}}{2}\right\}$
- $|\Delta_{4_1}(-1)| = 5$, $\frac{|\Delta_{4_1}(-1)|-1}{2} = \frac{5-1}{2} = 2$

The trace-free slice $S_0(5_2)$

- $X(5_2) = \left\{(x,y) \in \mathbb{C}^2 \,\middle|\, (y-2)(x^2y^2 - x^2y - y^3 - y^2 + 2y + 1) = 0\right\}$



- $S_0(5_2) = X(5_2) \cap \{t_{m_1} = 0\} = \{2, -1.8019..., \ -0.44504..., \ 1.2470...\}$
- $|\Delta_{5_2}(-1)| = 7$, $\frac{|\Delta_{5_2}(-1)|-1}{2} = \frac{7-1}{2} = 3$

- This can be done because we have the defining poly of $X(K_m)$.

We want to calculate $S_0(K)$ directly w/o the calculation of $X(K)$.

Theorem ([N1], cf.[N2])

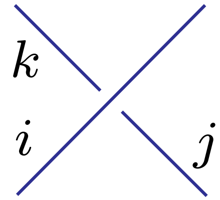
- $G(K)=\langle m_1,\cdots,m_n \mid r_1=1,\cdots,r_n=1\rangle$: **a Wirtinger presentation**
- $t:\mathcal{S}_0(K)\to\mathbb{C}^{\binom{n}{2}+\binom{n}{3}}$, $t(\chi_\rho)=(x_{ij};x_{ijk})=\left(-t_{m_im_j}(\chi_\rho);-t_{m_im_jm_k}(\chi_\rho)\right)$

Then $t(\mathcal{S}_0(K))=S_0(K)$ is realized as the following algebraic set:

$$\left\{ (x_{ab};x_{pqr})\in\mathbb{C}^{\binom{n}{2}+\binom{n}{3}} \;\middle|\; \begin{array}{l} \textbf{(F2)}\; x_{ak}=x_{ij}x_{ai}-x_{aj}\quad (x_{aa}=2,x_{st}=x_{ts}) \\ \left(a\in\{1,\cdots,n\},\ \forall\ \text{Wirtinger triple } (i,j,k) \right) \\ \textbf{(H)}\; x_{i_1i_2i_3}\cdot x_{j_1j_2j_3}=\frac{1}{2}\begin{vmatrix} x_{i_1j_1} & x_{i_1j_2} & x_{i_1j_3} \\ x_{i_2j_1} & x_{i_2j_2} & x_{i_2j_3} \\ x_{i_3j_1} & x_{i_3j_2} & x_{i_3j_3} \end{vmatrix} \\ (1\le i_1<i_2<i_3\le n,\ 1\le j_1<j_2<j_3\le n) \end{array} \right\}$$

$(1\le a<b\le n)$

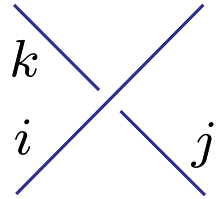
$(1\le p<q<r\le n)$



[N1] F. Nagasato, *Trace-free characters and abelian knot contact homology I*, available on ArXiv. (will be updated)

[N2] F. Nagasato, *Varieties via a filtration of the KBSM and knot contact homology*, Topology Appl. **264** (2019), 251-275.

▶ $\boxed{x_{ak}} = x_{ij}x_{ai} - x_{aj}$ **(F2)**

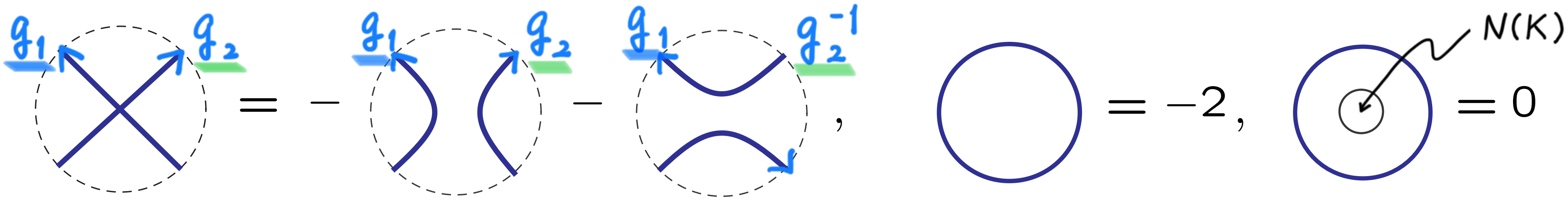


➡ $m_k = m_i m_j m_i^{-1}$

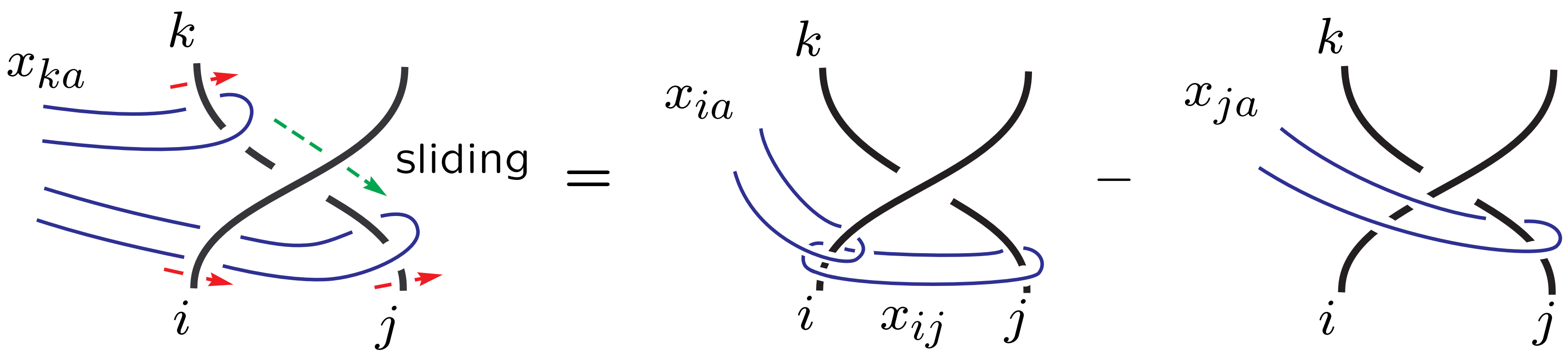
$$\boxed{-\mathrm{tr}\rho(m_a m_k)} = -\mathrm{tr}\rho(m_a m_i m_j m_i^{-1}) = -\mathrm{tr}\rho(m_a m_i)\,\mathrm{tr}\rho(m_j m_i^{-1}) + \mathrm{tr}\rho(m_a m_i^2 m_j^{-1})$$

$$= \cdots = \mathrm{tr}\rho(m_i m_j)\,\mathrm{tr}\rho(m_a m_i) + \mathrm{tr}\rho(m_a m_j)$$ (by trace identity)

trace-free Kauffman brakcet skein relation (at $t = -1$)



$$-\mathrm{tr}\rho(g_1 g_2) = -\mathrm{tr}\rho(g_1)\,\mathrm{tr}\rho(g_2) + \mathrm{tr}\rho(g_1 g_2^{-1}), \quad -\mathrm{tr}(\mathbf{E}) = -2, \quad -\mathrm{tr}\rho(\mu) = 0$$



- $x_{ak} = x_{ij}x_{ai} - x_{aj}$ **(F2)**, $\left(x_{abk} = x_{ij}x_{abi} - x_{abj}\right.$ **(F3)**$\left.\right)$

the fundamental relations (F)

- $$x_{i_1i_2i_3} \cdot x_{j_1j_2j_3} = \frac{1}{2}\begin{vmatrix} x_{i_1j_1} & x_{i_1j_2} & x_{i_1j_3} \\ x_{i_2j_1} & x_{i_2j_2} & x_{i_2j_3} \\ x_{i_3j_1} & x_{i_3j_2} & x_{i_3j_3} \end{vmatrix}$$

the hexagon relations (H)

coming from trace identity

$$\left(\begin{vmatrix} 2 & x_{12} & x_{1a} & x_{1b} \\ x_{21} & 2 & x_{2a} & x_{2b} \\ x_{a1} & x_{a2} & 2 & x_{ab} \\ x_{b1} & x_{b2} & x_{ba} & 2 \end{vmatrix} = 0 \qquad \text{(these appear for } n \geq 4\text{)} \right)$$

