An alternative proof of Ivrii-Petkov’s necessary condition for $C^\infty$ well-posedness of the Cauchy problem

Seiichiro Wakabayashi

May 17, 2006

Let $P(x, \xi)$ be a polynomial of $\xi = (\xi_1, \cdots, \xi_n)$ whose coefficients are $C^\infty$ functions of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We write

$$P(x, \xi) = \sum_{j=0}^{m} P_j(x, \xi),$$

where $m = \deg \xi P(x, \xi)$ and $P_j(x, \xi)$ is a homogeneous polynomial of degree $j$. Let us consider the Cauchy problem

(CP) $\begin{cases} P(x, D)u(x) = f(x) & \text{in } \mathbb{R}^n, \\ \text{supp } u \subset \{ x \in \mathbb{R}^n; x_1 \geq 0 \}, \end{cases}$

where $D = (D_1, \cdots, D_n) = -i(\partial / \partial x_1, \cdots, \partial / \partial x_n)$ and $f \in C^\infty(\mathbb{R}^n)$ satisfies $\text{supp } f \subset \{ x \in \mathbb{R}^n; x_1 \geq 0 \}$. We say that the Cauchy problem (CP) is $C^\infty$ well-posed if the following two conditions are satisfied:

(E) For any $f \in C^\infty(\mathbb{R}^n)$ with $\text{supp } f \subset \{ x \in \mathbb{R}^n; x_1 \geq 0 \}$ there is $u \in C^\infty(\mathbb{R}^n)$ satisfying (CP).

(U) If $t > 0, u \in C^\infty(\mathbb{R}^n)$, $\text{supp } u \subset \{ x \in \mathbb{R}^n; x_1 \geq 0 \}$ and $\text{supp } P(x, D)u \subset \{ x \in \mathbb{R}^n; x_1 \geq t \}$, then $\text{supp } u \subset \{ x \in \mathbb{R}^n; x_1 \geq t \}$.

We assume that $P_m(x, \vartheta) \neq 0$, where $\vartheta = (1,0, \cdots, 0) \in \mathbb{R}^n$. Then $C^\infty$ well-posedness implies that $P_m(x, \xi)$ is hyperbolic with respect to $\vartheta$ for each $x \in \mathbb{R}^n$ with $x_1 \geq 0$, i.e., $P_m(x, \xi - i\vartheta) \neq 0$ for each $x \in \mathbb{R}^n$ with $x_1 \geq 0$ and $\xi \in \mathbb{R}^n$ (see [Mi]). Therefore, we assume that $P_m(x, \xi)$ is hyperbolic with respect to $\vartheta$ for each $x \in \mathbb{R}^n$ with $x_1 \geq 0$.

Ivrii and Petkov gave a necessary condition for $C^\infty$ well-posedness in [IP], and Ivrii improved the result and gave the following theorem in [I] (see, also Mandai [Ma]).
Theorem 1. Assume that the Cauchy problem (CP) is $C^\infty$ well-posed. Let $x^0 \in \mathbb{R}^n$ satisfy $x^0_j \geq 0$, and assume that there are $r \in \mathbb{Z}_+$ ($:= \mathbb{N} \cup \{0\}$) and $q_j \in \mathbb{Q}$ ($1 \leq j \leq n$) such that $q_j > 0$ ($1 \leq j \leq n$), $1 + q_1 > q_j$ ($2 \leq j \leq n$) and

\begin{align*}
P_m^{(r_1)}(x^0, e_n) \neq 0, \\
P_m^{(s)}(x^0, e_n) = 0 & \quad \text{if} \quad (1 + q_1)|\alpha| + \langle q, \beta - \alpha \rangle < r,
\end{align*}

where $e_j$ denotes the $n$-tuple vector whose $k$-th component is equal to $\delta_{j,k}$ ($1 \leq k \leq n$), $P_m^{(s)}(x, \xi) = \delta^{\alpha} D_{\xi}^\beta P(x, \xi)$, $q = (q_1, \cdots, q_n)$ and $\langle q, \beta - \alpha \rangle = \sum_{j=1}^n q_j (\beta_j - \alpha_j)$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n) \in (\mathbb{Z}_+)^n$. Then

\begin{align*}
P_m^{(s)}(x^0, e_n) = 0 & \quad \text{if} \quad (1 + q_1)(s + |\alpha|) + \langle q, \beta - \alpha \rangle < r.
\end{align*}

