proof $x \in I$, $y = f(x) \in J$ をとる. 定理 1.2 より.
$\exists \varphi_1(t)$, $\exists \varphi_2(t)$ $\exists \delta > 0$ s.t. $|h| < \delta$ の範囲で

$$f(x+h) = f(x) + f'(x)h + \varphi_1(h), \quad \cdots ①$$
$$g(y+h) = g(y) + g'(y)h + \varphi_2(h), \quad \cdots ②$$

$$\varphi_1(h) = o(h), \quad \varphi_2(h) = o(h).$$

一方, Δx が十分小さいとき. ① より

$$f(x+\Delta x) = f(x) + \underbrace{f'(x)(\Delta x) + \varphi_1(\Delta x)}_{= \Delta y}$$

とおくと.

$\Delta x \to 0$ のとき $\Delta y \to 0$ より. $|\Delta x|$ を十分小さくして.
$|\Delta y| < \delta$ とすることができる.

このとき

$$(g \circ f)(x+\Delta x) = g(f(x+\Delta x)) = g(f(x) + \Delta y)$$
$$= g(f(x)) + g'(f(x))\Delta y + \boxed{\Delta y \cdot \varepsilon}(\Delta y)$$
$$= g(f(x)) + g'(f(x))(f'(x)(\Delta x) + \varphi_1(\Delta x)) + (f'(x)(\Delta x) + \varphi_1(\Delta x)) \times \varepsilon(\Delta y)$$
$$= g(f(x)) + g'(f(x))f'(x)(\Delta x) + \underbrace{g'(f(x))\varphi_1(\Delta x)}_{③} + \underbrace{(f'(x)(\Delta x) + \varphi_1(\Delta x))\varepsilon(\Delta y)}_{④}.$$

ここで, 関数 $\varepsilon(h)$ を

$$\varepsilon(h) = \begin{cases} \dfrac{\varphi_2(h)}{h} & (h \neq 0) \\ 0 & (h = 0) \end{cases}$$

とおくと.

$$\begin{cases} \lim_{h \to 0} \varepsilon(h) = 0, \quad \varepsilon(0) = 0. \\ g(y+h) = g(y) + g'(y)h + h\varepsilon(h) \quad (|h| < \delta) \end{cases}$$

が成り立つ.