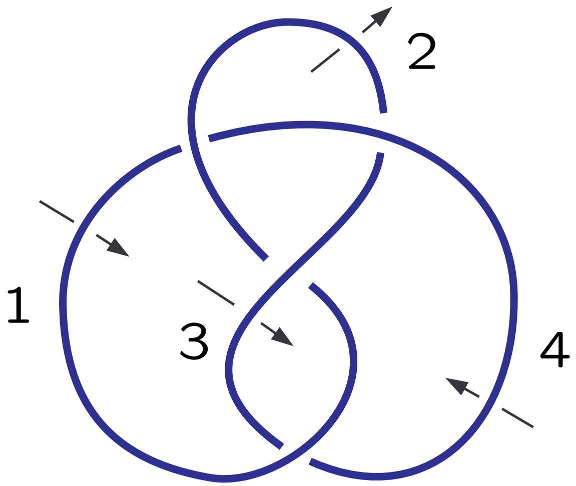
the rectangle relations (R)

NOTE **(H)** and **(R)** are coming from **the** $\mathrm{SL}_2(\mathbb{C})$**-trace identity**

i.e., the trace-free $\mathrm{SL}_2(\mathbb{C})$-character variety $X(F_n)$ by bf [GM].

([GM] F. González-Acuña and J.M. Montesinos: *On the character variety of group representations in* SL(2, ℂ) *and* PSL(2, ℂ), Math. Z., **214** (1993), 627–652.)

EX. **The case of** $K = 4_1$:



$$\left\langle m_1, m_2, m_3, m_4 \;\middle|\; \begin{array}{l} m_3 m_1 m_3^{-1} = m_2 \\ m_4 m_3 m_4^{-1} = m_2 \\ m_1 m_3 m_1^{-1} = m_4 \\ m_2 m_1 m_2^{-1} = m_4 \end{array} \right\rangle$$

All **(F2):**

$$\left\{ \begin{array}{c} x_{13}x_{23} - x_{12} = 2,\ x_{12}x_{24} - x_{14} = 2,\ x_{13}x_{14} - x_{34} = 2,\ x_{24}x_{34} - x_{23} = 2 \\ x_{13} = x_{23},\ \boxed{x_{12} = x_{24}},\ \boxed{x_{13} = x_{14}},\ x_{23} = x_{24} \\ \boxed{x_{12} = x_{13}^2 - 2},\ x_{24} = x_{13}^2 - 2,\ x_{34} = x_{13}^2 - 2 \\ \boxed{x_{13} = x_{14}x_{24} - x_{12}},\ x_{14} = x_{23}x_{34} - x_{24} \\ x_{13} = x_{23}x_{24} - x_{34},\ x_{23} = x_{12}x_{14} - x_{24} \end{array} \right\}$$

All **(F2):**

$$\left\{\begin{array}{c} x_{13}x_{23}-x_{12}=2,\ x_{12}x_{24}-x_{14}=2,\ x_{13}x_{14}-x_{34}=2,\ x_{24}x_{34}-x_{23}=2 \\ x_{13}=x_{23},\ \boxed{x_{12}=x_{24}},\ \boxed{x_{13}=x_{14}},\ x_{23}=x_{24} \\ \boxed{x_{12}=x_{13}^2-2},\ x_{24}=x_{13}^2-2,\ x_{34}=x_{13}^2-2 \\ \boxed{x_{13}=x_{14}x_{24}-x_{12}},\ x_{14}=x_{23}x_{34}-x_{24} \\ x_{13}=x_{23}x_{24}-x_{34},\ x_{23}=x_{12}x_{14}-x_{24} \end{array}\right\}$$

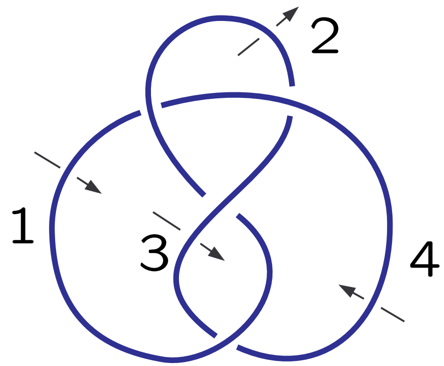
- $F_2(4_1) := \left\{ (x_{12}, \cdots, x_{45}) \in \mathbb{C}^{10} \;\middle|\; \begin{array}{l} x_{ka} = x_{ik}x_{ia} - x_{ja} \ \textbf{(F2)} \\ \text{for any Wirtinger triple } (i,j,k) \\ a \in \{1, \cdots, 4\} \ (x_{aa} = 2) \end{array} \right\}$

$F_2(4_1)$ is parametrized by x_{13} and

$$x_{13} = x_{14}x_{24} - x_{12} \Rightarrow x_{13} = x_{13}(x_{13}^2 - 2) - (x_{13}^2 - 2)$$

$$\Rightarrow (x_{13} - 2)(x_{13}^2 + x_{13} - 1) = 0$$

Hence we get $F_2(4_1) = \left\{2, \frac{-1\pm\sqrt{5}}{2}\right\} = S_0(4_1)$.



$$\Longrightarrow \begin{cases} x_{123} = 0, \; x_{124} = 0 \\ x_{134} = 0, \; x_{234} = 0 \end{cases}$$

Indeed, we can check this by **the hexagon relation**:

(H) $$x_{i_1 i_2 i_3} \cdot x_{j_1 j_2 j_3} = \frac{1}{2} \begin{vmatrix} x_{i_1 j_1} & x_{i_1 j_2} & x_{i_1 j_3} \\ x_{i_2 j_1} & x_{i_2 j_2} & x_{i_2 j_3} \\ x_{i_3 j_1} & x_{i_3 j_2} & x_{i_3 j_3} \end{vmatrix} \qquad \begin{matrix} (1 \le i_1 < i_2 < i_3 \le 4) \\ (1 \le j_1 < j_2 < j_3 \le 4) \end{matrix}$$

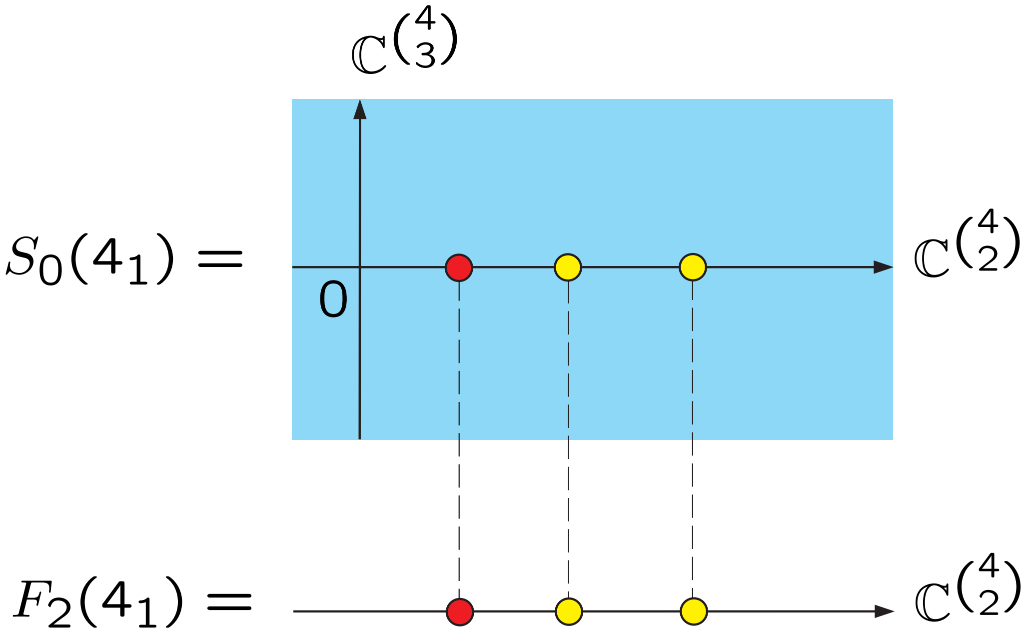
EX.

$$x_{123}^2 = \frac{1}{2} \begin{vmatrix} 2 & x_{12} & x_{13} \\ x_{21} & 2 & x_{23} \\ x_{31} & x_{32} & 2 \end{vmatrix} = x_{12}x_{13}x_{23} - x_{12}^2 - x_{13}^2 - x_{23}^2 + 4$$

$$= (x_{13}^2 - 2)x_{13}^2 - (x_{13}^2 - 2)^2 - x_{13}^2 - x_{13}^2 + 4 = 0$$

Hence, · · · · · · · · · · all points in $F_2(4_1)$ **lift** to $S_0(4_1)$.