Remark. We note that $P_m^{(s)}(x^0, e_n) \neq 0$ if $\alpha = (\alpha', \alpha_n) \in (\mathbb{Z}_+)^n$, and $P_m^{(s)}(x^0, e_n) \neq 0$ where $P_m^{(s)}(x, \xi) = P_m^{(s)}(x^0, e_n)$. We shall prove the above theorem, repeating the same argument as in the first part of the proof in [Ma] and, then, applying the idea used in [W].

Now we assume that the hypotheses of Theorem 1 are fulfilled. From Banach's closed graph theorem or the Baire category theorem we have the following lemma (see, e.g., [IP]).

Lemma 2. Let $K$ be a compact subset of $\{x \in \mathbb{R}^n; x_1 \geq 0\}$. Then there are $\ell = \ell_K \in \mathbb{Z}_+$ and $C = C_K > 0$ such that

\[ |u(x^1)| \leq C \sup_{|\beta| \leq \ell} \sup_{x_1 \leq x^1} |D^\beta (P(x, D)u(x))| \]

if $x^1 \in K$, $u \in C^0_0(\mathbb{R}^n)$ and supp $u \subset K$.

Let $\delta > 0$. We make an asymptotic change of variables

\[ y = \rho^{\delta q}(x - x^0) = (\rho^{\delta q_1}(x_1 - x_1^0), \cdots, \rho^{\delta q_n}(x_n - x_0^0)) \quad (\rho \gg 1). \]

Put $P_\rho(y, \eta) = P(x^0 + \rho^{-\delta q} y, \rho^{\delta q} \eta)$. Then we have

\begin{align*}
P_\rho(y, \eta) = \sum_{0 \leq s \leq m, \alpha \in (\mathbb{Z}_+)^{n-1} \atop \beta \in (\mathbb{Z}_+)^n, \mu(s, \alpha, \beta) > -N} \rho^{\mu(s, \alpha, \beta)} \frac{y^\beta}{\alpha! \beta!} P_m^{(s)}(x^0, e_n) \eta^\alpha \eta_n^{m-s-|\alpha'|} + \rho^{-N} R_N(y, \eta; \rho) := Q_N(y, \eta; \rho) + \rho^{-N} R_N(y, \eta; \rho),
\end{align*}
where $N \in \mathbb{N}, \eta^\alpha = \eta_1^\alpha \cdots \eta_{n-1}^\alpha$ for $\alpha = (\alpha_1, \cdots, \alpha_{n-1}), q' = (q_1, \cdots, q_{n-1})$ and

$$
\mu(s, \alpha', \beta) = \delta \langle q', \alpha' - \beta' \rangle + \delta q_n(m - s - |\alpha'| - \beta_n).
$$

Write

$$
R_N(y, \eta; \rho) = \sum_{|\alpha| \leq m} R_{N, \alpha}(y; \rho) \eta^\alpha.
$$

Then for any $N \in \mathbb{N}$ and $W \subseteq \mathbb{R}^n$ there are $C_{N,W, \beta} > 0 \ (\beta \in (\mathbb{Z}_+)^n)$ such that

$$
|R_{N, \alpha(\beta)}(y; \rho)| \leq C_{N,W, \beta} \quad \text{for } y \in W \text{ and } \rho \geq 1.
$$