$S_0(4_1) = F_2(4_1)$ and thus the theorem holds for 4_1.



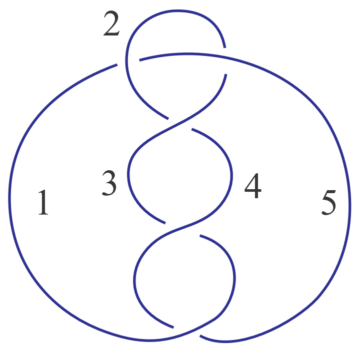
All points in $F_2(4_1)$ **lift** to $S_0(4_1)$

$|\Delta_{4_1}(-1)| = 5$, $\frac{5-1}{2} = 2$

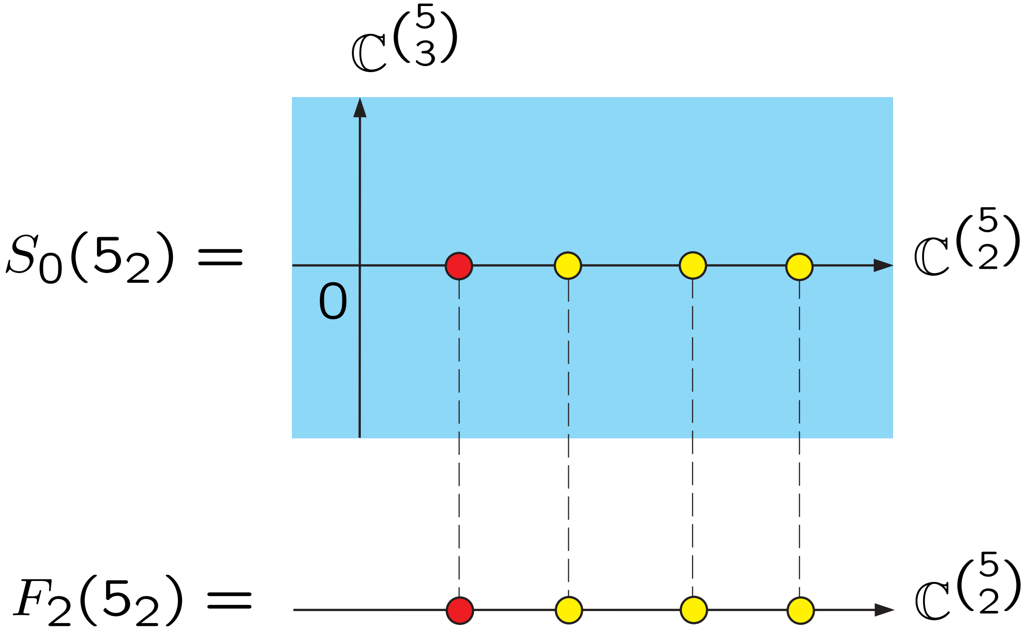
- Calculate $F_2(4_1)$ first, check the liftability second.

(In this case, **(H)** are trivial i.e. $\forall\ x_{ijk}^2 = 0$, **"1 to 1" lift**)

The case of $K = 5_2$



$$S_0(5_2) = F_2(5_2) = \left\{ x_{14} \in \mathbb{C} \,\middle|\, (x_{14} - 2)\left(x_{14}^3 + x_{14}^2 - 2x_{14} - 1\right) = 0 \right\}$$



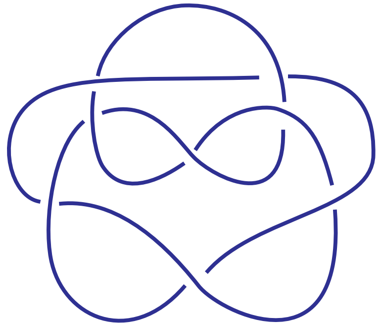
All points in $F_2(5_2)$ also **lift** to $S_0(5_2)$

$|\Delta_{4_1}(-1)| = 7$, $\frac{7-1}{2} = 3$

- Calculate $F_2(5_2)$ first, check the liftability second.

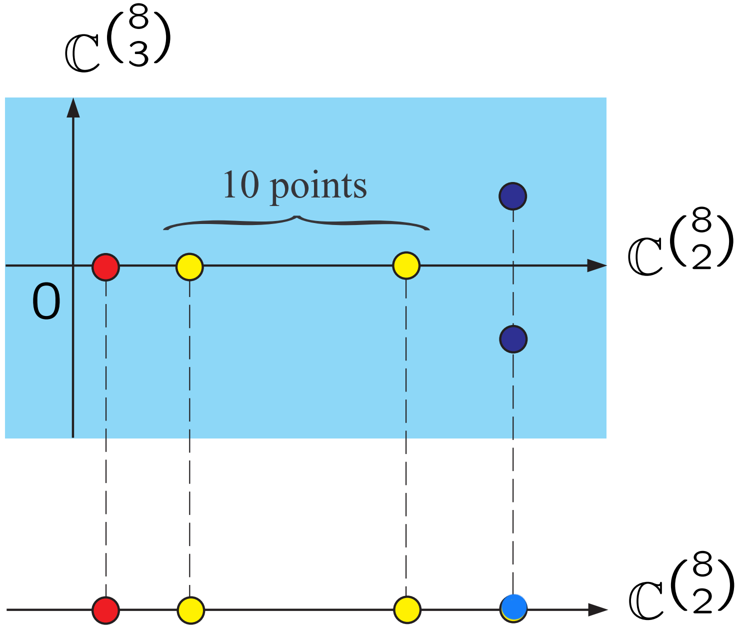
  (In this case, **(H)** are trivial i.e. $\forall\ x_{ijk}^2 = 0$, **"1 to 1" lift**)

The case of $K = 8_5$



By **Maple**, $F_2(8_5) = \{12 \text{ points}\}$, $S_0(8_5) = \{13 \text{ points}\}$

$S_0(8_5) =$



$F_2(8_5) =$

All points in $F_2(8_5)$ also **lift** to $S_0(8_5)$

$|\Delta_{8_5}(-1)| = 21, \ \frac{21-1}{2} = 10$

- Calculate $F_2(8_5)$ first, check the liftability second.

(In this case, **(H)** are non-trivial i.e. $\exists\ x_{ijk}^2 = c\ (\neq 0)$, **2-fold**)

MEMO: On non-metabelian representations of $G(K)$

- **[N-Yamaguchi] (2012)** shows that

$$S_0(S(p,q)) = \left\{1 \text{ abel} + \binom{\frac{p-1}{2}} \text{ irr. metabelian characters}\right\}$$

• $|\Delta_{S(p,q)}| = p$ ⇒ $\sharp S_0(S(p,q)) = 1 + \frac{p-1}{2}$.

- **[N] (2007)** shows that

∃ **an irreducible non-metabelian representation** for $K = 8_{20}$.

- **[Zentner] (2011)** shows that

∃ **a non-binary dihedral representation** for some alternating pretzel knots.

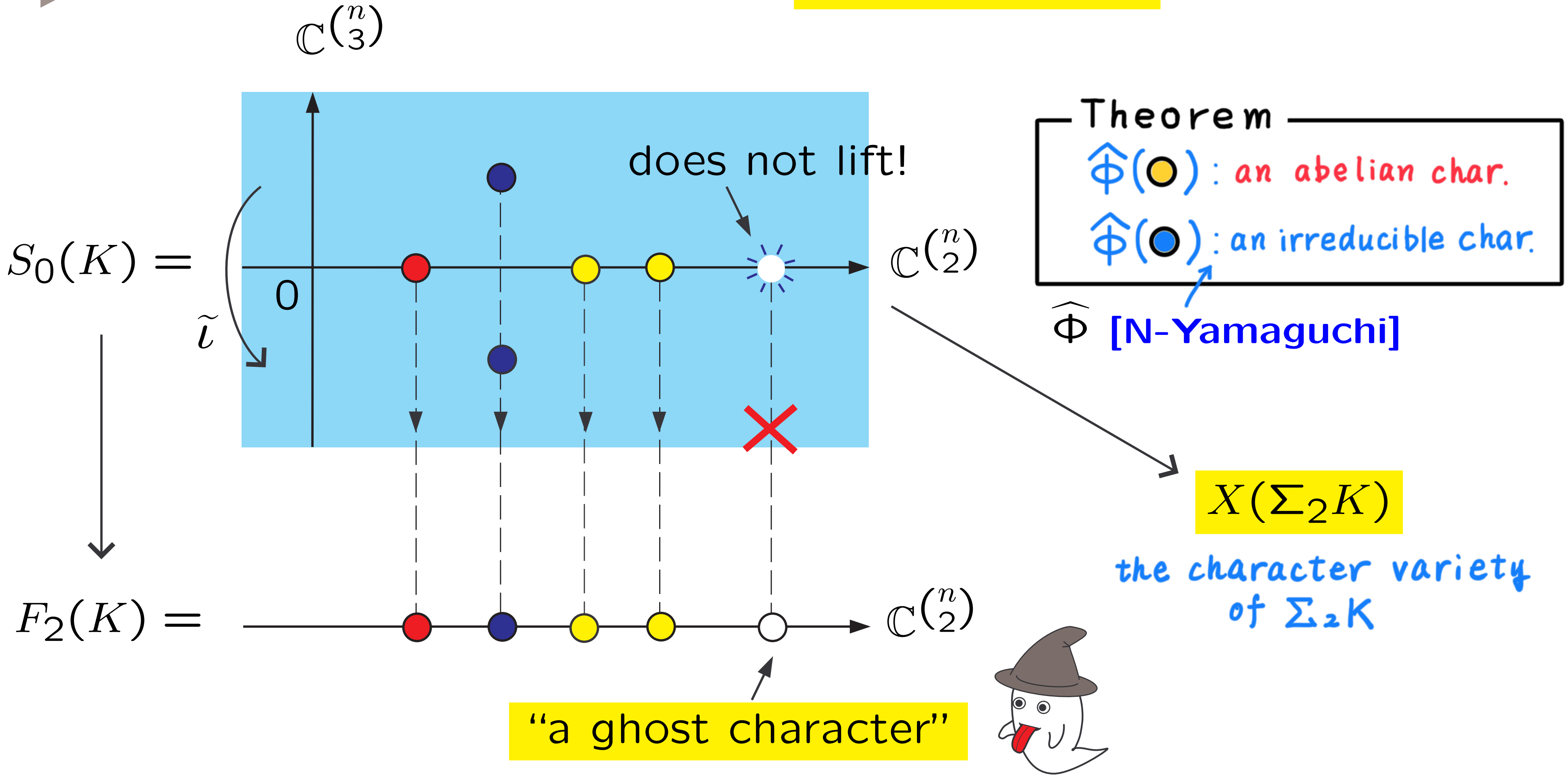
(**[N-Yamaguchi]** *On the geometry of the slice of trace-free* $\mathrm{SL}_2(\mathbb{C})$*-characters of a knot group*, Math. Ann. **354** (2012), 967-1002.

[N] *Finiteness of a section of the* $\mathrm{SL}(2,\mathbb{C})$*-character variety of knot groups*, Kobe J. Math. **24** (2007), 125-136.

[Zentner] R. Zentner, *Representation spaces of pretzel knots*, Algebr. Geom. Topol. **11** (2011), 2941-2970.)

S3. A symmetric structure of $S_0(K)$ with involution $\widetilde{\iota}$

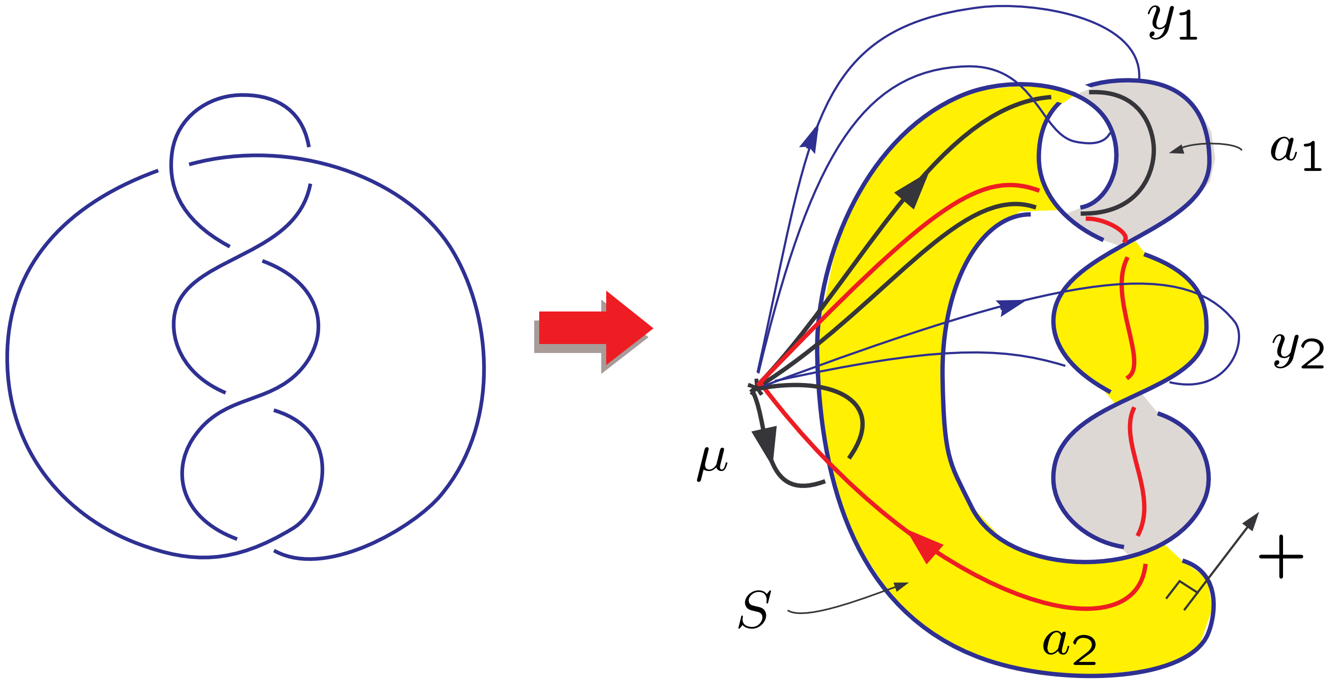
- $\Sigma_2 K$: the 2-fold branched cover of $\mathbf{S}^3$ branched along K
- **involution** $\iota : \mathcal{R}(K) \to \mathcal{R}(K)$, "$\iota(\rho)(\mu) = -\rho(\mu)$" gives $\widetilde{\iota}$.



([N-Yamaguchi] *On the geometry of the slice of trace-free* $\mathrm{SL}_2(\mathbb{C})$*-characters of a knot group*, Math. Ann. **354** (2012), 967-1002.)

EX.

- $\iota : \mathcal{R}(K) \to \mathcal{R}(K)$, $\iota(\rho)(g) = (-1)^{[g]}\rho(g)$ $([g] \in H_1(\mathbf{E}_K; \mathbb{Z}))$.
- Take a Seifert surface S via the Seifert algorithm.