Put

$$
\mathcal{M} = \{(s, \alpha', \beta); 1 \leq s \leq m, (1 + q_1)(s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle < r
$$

and

$$
p_{(\alpha')^*}(x^0, e_n) \neq 0\}.
$$

By assumption $\mathcal{M}$ is a finite set. Note that $(s, \alpha', \beta) \in \mathcal{M}$ if $\alpha = (\alpha', \alpha_n), 1 \leq s \leq m, (1 + q_1)(s + |\alpha|) + \langle q, \beta - \alpha \rangle < r$ and $p_{(\alpha')^*}(x^0, e_n) \neq 0$. So, in order to prove Theorem 1 it suffices to show that $\mathcal{M} = \emptyset$. Now suppose that $\mathcal{M} \neq \emptyset$. Define

$$
\varepsilon_0 = \max\{\varepsilon; \varepsilon > 0 \text{ and } (1 + q_1)(\varepsilon s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle = r
$$

for some $(s, \alpha', \beta) \in \mathcal{M}\} \quad (\beta > 1),
$$
$$
\mathcal{M}_0 = \{(s, \alpha', \beta) \in \mathcal{M}; (1 + q_1)(\varepsilon_0 s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle = r\}.
$$

Then we have

$$
\tilde{\mu}(s, \alpha', \beta) \equiv \mu(s, \alpha', \beta) + m - s - |\alpha'| + |\sigma| \alpha'
$$

$$
= \mu_0 + (r - |\alpha'|)(1 - \sigma - \delta(1 + q_1 - q_n)) + s\{\delta((1 + q_1)e_0 - q_n) - 1\}
$$

for $(s, \alpha', \beta) \in \mathcal{M}_0,$

where $\mu_0 = (\delta q_1 + \sigma)r + (1 + \delta q_n)(m - r)$. We choose $\delta > 0$ and $\sigma > 0$ so that

$$
\delta(1 + q_1 - q_n) = 1 - \sigma, \quad \delta((1 + q_1)e_0 - q_n) = 1,
$$

i.e.,

$$
\delta = ((1 + q_1)e_0 - q_n)^{-1}, \quad \sigma = (1 + q_1)(e_0 - 1)((1 + q_1)e_0 - q_n)^{-1}.
$$

Note that $0 < \sigma < 1$. By this choice we have the following:

(i) $\tilde{\mu}(s, \alpha', \beta) = \mu_0 \text{ for } (s, \alpha', \beta) \in \mathcal{M}_0.$

(ii) $\tilde{\mu}(s, \alpha', \beta) < \mu_0 \text{ for } (s, \alpha', \beta) \in \mathcal{M} \setminus \mathcal{M}_0.$
(iii) \( \hat{\mu}(s, \alpha', \beta) < \mu_0 \)
if \( 1 \leq s \leq m \) and \((1 + q_1)(s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle \geq r \).
(iv) \( \hat{\mu}(0, \alpha', \beta) = \mu_0 \) if \((1 + q_1)|\alpha'| + \langle q, \beta \rangle - \langle q', \alpha' \rangle = r \).
(v) \( \hat{\mu}(0, \alpha', \beta) < \mu_0 \) if \((1 + q_1)|\alpha'| + \langle q, \beta \rangle - \langle q', \alpha' \rangle > r \).

Put
\[
\mathcal{M}_1 = \mathcal{M}_0 \cup \{(0, \alpha', \beta); \ (1 + q_1)|\alpha'| + \langle q, \beta \rangle - \langle q', \alpha' \rangle = r \text{ and } P_{m(\beta)}(x^0, e_n) \neq 0 \}.
\]
Then there is \( \delta_0 > 0 \) such that for \( N \gg 1 \) and \( \gamma \in \mathbb{R}^n \setminus \{0\} \)
\[
Q_N(y, \gamma e_n + \rho^\sigma \eta; \rho) = \rho^{\mu_0} \{ r^{-|\alpha'| - s} \rho^\beta \}
\]
where
\[
\Phi(y, \eta'; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}_1} \frac{\gamma^{-|\alpha'| - s} \rho^\beta}{\alpha'|\beta|!} P_{m(\beta)}(x^0, e_n) \eta^\alpha'.
\]
Here \( R_N(y, \eta; \rho; \gamma) \) is a polynomial of \( (y, \eta, \gamma) \) and its coefficients are bounded for \( \rho \geq 1 \).