- $G(5_2) = \left\langle y_1, y_2, \mu \;\middle|\; \mu a_1^+ \mu^{-1} = a_1^-,\ \mu a_2^+ \mu^{-1} = a_2^- \right\rangle$

  * $y_1,\ y_2 \in [G(5_2), G(5_2)]$ by **Lin's presentations** of $G(K)$

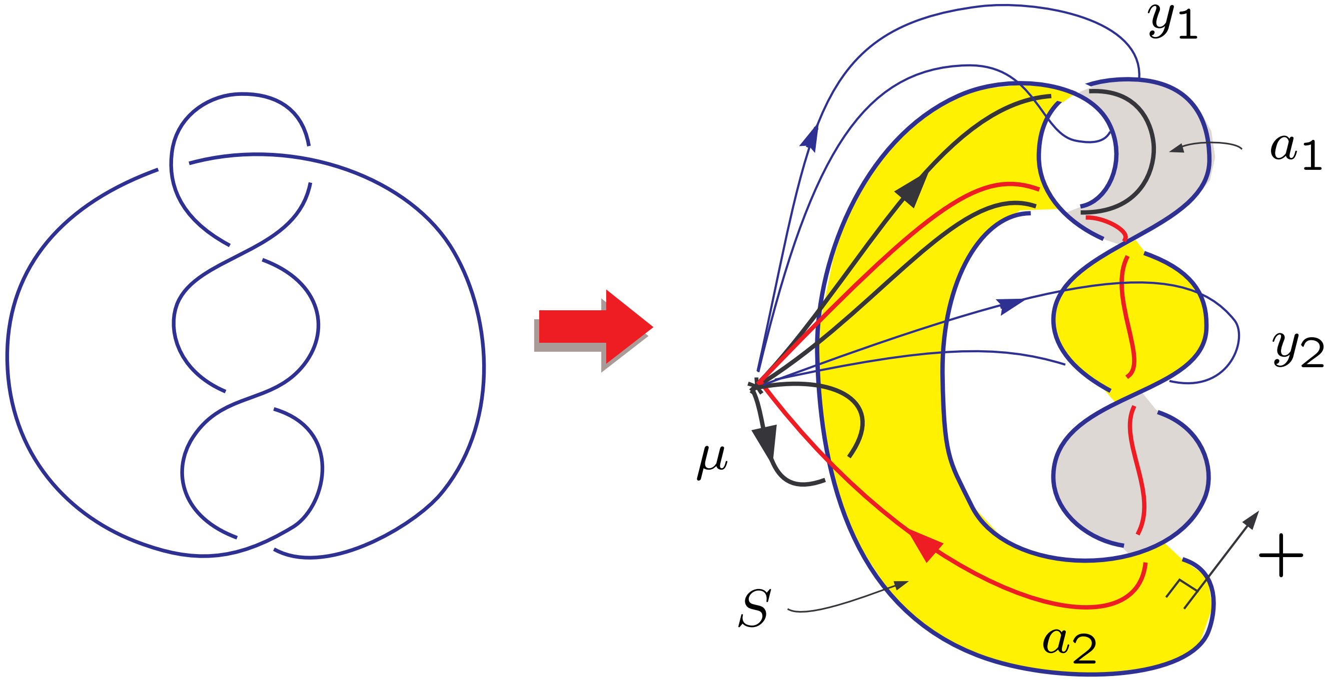
- Every **irr. metabelian repres** $\rho \in \mathcal{S}_0(5_2)$ satisfies

$$\big(\rho(y_1), \rho(y_2), \rho(\mu)\big) \sim \left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{bmatrix}, \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{bmatrix}, \begin{bmatrix} 0 & b \\ -b^{-1} & 0 \end{bmatrix} \right)$$

$$\sim \big(\iota(\rho)(y_1), \iota(\rho)(y_2), \iota(\rho)(\mu),\big)$$

EX.

- $\iota : \mathcal{R}(K) \to \mathcal{R}(K)$, $\iota(\rho)(g) = (-1)^{[g]}\rho(g)$ $([g] \in H_1(\mathbf{E}_K; \mathbb{Z}))$.
- Take a Seifert surface S via the Seifert algorithm.



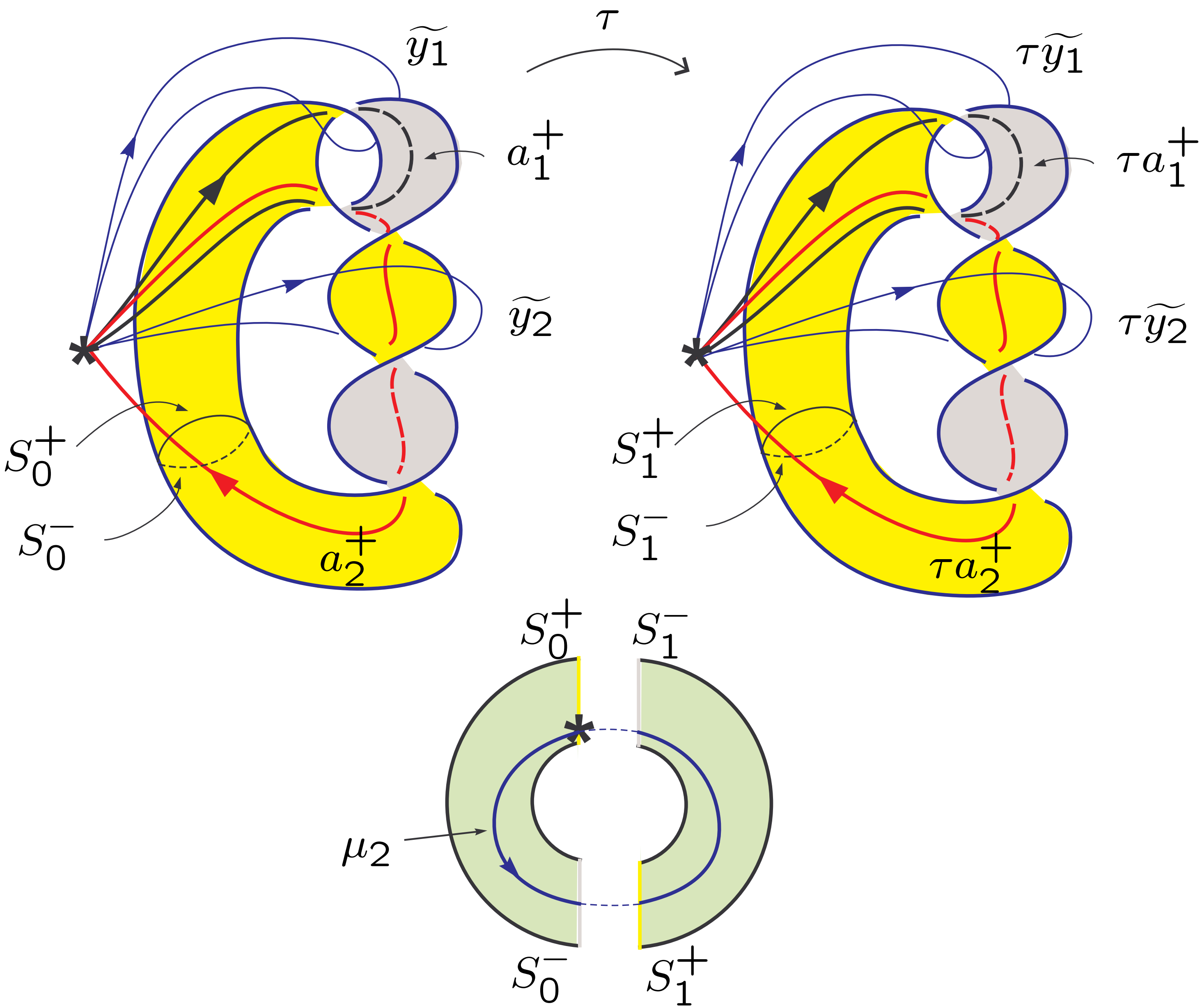
- $G(5_2) = \left\langle y_1, y_2, \mu \;\middle|\; \mu a_1^{+} \mu^{-1} = a_1^{-},\ \mu a_2^{+} \mu^{-1} = a_2^{-} \right\rangle$

  * $y_1,\ y_2 \in [G(5_2), G(5_2)]$ by **Lin's presentations** of $G(K)$

- By applying the argument for any knot K, we have

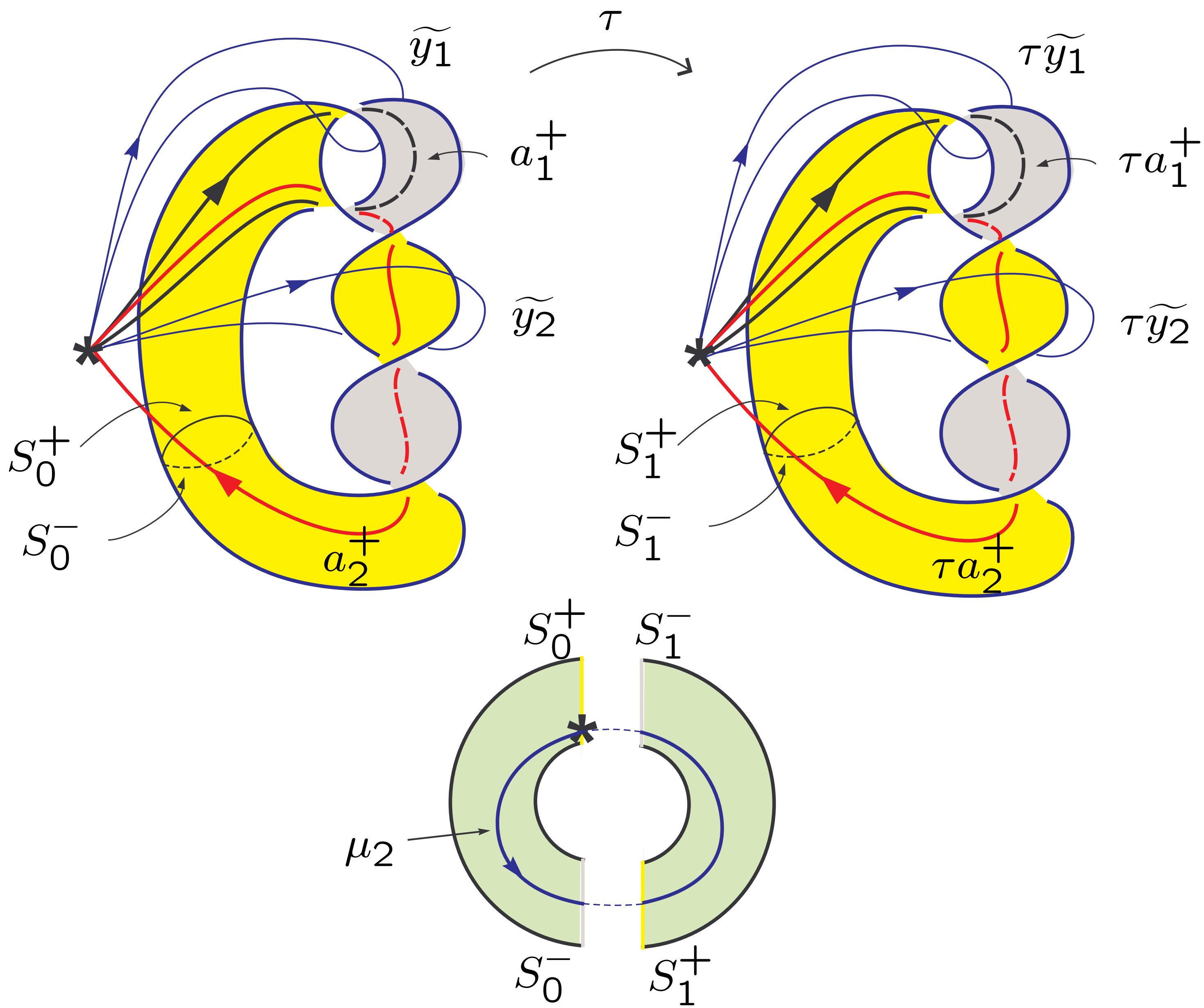
  $\mathrm{Fix}(\tilde{\iota}) = \{$the abelian and the irreducible metabelian characters$\}$

$\rightsquigarrow S_0(K) - \mathrm{Fix}(\tilde{\iota}) = \{$the irr. non-metabelian characters$\}$



- $\mathbf{C}_2K$: the 2-fold cyclic cover of $\mathbf{E}_K$

$$\pi_1(\mathbf{C}_2 5_2) = \left\langle \widetilde{y_1}, \widetilde{y_2}, \tau\widetilde{y_1}, \tau\widetilde{y_2}, \mu_2 \,\middle|\, \tau a_i^+ = a_i^-, \mu_2 a_i^+ \mu_2^{-1} = \tau a_i^- \ (i = 1, 2) \right\rangle$$



▶ $\pi_1(\Sigma_2 5_2) \cong \pi_1(\mathbf{C}_2 K)/\langle\langle \mu_2 \rangle\rangle$

$$= \left\langle \widetilde{y_1}, \widetilde{y_2}, \tau\widetilde{y_1}, \tau\widetilde{y_2} \,\middle|\, \tau a_i^+ = a_i^-, a_i^+ = \tau a_i^- \ \ (i = 1, 2) \right\rangle$$

▶ Define $\Phi : \mathcal{R}_0(K) := \{\rho \in \mathcal{R}(K) \mid \mathrm{tr}\rho(\mu) = 0\} \to \mathcal{R}(\Sigma_2 K)$

$$\Phi(\rho)(\gamma) := \sqrt{-1}^{\,p'_*[\gamma]}\rho(p_*\gamma), \quad \boxed{\gamma \in \pi_1(\mathbf{C}_2 K)}$$

where (1) $p_* : \pi_1(\mathbf{C}_2 K) \to G(K)$ (induced by the projection):

$$p_*(\mu_2) := \mu^2, \quad p_*(\tilde{y}_i) := y_i, \quad \boxed{p_*(\tau\tilde{y}_i) := \mu y_i \mu^{-1}}.$$

τ-equivariant

(2) $p'_* : H_1(\mathbf{C}_2 K;\mathbb{Z}) \to 2\mathbb{Z}\langle\mu\rangle \subset \mathbb{Z}\langle\mu\rangle = H_1(\mathbf{E}_K;\mathbb{Z})$

EX. $\Phi : \mathcal{R}_0(K) \to \mathcal{R}(\Sigma_2 K)$,

$$\begin{aligned} \Phi(\rho)(\mu_2) &= \sqrt{-1}^{\,p'_*[\mu_2]}\rho(p_*\mu_2) \\ &= \sqrt{-1}^{\,2}\rho(\mu^2) = -(-\mathbf{E}) = \mathbf{E} \ \textbf{(well-defined)} \end{aligned}$$

$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ = "$E^{\frac{1}{4}}$"

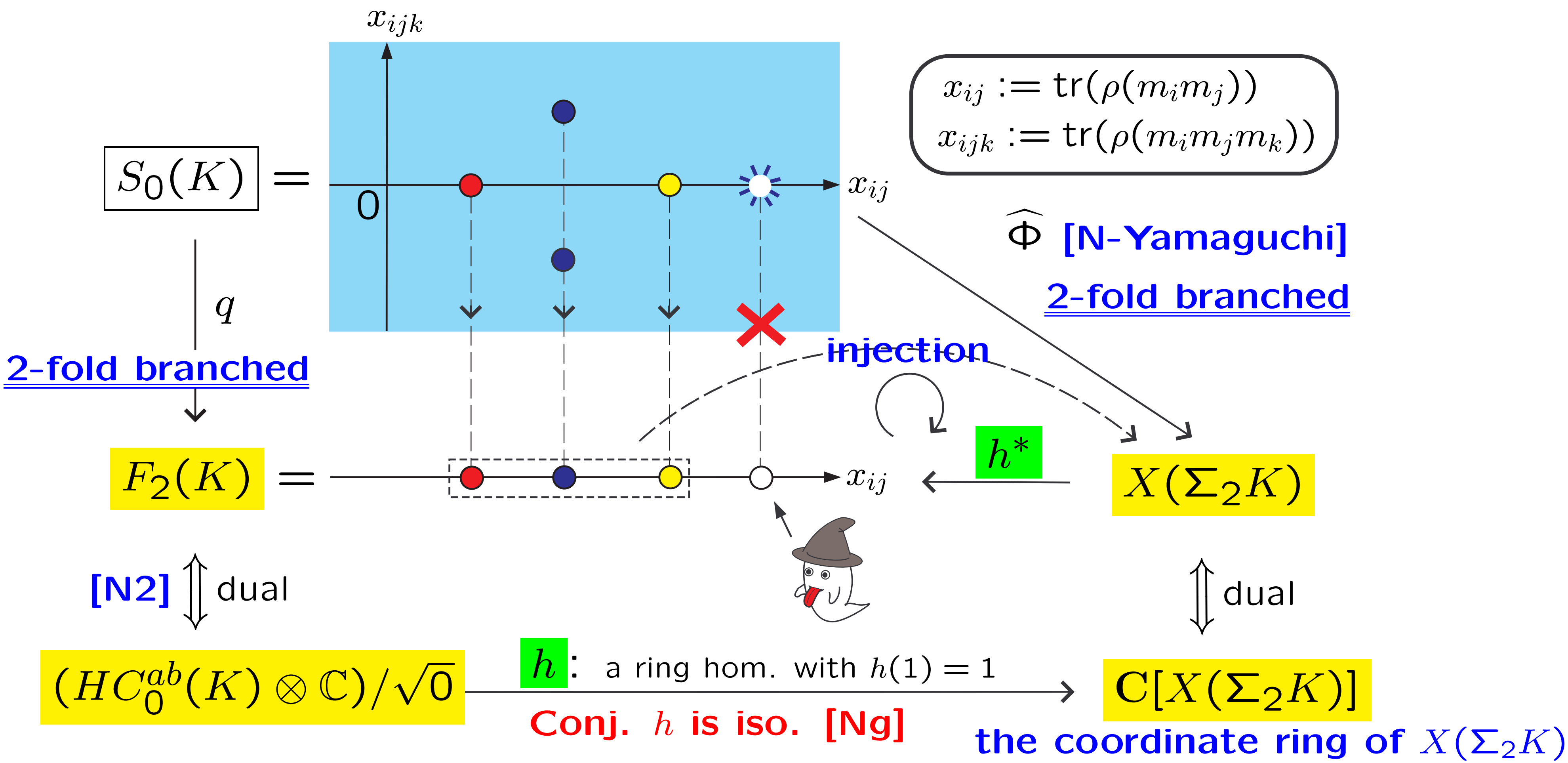
The map Φ gives $\widehat{\Phi} : \mathcal{S}_0(K) \to \mathcal{X}(\Sigma_2 K)$,