**Lemma 3.** (i) If \( (s, \alpha', \beta) \in \mathcal{M}_1 \), then \( s, \alpha', \beta) = (0, e_1', 0) \) or \( \alpha_1 + s < r \),
where \( e_1' = (1, 0, \ldots, 0) \in (\mathbb{Z}_+)^{n-1} \). (ii) There are \( \hat{\gamma} \in \mathbb{R}^n \), \( \hat{\eta}' \in \mathbb{C}^n \times (\mathbb{R}^{n-2} \setminus \{0\}) \) and \( \hat{\gamma} \in \mathbb{R}^n \setminus \{0\} \) such that \( \hat{\gamma}_1 > 0 \) if \( x_1 = 0 \), \( \mathrm{Im} \hat{\eta}_1 < 0 \) and \( \Phi(\hat{\gamma}, \hat{\eta}; \hat{\gamma}) = 0 \).

**Proof.** (i) Let \((0, \alpha', \beta) \in \mathcal{M}_1 \). Then we have
\[
\alpha_1 + \langle q, \beta \rangle + \sum_{j=2}^{n-1} (1 + q_1 - q_j) \alpha_j = r
\]
Since \( 1 + q_1 > q_j \ (2 \leq j \leq n - 1) \), we have \( \alpha_1 < r \) if \( \sum_{j=2}^{n-1} \alpha_j + |\beta| \neq 0 \). By assumption we have \((0, e_1', 0) \in \mathcal{M}_1 \). Moreover, if \((s, \alpha', \beta) \in \mathcal{M}_0 \), we have
\[
\alpha_1 + s \leq (1 + q_1)(s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle < r.
\]
(ii) Put
\[
\theta = \max\{(e_0 - 1)(1 + q_1)s/(r - \alpha_1); \ (s, \alpha', \beta) \in \mathcal{M}_0 \} \ ( > 0),
\]
\[
\mathcal{M}' = \{(s, \alpha', \beta) \in \mathcal{M}_1; \ \theta(r - \alpha_1) = (e_0 - 1)(1 + q_1)s\}.
\]
Note that
\[
(2) \quad (0, \alpha', \beta) \in \mathcal{M}' \ \text{if and only if} \ \alpha' = re_1' \text{ and } \beta = 0.
\]
For \( \omega \gg 1 \) we have
\[
\Phi(\omega^{-q} y, \omega^\hat{\eta}; \omega^{1+q_1} \gamma)
\]
Choose \( \varepsilon \) sufficiently small, then the equation
\[
\Phi_1(y, \eta'; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha'},
\]
where \( \tilde{\eta} = (q_1 + \theta, q_2, \ldots, q_n) \in \mathbb{R}^{n-1} \) and \( \nu(s, \alpha', \beta) = (1 + q_1)(r - |\alpha'| - s) - \langle q, \beta \rangle + \langle q', \alpha' \rangle + \theta \alpha_1 \). Since
\[
\nu(s, \alpha', \beta) = \left\{ \begin{array}{ll}
(q_1 + \theta) & \text{if } (s, \alpha', \beta) \in \mathcal{M}', \\
< (q_1 + \theta) & \text{if } (s, \alpha', \beta) \in \mathcal{M}_1 \setminus \mathcal{M}'.
\end{array} \right.
\]
we have
\[
\Phi_1(y, \eta; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha'},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
where \( \eta'' = (\eta_2, \ldots, \eta_{n-1}) \), \( \alpha' = (\alpha_1, \alpha'') \in (\mathbb{Z}_+)^{n-1} \) and \( \eta^{\alpha''} = \eta^{(0, \alpha'', 0)} \). It follows from the assertion (i) and (2) that
\[
\Phi_1(y, \eta, \gamma^{\alpha''}; \gamma) = \sum_{(s, \alpha', \beta) \in \mathcal{M}'} \frac{\gamma^{r-|\alpha'|-s} \eta}{\alpha'!} \Phi_{m-s}(\beta)(x^0, e_n) \eta^{\alpha''} \eta_1^{\alpha_1},
\]
that $1 \leq p \leq r$, $\Im \tau(\eta'') < 0$, $\Phi'(\eta')$ is a polynomial of $\eta_1$, $\Phi'(\tau''', \eta''') \neq 0$ and

$$\Phi(\tilde{\eta}, \eta_1, \eta'''; \tilde{\eta}) = (\eta_1 - \tau(\eta'''))^p \Phi(\eta').$$

Let $\varphi(x)$ be a solution of

$$\frac{\partial \varphi}{\partial y_1} = \tau(\nabla y_1 \varphi(y)), \quad \varphi(y_1, y'') = (y'' - \tilde{y}'') \cdot \eta''' + i|y'' - \tilde{y}''|^2,$$

in an open neighborhood $V$ of $\tilde{y}$, where $y = (y', y_1) = (y_1, y'', y_n) = (y_1, y'')$. We may assume that $x_0^1 + \rho^{-\delta q_1} y_1 > 0$ if $y \in V$ and $\rho \geq 1$. Up to this point the proof is the same as in [Ma]. It follows from §3 of Chapter VI of [T] that

$$\Phi(\tilde{y}, \rho^{-\sigma} D' ; \tilde{y})(\exp\{i \rho^\sigma \varphi(y)\} u(y))$$

$$= \exp\{i \rho^\sigma \varphi(y)\} \sum_{|\alpha'| \geq p} \Phi(\alpha')(\tilde{y}, \nabla y_1 \varphi(y); \tilde{y}) \mathfrak{H}(y, D'; \rho) u(y),$$

where $D' = (D_1, \ldots, D_{n-1})$, $D'' = D^2_{w_1} \cdots D^2_{w_{n-1}}$ and $\Psi(y, w') = \varphi(w', y_n) - \varphi(y) - (w' - y') \cdot (\nabla y_1 \varphi)(y)$. It is easy to see that

$$\mathfrak{H}(y, \eta'; \rho) = \rho^{-|\alpha'|} \eta^{\alpha'/\alpha'} + \sum_{\beta < \alpha'} \rho^{-|\beta'|} b_{\alpha', \beta}(y; \rho) \eta^\beta,$$

$$|b_{\alpha', \beta}(y; \rho)| \leq C_{\alpha', \beta} \rho^{-|\alpha'| - |\beta'| + ((|\alpha'| - |\beta'|)/2)} ,$$

$$b_{\alpha', \beta}(y; \rho) \equiv 0 \quad \text{if } |\alpha'| - |\beta'| = 1 \text{ and } \beta' < \alpha'$$

for $y \in V$, where $[a]$ denotes the largest integer $\leq a$. Now we make an asymptotic change of variables, again:

$$z_1 = \rho^{\sigma_1}(y_1 - \tilde{y}_1), \quad z'' = \rho^{\sigma/3}(y'' - \tilde{y}''),$$

where $\sigma/2 < \sigma_1 < \sigma$. Put $y(z; \rho) = \tilde{y} + (\rho^{-\sigma_1} z_1, \rho^{-\sigma/3} z'')$ and $\varphi(z; \rho) = \varphi(y(z; \rho))$. A simple calculation yields

$$\Phi(\tilde{y}, \rho^{-\sigma + \sigma_1} D_1, \rho^{-2\sigma/3} D'' ; \tilde{y})(\exp\{i \rho^\sigma \varphi(z; \rho)\} v(z))$$

$$= \exp\{i \rho^\sigma \varphi(z; \rho)\} \sum_{|\alpha'| \geq p} \Phi(\alpha')(\tilde{y}, \nabla y_1 \varphi)(y(z; \rho); \tilde{y})$$

$$\times \{\rho^{-|\alpha'| + \sigma_1 \alpha_1 + |\alpha''|/3} D''^2 v(z) / \alpha'!$$

$$+ \sum_{\beta' < \alpha'} \rho^{-|\beta'| + \sigma_1 \beta_1 + |\beta''|/3} b_{\alpha', \beta'}(y(z; \rho); \rho) D^{\beta'} v(z)\}$$