$$\widehat{\Phi}(\chi_\rho) := \chi_{\Phi(\rho)}$$

2-fold ⇝ "$E^{\frac{1}{2\cdot 2}}$"
n-fold ⇝ "$E^{\frac{1}{2n}}$"

*[N-Yamaguchi] constructs $\widehat{\Phi} : \mathcal{S}_c(K) \to X(\Sigma_n K)$ for $\forall n \in \mathbb{N}_{\geq 2}$.

S4. A deep inside $S_0(K)$ from $HC_0(K)$ and ghost characters



[N1] F. Nagasato, *Trace-free characters and abelian knot contact homology I*, available on ArXiv. (will be updated)

[N2] F. Nagasato, *Varieties via a filtration of the KBSM and knot contact homology*, Topology Appl. **264** (2019), 251-275.

[Ng] L. Ng, *Knot and braid invariants from contact homology II*, Geom. Topol. **9** (2005), 1603-1637.

Description of $\widehat{\Phi} : S_0(K) \to X(\Sigma_2 K)$ as a polynomial map

- For $G(K) = \langle m_1, \cdots, m_n \mid r_1, \cdots, r_n \rangle$ (m_i are **meridians** of K)
  - short exact sequence: $1 \to \pi_1(C_2K) \xrightarrow{p_*} G(K) \to \mathbb{Z}/2\mathbb{Z} \to 1$
  - coset decomposition: $G(K) = \mathrm{Im}(p_*) \cup \mathrm{Im}(p_*) m_1$

⇒ $\pi_1(C_2K) \cong \mathrm{Im}(p_*)$

the word given by interpreting r_j, $m_1 r_j m_1^{-1}$ with $m_1 m_i$'s, $m_i m_1^{-1}$'s

$$= \left\langle m_1 m_i, m_i m_1^{-1} (1 \le i \le n) \mid w(r_j), w(m_1 r_j m_1^{-1}) \ (1 \le j \le n) \right\rangle$$

- Taking the quotient of $\mathrm{Im}(p_*)$ by $\langle\langle p_* \mu_2 \rangle\rangle = \left\langle\left\langle m_1^2 \right\rangle\right\rangle$

⇒ $\pi_1(\Sigma_2 K) \cong \left\langle m_1 m_2, \cdots, m_1 m_n \mid w(r_j), w(m_1 r_j m_1^{-1}), m_j^2 \ (1 \le j \le n) \right\rangle$

- "**embedding**" $\widetilde{t} : \mathcal{X}(\Sigma_2 K) \to \mathbb{C}^{\binom{n}{2} + \binom{n-1}{3}}$,

$$t(\chi_{\rho_*}) = \left(t_{(m_1 m_i)}(\chi_{\rho_*}); t_{(m_1 m_i)(m_1 m_j)}(\chi_{\rho_*}); t_{(m_1 m_i)(m_1 m_j)(m_1 m_k)}(\chi_{\rho_*}) \right)$$

⇒ $\widetilde{t}(\chi_{\rho_*}) = \left(t_{(m_i m_j)}(\chi_{\rho_*}); t_{(m_1 m_i)(m_j m_k)}(\chi_{\rho_*}) \right)$ (by trace identity)

the coordinates for $X(\Sigma_2 K)$

- “**embedding**” $\widetilde{t} : \mathcal{X}(\Sigma_2 K) \to \mathbb{C}^{\binom{n}{2}+\binom{n-1}{3}}$,

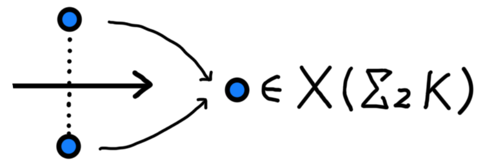
$$\widetilde{t}(\chi_{\rho_*}) = \left(t_{(m_i m_j)}(\chi_{\rho_*}); t_{(m_1 m_i)(m_j m_k)}(\chi_{\rho_*})\right)$$

- $\widehat{\Phi} : S_0(K) \to X(\Sigma_2 K)$ as **a polynomial map** is given by

$$\widehat{\Phi}((x_{ij}; x_{ijk})) := \left(t_{(m_i m_j)}\left(\chi_{\Phi(\rho)}\right); t_{(m_1 m_i)(m_j m_k)}\left(\chi_{\Phi(\rho)}\right)\right)$$

$$= \left(x_{ij}; \tfrac{1}{2}(x_{1i}x_{jk} + x_{1k}x_{ij} - x_{1j}x_{ik})\right)$$

- **1 to 1** on **the metabelian characters**
- **2 to 1** on **the non-metabelian characters**



- More arguments show that

$\widehat{\Phi}$ is **surjective** for 2- and 3-bridge knots.

That is, $X(\Sigma_2 K)$ does not have non **τ-equivariant characters** ($\nexists$ **proper representations** in $R(\Sigma_2 K)$) for there knots.

Definition of the map h^*

- By the "**embedding**" $\widetilde{t}: \mathcal{X}(\Sigma_2 K) \to \mathbb{C}^{\binom{n}{2}+\binom{n-1}{3}}$,

$$\widetilde{t}(\chi_{\rho*}) = \left(t_{(m_i m_j)}(\chi_{\rho*}); t_{(m_1 m_i)(m_j m_k)}(\chi_{\rho*}) \right)$$

the map $h^*: X(\Sigma_2 K) \to F_2(K)$ is defined as just a **projection**:

$$h^* \left(t_{(m_i m_j)}(\chi_{\rho*}); t_{(m_1 m_i)(m_j m_k)}(\chi_{\rho*}) \right) := \left(t_{(m_i m_j)}(\chi_{\rho*}) \right)$$

h^* is well-defined

Actually, $\left(t_{(m_i m_j)}(\chi_{\rho*}) \right)$ satisfies **(F2)**:

for any **Wirtinger triple** (i, j, k) and any $1 \le a \le n$,

⇒ $m_a m_k = (m_a m_i)(m_j m_i)$ and $m_\ell^2 = 1$ $(1 \le \ell \le n)$

⇒ $t_{m_a m_k}(\chi_{\rho*}) = t_{m_a m_i}(\chi_{\rho*}) t_{m_i m_j}(\chi_{\rho*}) - t_{m_a m_j}(\chi_{\rho*})$

⇒ $x_{ak} = x_{ai} x_{ij} - x_{aj}$ **(F2)**.