$$= \exp\{i \rho^\sigma \varphi(z; \rho)\} \rho^{-p(\sigma - \sigma_1)} \{\Phi^{(p')}(\tilde{y}, \tau(\eta'''); \tilde{y}) D^p v(z) / p!\}$$
\[ + \rho^{-\delta_1} \sum_{|\alpha'| \leq p_0} c_{\alpha'}(z; \rho) D^{\alpha'} \nu(z), \]

\[ |c_{\alpha'}(\beta; \rho)| \leq C_{\alpha', \beta} \]

for \( z \in \mathbb{R}^n \) with \( y(z; \rho) \in V \), where \( p_0 = \deg \eta \Phi(\tilde{y}, \eta'; \tilde{\gamma}) \) and \( \delta_1 = \min\{\sigma/3, \sigma - \sigma_1, 2\sigma_1 - \sigma, \sigma_1 - \sigma/3\} \). Indeed, we have

\[ p(\sigma - \sigma_1) - \sigma|\alpha'| + \alpha_1 \beta_1 + \alpha_1 \beta'' + \sigma/3 + \sigma(\alpha' - \beta')/2 \]

\[ \leq - (|\alpha'| - p)(\sigma - \sigma_1) - (|\alpha'| - \beta')(\sigma_1 - \sigma/2) - (\sigma_1 - \sigma/3)|\beta''| \]

\[ \leq - (2\sigma_1 - \sigma) \text{ if } |\alpha'| \geq p, \beta' < \alpha' \text{ and } |\beta'| \leq |\alpha'| - 2, \]

\[ p(\sigma - \sigma_1) - \sigma|\alpha'| + \alpha_1 \beta_1 + \sigma(\alpha'' + \sigma)/3 \]

\[ = - (|\alpha'| - p)(\sigma - \sigma_1) - (\sigma_1 - \sigma/3)|\alpha''|. \]

Put \( E(z, \rho) = \exp[i \tilde{y} \rho^{1-\sigma/3} z_n] \) and

\[ \tilde{P}_\rho(z, \zeta) = P(x_0 + \rho^{-\delta_2} y(z; \rho), \rho^{\delta_2} (\rho^{\alpha_1} \zeta_1, \rho^{\sigma/3} \zeta'')) \]

\[ = P_\rho(y(z; \rho), \rho^{\alpha_1} \zeta_1, \rho^{\sigma/3} \zeta''). \]

Then we can write

\[ \tilde{P}_\rho(z, \zeta) = \tilde{Q}_N(z, \rho^{\alpha_1} \zeta_1, \rho^{\sigma/3} \zeta''); \rho) + \rho^{-N} \tilde{R}_N(z, \rho^{\alpha_1} \zeta_1, \rho^{\sigma/3} \zeta''); \rho) \]

Here \( \tilde{R}_N(z, \zeta; \rho) = \sum_{|\alpha| \leq m} \tilde{R}_N, \alpha(z; \rho) \zeta^\alpha \) and for any \( W \in \mathbb{R}^n \) there are \( C_{N,W, \beta} > 0 \) \(( \beta \in (\mathbb{Z}_+)^n \) such that

\[ |\tilde{R}_N, \alpha(\beta)(z; \rho)| \leq C_{N,W, \beta} \text{ for } z \in W \text{ and } \rho \geq 1. \]

Let \( W \) be an open neighborhood of 0 in \( \mathbb{R}^n \), and choose \( \rho(W) \geq 1 \) so that \( y(z; \rho) \in V \) for \( z \in W \) and \( \rho \geq \rho(W) \). Then we have

\[ \tilde{P}_\rho(z, D)(E(z; \rho) \exp[i \rho^\sigma \varphi(z; \rho)] \nu(z)) \]

\[ = E(z; \rho) \{ \tilde{Q}_N(z, \gamma \rho \epsilon_n + (\rho^{\alpha_1} D_1, \rho^{\sigma/3} D'') \rho) \]

\[ + \rho^{-N} \tilde{R}_N(z, \gamma \rho \epsilon_n + (\rho^{\alpha_1} D_1, \rho^{\sigma/3} D'') \rho) \exp[i \rho^\sigma \varphi(z; \rho)] \nu(z) \}

for \( z \in W \) and \( \rho \geq \rho(W) \). By (1) we have

\[ \tilde{Q}_N(z, \gamma \rho \epsilon_n + (\rho^{\alpha_1} \eta; \rho) = \gamma^{\sigma-\rho} \rho^\mu_0 \{ \Phi(\tilde{y}, \eta'; \tilde{\gamma}) + \rho^{-\delta_0} \tilde{Q}_N(z, \eta; \rho) \}, \]

where \( \delta_0 = \min\{\delta_0, \sigma/3\} \). Since \( 0 < \sigma_1 < \sigma \), this, together with (3), gives

\[ \tilde{Q}_N(z, \gamma \rho \epsilon_n + (\rho^{\alpha_1} D_1, \rho^{\sigma/3} D'') \rho) \exp[i \rho^\sigma \varphi(z; \rho)] \nu(z) \]
\[
= \exp[i\rho^\sigma \phi(z;\rho)] \hat{\gamma}_m \rho^{-p(\sigma-\sigma_1)+\mu_0} \{a_0 D_\rho^\sigma v(z) / p! \\
+ \rho^{-\delta_1} \sum_{|\alpha'| \leq \rho_0} c_{\alpha'}(\tau) D_{\alpha'} v(z) + \rho^{-\delta_0+\mu_0} \hat{r}(z,\rho)v(z) \}
\]

for \( z \in W \) and \( \rho \geq \rho(W) \), where \( a_0 = \Phi(\tau') = (\hat{\gamma}, \tau(\eta^{0''}), \eta^{0''}) \) (\( \neq 0 \)). Here \( \hat{r}(z, \zeta; \rho) \) is a polynomial of \( \zeta \) of degree \( m \) and has the same properties for \( \rho \geq \rho(W) \) as \( R_N(z, \zeta; \rho) \). Now let us choose \( N \in \mathbb{N} \) and \( \sigma_1 > 0 \) so that

\[
N \geq \rho(\sigma-\sigma_1) - \mu_0 + m + \delta_1, \\
\sigma_1 = \max\{2\sigma/3, \sigma - \delta_0/(p+1)\}.
\]

From Lemma 2 it follows that there are \( \ell \in \mathbb{Z}_+ \) and \( C > 0 \) such that

\[
|u(0)| \leq C \sup_{|\beta| \leq \ell} |z_1| \leq 0 \rho(\delta_{\eta_1+\sigma_1} + \delta(\eta', \eta'') + \sigma) |D_{\beta} \hat{P}(z,\rho)|u(z)|
\]

for \( u \in C_0^\infty(W) \) and \( \rho \geq \rho(W) \). Since \( \delta_1 \leq \sigma - \sigma_1 \leq \delta_0 - p(\sigma - \sigma_1) \), we have

\[
\hat{P}(z,\rho)(E(z;\rho) \exp[i/\rho summary continues...
for $\rho \geq \rho(W)$. Then, by standard arguments we have

$$
\sup_{z_1 \leq 0, |\beta| \leq \ell} \rho^{(\delta q_1 + \sigma_1)\beta_1 + \delta(q''\beta'') + \sigma|\beta''|/3}
\times |D\beta \{ \bar{P}_\rho(z; D)(E(z; \rho) \exp[i\rho^\sigma \varphi(z; \rho)]u_M(z; \rho)) \} |
\leq C_M \rho^{v(\ell) - M\delta_1} \leq C_M \rho^{-1} \quad \text{if } M\delta_1 \geq 1 + v(\ell) \text{ and } \rho \geq \rho(W),
$$

where $v(\ell) = -p(\sigma - \sigma_1) + \delta(1 + q_1)(r + \ell) + \sigma r + \sigma_1 \ell + (1 + \delta q_n)(m - r)$. On the other hand, $u_M(0; \rho) = 1$, which contradicts (4). This proves Theorem 1.

References


[Ma] T. Mandai, General Levi conditions for weakly hyperbolic equations - An attempt to treat the degeneracy with respect to the space variables -, Publ. RIMS, Kyoto Univ. 22 (1986), 1–23.