S5. Ghost characters as an obstructions for $F_2(K) \cong X(\Sigma_2 K)$

Theorem [N1], [N2]

- K does not have  & $\widehat{\Phi} : S_0(K) \to X(\Sigma_2 K)$ is **surjective**.

  ⇒ $F_2(K) \cong X(\Sigma_2 K)$ (**Ng's conj is true**.)

- K has  $\in F_2(K)$ s.t. $(h^*)^{-1}\Big($  $\Big) \neq \emptyset$

  ⇒ $\widehat{\Phi} : S_0(K) \to X(\Sigma_2 K)$ is **not surjective**.

- K has  $\in F_2(K)$ s.t. $(h^*)^{-1}\Big($  $\Big) = \emptyset$

  ⇒ $h^* : X(\Sigma_2 K) \to F_2(K)$ is **not an iso.**

  (**Ng's conj does not hold**.)

[N1] F. Nagasato, *Trace-free characters and abelian knot contact homology I*, available on ArXiv. (will be updated)

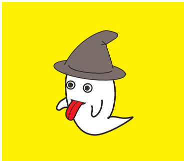
[N2] F. Nagasato, *Varieties via a filtration of the KBSM and knot contact homology*, Topology Appl. **264** (2019), 251-275.

S6. Finding ghost characters (using Maple)

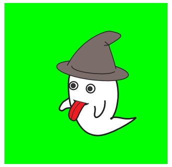
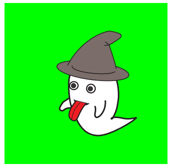
Theorem [NS1], [NS2]+α

- $\nexists$  & $\widehat{\Phi}$ is surjective for **2- and 3-bridge knots**

  ➡ $F_2(K) \cong X(\Sigma_2 K)$ (**Ng's conj is true**) for these knots.

- $T_{4,5}$ has a  and $(h^*)^{-1}(\;) \neq \emptyset$.

  ➡ $\widehat{\Phi}$ is **not surjective**. ($\exists$ a **proper** repres. in $R(\Sigma_2 T_{4,5})$)

- $T_{5,6}$ has , ,  and $(h^*)^{-1}(\;) = \emptyset$.

  ➡ $h^* : X(\Sigma_2) \to F_2(K)$ is **not an iso.**

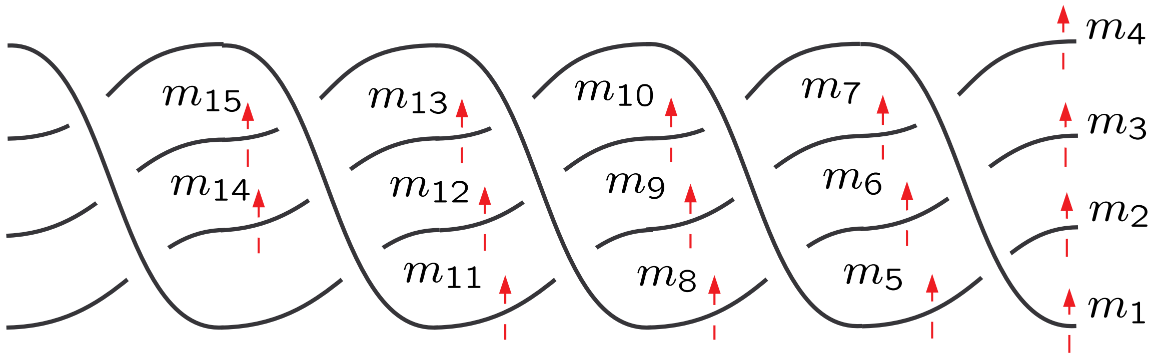
  (**a counter example of Ng's conj**)

[NS1] F. Nagasato and S. Suzuki: *Ghost characters and character varieties of 2-fold branched covers*, available on ArXiv. (will be updated)

[NS2] F. Nagasato and S. Suzuki: *Trace-free characters and abelian knot contact homology II*, available on ArXiv. (will be updated)

S7. Finding non τ-equivariant (proper) representation

EX. Let's take a look at the (4,5)-torus knot $T_{4,5}$:

$T_{4,5} =$ 

⇒ $G(T_{4,5}) = \langle m_1, m_2, m_3, m_4 \mid w_1, w_2, w_3 \rangle$,

where w_1, w_2 and w_3 are the following words:

$$w_1 = m_4m_1m_2m_3m_4m_2m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_4^{-1}m_1^{-1},$$
$$w_2 = m_4m_1m_2m_3m_4m_3m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_4^{-1}m_2^{-1},$$
$$w_3 = m_4m_1m_2m_3m_4m_3^{-1}m_2^{-1}m_1^{-1}m_4^{-1}m_3^{-1}.$$

⇒ $\pi_1(\Sigma_2T_{4,5}) \cong \langle m_1m_2, m_1m_3, m_1m_4 \mid w_i \ (1 \le i \le 6) \rangle$,

where $w_4 = m_1w_1m_1^{-1}, w_5 = m_1w_2m_1^{-1}, w_6 = m_1w_3m_1^{-1}$.

▶ $\exists\ \rho_* : \pi_1(\Sigma_2 T_{4,5}) \to \mathrm{SL}_2(\mathbb{C})$ satisfying

$$(\rho_*(m_1m_2), \rho_*(m_1m_3), \rho_*(m_1m_4))$$
$$= \left(\begin{pmatrix} e^{\frac{2}{3}\pi i} & 0 \\ 0 & e^{-\frac{2}{3}\pi i} \end{pmatrix}, \begin{pmatrix} -\frac{i}{\sqrt{3}}e^{\frac{\pi}{3}i} & -\frac{2}{3} \\ 1 & \frac{i}{\sqrt{3}}e^{-\frac{\pi}{3}i} \end{pmatrix}, \begin{pmatrix} \frac{i}{\sqrt{3}}e^{\frac{\pi}{3}i} & \frac{1+2\alpha}{3} \\ \alpha & -\frac{i}{\sqrt{3}}e^{-\frac{\pi}{3}i} \end{pmatrix} \right),$$

where α is a root of $2\alpha^2 + \alpha + 2 = 0$.

▶ Actually, the image $h^*(\chi_{\rho_*})$ gives  of $T_{4,5}$:

$$x_{12} = \mathrm{tr}\rho_*(m_1m_2) = 2\cos\left(\frac{2}{3}\pi\right) = -1,$$
$$x_{13} = \mathrm{tr}\rho_*(m_1m_3) = -\frac{i}{\sqrt{3}} \cdot 2i\sin\left(\frac{\pi}{3}\right) = 1,$$
$$x_{14} = \mathrm{tr}\rho_*(m_1m_4) = \frac{i}{\sqrt{3}} \cdot 2i\sin\left(\frac{\pi}{3}\right) = -1.$$

* This does not satisfies **the rectangle relation (R)**.

(**[NS]** F. Nagasato and S. Suzuki, *Ghost characters and character varieties of 2-fold branched covers*, available on ArXiv. (will be updated))

Remarks

212 *Matthew Hedden, Christopher M Herald and Paul Kirk*

*The pillowcase and perturbations of traceless representations of knot groups, Geometry & Topology 18 (2014) 211–287

a traceless matrix. For alternating knots, he showed that the Khovanov homology is isomorphic to the homology of the subvariety of binary dihedral representations.

In fact for alternating knots the Khovanov and singular instanton homology groups are isomorphic (with the Khovanov bigrading appropriately collapsed to a $\mathbb{Z}/4$ grading), a fact implied by the collapse of Kronheimer and Mrowka's spectral sequence at the E_2 page. In contrast, they show that there are nontrivial higher differentials in the spectral sequence associated to the $(4,5)$ torus knot [21, Section 11], and hence Khovanov and instanton homology do not have the same rank, in general. (Rasmussen noticed that the Khovanov homology of the $(4,5)$ torus knot also has larger rank than its Heegaard knot Floer homology groups. It is conjectured that there is a similar spectral sequence in that context.) Zentner [41] showed that for some alternating pretzel knots there are nonbinary dihedral traceless representations (in contrast to 2–bridge knots), so that for these families one expects there to be nontrivial differentials in the singular instanton chain complex.

I'm now speculating on how $F_2(K)$ and the ghost characters work for $HC_0(K)$ and these homologies.

THANK YOU!